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Local separation, closedness and zero-dimensionality in quantale-valued reflexive spaces

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Abstract. In this paper, first, we introduce the category **Q-RRel** consisting of quantale-valued reflexive spaces and $\mathcal{Q}$-monotone mappings, and prove that it is a normalized topological category over **Set**, the category of sets and functions. Furthermore, we characterize explicitly each of local T_i, $i = 0, 1, 2$ and $PreT_2$ $\mathcal{Q}$-reflexive spaces and examine the relationships among them. Finally, we give the characterizations of (strongly) closed subsets and zero-dimensional objects in this category.

1. Introduction

Order theory is an area of mathematics which deals with different types of binary relations. These relations comprehend the instinctive concept of mathematical ordering and its related areas. Domain theory as a subject of order theory has major applications in computer science. It was firstly studied in the 1960s by Dana Scott and used to specify denotational semantics, especially for functional programming languages (cf. [28]). Therefore, it can be considered as an interface between computer science and mathematics.

In 1991, Baran [2] extended the classical separation axioms of topology to an arbitrary set-based topological category in terms of initial, final structures and discreteness. He defined these axioms first locally and then point free. For arbitrary set-based topological categories, the concepts of closedness and strong closedness is also presented by Baran [2, 3] and he used these notions to generalize some fundamental topological concepts to topological categories. Moreover, it is shown that they form suitable closure operators defined by Dikranjan and Giuli [15] in some considerable topological categories [6, 7, 9].

Zero-dimensionality for a topological space (X, τ) is defined as X has a basis comprising of clopen (both closed and open) sets [18] and it has been used to construct many useful classes of topological spaces (cf. [23]). Sierpinski [27] defined the zero-dimensional spaces, and this notion has been extended to an arbitrary topological category by Stine [29].

With the progress of lattice theory, distinct mathematical frameworks have been studied with lattice structures including lattice-valued topology [14], quantale-valued approach space [20, 21, 26], quantale-valued metric space [22], lattice-valued convergence space [19] and lattice-valued preordered space [14]. This motivates us to study local separation axioms, (strong) closedness and zero-dimensionality in quantale-valued reflexive spaces, which is a generalization of quantale-valued preordered spaces.

The purposes of this paper are stated below:

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(i) to introduce the category **Q-RRel** consisting of quantale-valued reflexive spaces and Q-monotone maps, and to show that it is a normalized topological category over **Set**,

(ii) to give the characterizations of each of local $\overline{T_0}$, T_0', T_1, $Pre\overline{T_2}$, $PreT_2'$, $\overline{T_2}$ and T_2' Q-reflexive spaces and to investigate how these characterizations are related, and

(iii) to characterize a closed point, (strongly) closed subsets and zero-dimensional objects in **Q-RRel**.

2. Preliminaries

Let $(\mathcal{L}, \leq)$ be a partially ordered set (poset). If all subsets of $\mathcal{L}$ have both infimum ($\bigwedge$) and supremum ($\bigvee$), then $(\mathcal{L}, \leq)$ is named as a *complete lattice*. The bottom and top elements are represented by $\bot$ and $\top$, respectively, for any complete lattice [21].

In a complete lattice $(\mathcal{L}, \leq)$, a *well-below relation* was defined by $\alpha \lhd \beta$ (α is well-below β) if for all $A \subseteq \mathcal{L}$ such that $\beta \leq \bigvee A$ there is $\delta \in A$ such that $\alpha \leq \delta$. Similarly, a *well-above relation* was defined by $\alpha \prec \beta$ (β is well-above α) if for all $A \subseteq \mathcal{L}$ such that $\bigwedge A \leq \alpha$ there exists $\delta \in A$ such that $\delta \leq \beta$. Moreover, a complete lattice $(\mathcal{L}, \leq)$ is named as a *completely distributive* provided that we have $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$ for any $\alpha \in \mathcal{L}$.

The triple $(\mathcal{L}, \leq, *)$ is named as a *quantale* if $(\mathcal{L}, \leq)$ is a complete lattice, $(\mathcal{L}, *)$ is a semi group, and the operation $*$ satisfies the following:

$$\beta * (\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I}(\beta * \alpha_i) \text{ and } (\bigvee_{i\in I} \alpha_i) * \beta = \bigvee_{i\in I}(\alpha_i * \beta)$$

for all $\alpha_i, \beta \in \mathcal{L}$, i.e., $*$ is distributive over arbitrary joins.

A quantale $(\mathcal{L}, \leq, *)$ is named as *integral* if for all $\alpha \in \mathcal{L}$, $\alpha * \top = \top * \alpha = \alpha$ and it is called *commutative* if $(\mathcal{L}, *)$ is a commutative semi group.

In this paper we consider only integral and commutative quantales, denoted by $Q = (\mathcal{L}, \leq, *)$ with completely distributive lattices $\mathcal{L}$.

A quantale $Q = (\mathcal{L}, \leq, *)$ is named as a *value quantale* if $(\mathcal{L}, \leq)$ is completely distributive lattice such that $\forall \alpha, \beta \lhd \top$, $\alpha \vee \beta \lhd \top$ [16]. Note that in a quantale $Q = (\mathcal{L}, \leq, *)$, if $e \in \mathcal{L}$ and $e \neq \top$, then e is named as a *prime element* provided that $\alpha \wedge \beta \leq e$ implies $\alpha \leq e$ or $\beta \leq e$ for all $\alpha, \beta \in \mathcal{L}$.

Definition 2.1. Let $A \neq \emptyset$ be a set and $\Re : A \times A \longrightarrow Q = (\mathcal{L}, \leq, *)$ be a quantale-valued map. The map $\Re$ is called an Q-reflexive relation on A if it satisfies the reflexivity, i.e., $\Re(x, x) = \top$ for all $x \in A$. The pair $(A, \Re)$ is called a Q-reflexive space.

Definition 2.2. A mapping $f : (A, \Re) \to (B, \Re')$ is called a *Q-monotone mapping* if $\Re(x, y) \leq \Re'(f(x), f(y))$ for all $x, y \in A$.

Definition 2.3. The category of quantale-valued reflexive spaces, **Q-RRel** has the pairs $(A, \Re)$ as objects, where $\Re$ is a quantale-valued reflexive relation on the set A, and has Q-monotone mappings as morphisms.

Example 2.4. (i) For $Q = (\{0, 1\}, \leq, \wedge)$, **Q-RRel** $\cong$ **RRel**, where **RRel** is the category of reflexive relation spaces and monotone maps.

(ii) For $Q = ([0, \infty], \geq, +)$ (Lawvere's quantale), **Q-RRel** $\cong$ ∞**pqsMet**, where ∞**pqsMet** is the category of extended pseudo-quasi-semi metric spaces and nonexpansive mappings [25].

(iii) For $Q = (\triangle^+, \leq, *)$ (distance distribution functions quantale defined in [21]), then **Q-RRel** $\cong$ **ProbpqsMet**, where **ProbpqsMet** is the category of probabilistic pseudo-quasi-semi metric spaces and nonexpansive mappings.

Note that for the quantale $Q = (\mathcal{L}, \leq, *)$, a Q-reflexive space $(A, \Re)$ is a Q-preordered space if $\Re(x, y) * \Re(y, z) \leq \Re(x, z)$ for all $x, y, z \in A$ (transitivity). In some literature, a Q-preordered space is often called an $\mathcal{L}$-continuity space if Q is a value quantale (cf. [16]), an $\mathcal{L}$-metric space (cf. [22]) and an $\mathcal{L}$-category (cf. [17]).

3. Topological construct of quantale-valued reflexive spaces

Definition 3.1. ([1]) A functor $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ is said to be topological or $\mathcal{E}$ is a topological category over **Set** provided that the following conditions hold:

1. $\mathcal{U}$ is amnestic and faithful, i.e., concrete.
2. $\mathcal{U}$ has small (i.e., set) fibers.
3. Each $\mathcal{U}$-source has an initial lift or equivalently, every $\mathcal{U}$-sink has a final lift.

Theorem 3.2. *The category* ***Q-RRel*** *is topological over* ***Set****.*

Proof. Let $\mathcal{U} : \mathbf{Q\text{-}RRel} \to \mathbf{Set}$ be a forgetful functor. It is clear that $\mathcal{U}$ is concrete and has small fibers. Now, we prove that each $\mathcal{U}$-source has an initial lift. Suppose A is a nonempty set, $\{(A_i, \Re_i)\}_{i\in I}$ is a collection of Q-reflexive spaces and $\{f_i : A \to \mathcal{U}((A_i, \Re_i)) = A_i\}_{i\in I}$ is any $\mathcal{U}$-source in **Set**. We define the Q-reflexive relation $\Re$ on A by

$$\Re(x, y) \quad = \quad \bigwedge_{i\in I} \Re_i(f_i(x), f_i(y))$$

for all $x, y \in A$. Reflexivity holds trivially and it follows that $(A, \Re)$ is a Q-reflexive space. Since $\Re(x, y) \leq \Re_i(f_i(x), f_i(y))$, $f_i : (A, \Re) \to (A_i, \Re_i)$ is a Q-monotone mapping for each $i \in I$.

Suppose $f : (B, \Re_B) \to (A, \Re)$ is a mapping, then we prove that f is a Q-monotone mapping if and only if $f_i \circ f$ is a Q-monotone mapping. The necessity is obvious since compositions of Q-monotone mappings are Q-monotone. Conversely, let $x, y \in B$. Then,

$$\begin{aligned} \Re_B(x, y) &\leq \bigwedge_{i\in I} \Re_i(f_i \circ f(x), f_i \circ f(y)) \\ &= \bigwedge_{i\in I} \Re_i(f_i(f(x)), f_i(f(y))) \\ &= \Re(f(x), f(y)). \end{aligned}$$

So f is a Q-monotone mapping. Hence, the source $\{f_i : (A, \Re) \to (A_i, \Re_i)\}_{i\in I}$ is initial in **Q-RRel**.

Consequently, the functor $\mathcal{U} : \mathbf{Q\text{-}RRel} \to \mathbf{Set}$ is topological. □

Definition 3.3. Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be a topological functor. If the subterminals, i.e., constant objects, have a unique structure, then $\mathcal{U}$ is said to be normalized.

Remark 3.4. The topological functor $\mathcal{U} : \mathbf{Q\text{-}RRel} \to \mathbf{Set}$ is normalized since there is only one Q-reflexive relation on a point and on the empty set.

Lemma 3.5. *Let $\{(A_i, \Re_i)\}_{i\in I}$ be a collection of Q-reflexive spaces. A source $\{f_i : (A, \Re) \to (A_i, \Re_i)\}_{i\in I}$ is initial in* ***Q-RRel*** *iff for all $x, y \in A$,*

$$\Re(x, y) \quad = \quad \bigwedge_{i\in I} \Re_i(f_i(x), f_i(y)).$$

Proof. The proof is given in the proof of Theorem 3.2. □

Lemma 3.6. *Let $(B, \Re)$ be a Q-reflexive space. An epimorphism $f : (B, \Re) \to (A, \Re')$ is final in* ***Q-RRel*** *iff for all $x, y \in A$,*

$$\Re'(x, y) = \bigvee_{\substack{z_1, z_2 \in B \\ f(z_1) = x \\ f(z_2) = y}} \Re(z_1, z_2).$$

Proof. It is easy to show that $\Re'$ is reflexive. So $(A, \Re')$ is a $\mathcal{Q}$-reflexive space. Also $f : (B, \Re) \to (A, \Re')$ is a $\mathcal{Q}$-monotone mapping since for all $t_1, t_2 \in B$,

$$\Re'(f(t_1), f(t_2)) = \bigvee_{\substack{z_1, z_2 \in B \\ f(z_1) = f(t_1) \\ f(z_2) = f(t_2)}} \Re(z_1, z_2) \geq \Re(t_1, t_2).$$

Suppose $g : (A, \Re') \to (C, \Re_C)$ is a mapping, then we show g is a $\mathcal{Q}$-monotone mapping if and only if $g \circ f$ is a $\mathcal{Q}$-monotone mapping. Necessity is obvious since compositions of $\mathcal{Q}$-monotone mappings are $\mathcal{Q}$-monotone. Conversely, let $x, y \in A$, Then,

$$\begin{aligned} \Re'(x, y) &= \bigvee_{\substack{z_1, z_2 \in B \\ f(z_1) = x \\ f(z_2) = y}} \Re(z_1, z_2) \\ &\leq \bigvee_{\substack{z_1, z_2 \in B \\ g(f(z_1)) = g(x) \\ g(f(z_2)) = g(y)}} \Re_C(g \circ f(z_1), g \circ f(z_2)) \\ &= \Re_C(g(x), g(y)). \end{aligned}$$

So g is a $\mathcal{Q}$-monotone mapping. Hence, the epimorphism $f : (B, \Re) \to (A, \Re')$ is final in **Q-RRel**. $\square$

Definition 3.7. For a topological functor $\mathcal{U} : \mathcal{E} \to$ **Set**, an object A in $\mathcal{E}$ is discrete if and only if each mapping $\mathcal{U}(A) \to \mathcal{U}(B)$ lifts to a mapping $A \to B$ for each object B in $\mathcal{E}$, and an object A in $\mathcal{E}$ is indiscrete if and only if each mapping $\mathcal{U}(B) \to \mathcal{U}(A)$ lifts to a mapping $B \to A$ for each object B in $\mathcal{E}$.

Lemma 3.8. *Let $A \neq \emptyset$ be a set and $x, y \in A$.*

*(i) The discrete $\mathcal{Q}$-reflexive relation $\Re_D$ on A in **Q-RRel** is defined by*

$$\Re_D(x, y) = \begin{cases} \top, & x = y, \\ \bot, & x \neq y. \end{cases}$$

*(ii) The indiscrete $\mathcal{Q}$-reflexive relation $\Re_I$ on A in **Q-RRel** is defined by*

$$\Re_I(x, y) = \top.$$

Lemma 3.9. (cf. [17, p. 181]) *Let $\{(A_i, \Re_i)\}_{i \in I}$ be a collection of $\mathcal{Q}$-reflexive spaces and $A = \coprod_{i \in I} A_i$. Define*

$$\Re((i, x), (j, y)) = \begin{cases} \Re_i(x, y), & i = j, \\ \bot, & i \neq j. \end{cases}$$

for all $(i, x), (j, y) \in A$. $(A, \Re)$ is the coproduct of $\mathcal{Q}$-reflexive spaces $\{(A_i, \Re_i)\}_{i \in I}$. Particularly, $\{c_i : (A_i, \Re_i) \to (A, \Re)\}_{i \in I}$ is final lift of $\{c_i : A_i \to A\}_{i \in I}$ for the canonical injection maps c_i.

4. Local separation properties in Q-RRel

Let X be a set, p be a point in X and $X \vee_p X$ be the *wedge product* of X at p [2], i.e., two distinct copies of X identified at p, or in other words, the pushout of $p : 1 \to X$ along itself, where 1 is the terminal object in **Set**. More clearly, if i_1 and $i_2 : X \to X \vee_p X$ specify the inclusion of X as the first and second component, respectively, then $i_1 p = i_2 p$ is the pushout diagram [7].

In the wedge $X \vee_p X$, a point x is represented as x_k if it lies in the k-th component for $k = 1, 2$.

Definition 4.1. ([2]) Let $X \vee_p X$ be the wedge product at p and X^2 be the cartesian product of X.

1. The *principal p-axis mapping*, $\mathcal{A}_p : X \vee_p X \to X^2$ is stated by $\mathcal{A}_p(x_1) = (x,p)$ and $\mathcal{A}_p(x_2) = (p,x)$.
2. The *skewed p-axis mapping*, $\mathcal{S}_p : X \vee_p X \to X^2$ is stated by $\mathcal{S}_p(x_1) = (x,x)$ and $\mathcal{S}_p(x_2) = (p,x)$.
3. The *fold mapping at p*, $\nabla_p : X \vee_p X \to X$ is stated by $\nabla_p(x_1) = x = \nabla_p(x_2)$.

Definition 4.2. ([2]) Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be topological functor, $X \in Ob(\mathcal{E})$ with $\mathcal{U}(X) = B$ and $p \in B$.

(i) X is $\overline{T_0}$ at p provided that the initial lift of the $\mathcal{U}$-source $\{\mathcal{A}_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor that is a left adjoint to $\mathcal{U}$.

(ii) X is T_0' at p provided that the initial lift of the $\mathcal{U}$-source $\{id : B \vee_p B \to \mathcal{U}(X \vee_p X) = B \vee_p B$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $X \vee_p X$ is the wedge in $\mathcal{E}$, i.e., the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \vee_p B\}$ where i_1, i_2 represent the canonical injections.

(iii) X is T_1 at p provided that the initial lift of the $\mathcal{U}$-source $\{\mathcal{S}_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.

(iv) X is $Pre\overline{T_2}$ at p provided that the initial lift of the $\mathcal{U}$-source $\mathcal{S}_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and the initial lift of the $\mathcal{U}$-source $\mathcal{A}_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ agree.

(v) X is $PreT_2'$ at p provided that the initial lift of the $\mathcal{U}$-source $\mathcal{S}_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \vee_p B\}$ agree.

(vi) X is $\overline{T_2}$ at p provided that X is $\overline{T_0}$ at p and $Pre\overline{T_2}$ at p.

(vii) X is T_2' at p provided that X is T_0' at p and $PreT_2'$ at p.

Remark 4.3. (i) Separation axioms $\overline{T_0}$ at p and T_1 at p are used to define the notions of (strong) closedness in arbitrary set-based topological categories [2, 3].

(ii) If the topological functor $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ is normalized, then each of $\overline{T_0}$ at p and T_1 at p implies T_0' at p ([4] Corollary 2.11).

(iii) Let $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ be normalized and $X \in Ob(\mathcal{E})$ with $p \in \mathcal{U}(X)$. If X is $Pre\overline{T_2}$ object at p, then X is $\overline{T_0}$ at p if and only if T_1 at p [4, 8, 12].

Theorem 4.4. *A $\mathcal{Q}$-reflexive space $(A, \mathfrak{R})$ is $\overline{T_0}$ at p iff for all $x \in A$ with $x \neq p$, $\mathfrak{R}(x,p) \wedge \mathfrak{R}(p,x) = \bot$.*

Proof. Firstly, suppose $(A, \mathfrak{R})$ is $\overline{T_0}$ at p and $x \in A$ with $x \neq p$. Let $\mathfrak{R}_D$ be the discrete $\mathcal{Q}$-reflexive relation on A and $\rho_i : A^2 \to A$ $(i = 1, 2)$ be the projection maps. For $x_1, x_2 \in A \vee_p A$,

$$\begin{aligned}
\mathfrak{R}(\rho_1\mathcal{A}_p x_1, \rho_1\mathcal{A}_p x_2) &= \mathfrak{R}(\rho_1(x,p), \rho_1(p,x)) = \mathfrak{R}(x,p)\\
\mathfrak{R}(\rho_2\mathcal{A}_p x_1, \rho_2\mathcal{A}_p x_2) &= \mathfrak{R}(\rho_1(x,p), \rho_1(p,x)) = \mathfrak{R}(p,x)\\
\mathfrak{R}_d(\nabla_p x_1, \nabla_p x_2) &= \mathfrak{R}_d(x,x) = \top
\end{aligned}$$

Since $(A, \mathfrak{R})$ is $\overline{T_0}$ and $x_1 \neq x_2$, by Definition 4.2 and Lemmas 3.5, 3.8,

$$\begin{aligned}
\bot &= \bigwedge \{\mathfrak{R}(\rho_i\mathcal{A}_p x_1, \rho_i\mathcal{A}_p x_2)_{(i=1,2)}, \mathfrak{R}_d(\nabla_p x_1, \nabla_p x_2)\}\\
&= \bigwedge \{\mathfrak{R}(x,p), \mathfrak{R}(p,x), \top\}
\end{aligned}$$

Hence, we get $\mathfrak{R}(x,p) \wedge \mathfrak{R}(p,x) = \bot$.

Conversely, let $\mathfrak{R}^*$ be the initial $\mathcal{Q}$-reflexive relation on $A \vee_p A$ induced by $\mathcal{A}_p : A \vee_p A \to \mathcal{U}(A^2, \mathfrak{R}^2) = A^2$ and $\nabla_p : A \vee_p A \to \mathcal{U}(A, \mathfrak{R}_D) = A$, where $\mathfrak{R}^2$ is the product structure on A^2 induced by the projection maps ρ_i for $i = 1, 2$.

Assume that the condition is true, i.e., $\mathfrak{R}(x,p) \wedge \mathfrak{R}(p,x) = \bot$ for all $x \in A$ with $x \neq p$. Let m and n be any points in $A \vee_p A$.

(i) If $m = n$, then $\mathfrak{R}^*(m,n) = \top$.

(ii) If $m \neq n$ and $\nabla_p m \neq \nabla_p n$, then $\mathfrak{R}_D(\nabla_p m, \nabla_p n) = \bot$. By Lemma 3.5,

$$\mathfrak{R}^*(m,n) = \bigwedge \{\mathfrak{R}(\rho_i\mathcal{A}_p m, \rho_i\mathcal{A}_p n)_{(i=1,2)}, \mathfrak{R}_D(\nabla_p m, \nabla_p n)\} = \bot$$

(iii) Suppose $m \neq n$ and $\nabla_p m = \nabla_p n$. It follows that $\nabla_p m = x = \nabla_p n$ for some points $x \in A$ with $x \neq p$. We must have $m = x_1$ and $n = x_2$ or $m = x_2$ and $n = x_1$ since $m \neq n$.

(a) If $m = x_1$ and $n = x_2$, then

$$\begin{aligned}\mathfrak{R}(\rho_1\mathcal{A}_p m, \rho_1\mathcal{A}_p n) &= \mathfrak{R}(x, p)\\ \mathfrak{R}(\rho_2\mathcal{A}_p m, \rho_2\mathcal{A}_p n) &= \mathfrak{R}(p, x)\\ \mathfrak{R}_D(\nabla_p m, \nabla_p n) &= \mathfrak{R}_D(x, x) = \top\end{aligned}$$

and it follows that

$$\begin{aligned}\mathfrak{R}^*(m, n) &= \bigwedge \{\mathfrak{R}(\rho_i\mathcal{A}_p m, \rho_i\mathcal{A}_p n)_{(i=1,2)}, \mathfrak{R}_D(\nabla_p m, \nabla_p n)\}\\ &= \bigwedge \{\mathfrak{R}(x, p), \mathfrak{R}(p, x), \top\}\\ &= \mathfrak{R}(x, p) \wedge \mathfrak{R}(p, x)\end{aligned}$$

By the assumption, $\mathfrak{R}(x, p) \wedge \mathfrak{R}(p, x) = \perp$, we obtain $\mathfrak{R}^*(m, n) = \perp$.

(b) Similarly, if $m = x_2$ and $n = x_1$, then $\mathfrak{R}^*(m, n) = \perp$.

Consequently, for all m, n in the wedge $A \vee_p A$, we have

$$\mathfrak{R}^*(m, n) = \begin{cases}\top, & m = n\\ \perp, & m \neq n\end{cases}$$

By Lemma 3.8, $\mathfrak{R}^*$ is the discrete $\mathcal{Q}$-reflexive relation on $A \vee_p A$. Hence, by Definition 4.2, $(A, \mathfrak{R})$ is $\overline{T_0}$ at p. □

Theorem 4.5. *All $\mathcal{Q}$-reflexive spaces are T_0' at p.*

Proof. Let $(A, \mathfrak{R})$ be a $\mathcal{Q}$-reflexive space, $\mathfrak{R}'$ be the final $\mathcal{Q}$-reflexive relation on $A \vee_p A$ induced by $i_1, i_2 : \mathcal{U}(A, \mathfrak{R}) = A \to A \vee_p A$, where i_1 and i_2 are the canonical injection maps and $\overline{\mathfrak{R}}$ be the initial structure on $A \vee_p A$ induced by $id : A \vee_p A \to \mathcal{U}(A \vee_p A, \mathfrak{R}') = A \vee_p A$ and $\nabla_p : A \vee_p A \to \mathcal{U}(A, \mathfrak{R}_D) = A$, where id is the identity map and $\mathfrak{R}_D$ be the discrete $\mathcal{Q}$-reflexive relation on A.

By Definition 4.2, we need to show that $\overline{\mathfrak{R}}$ is discrete. Let m and n be any points in $A \vee_p A$.

(i) If $m = n$, then $\overline{\mathfrak{R}}(m, n) = \top$.

(ii) If $m \neq n$ and $\nabla_p m \neq \nabla_p n$, then $\mathfrak{R}_D(\nabla_p m, \nabla_p n) = \perp$, and by Lemma 3.5,

$$\overline{\mathfrak{R}}(m, n) = \bigwedge \{\mathfrak{R}'(m, n), \mathfrak{R}_D(\nabla_p m, \nabla_p n)\} = \mathfrak{R}'(m, n) \wedge \perp = \perp$$

(iii) Suppose $m \neq n$ and $\nabla_p m = \nabla_p n$. It follows that we must have $m = x_1$ and $n = x_2$ or $m = x_2$ and $n = x_1$ for some $x \in A$.

If $m = x_1$ and $n = x_2$, then by Lemma 3.5,

$$\begin{aligned}\overline{\mathfrak{R}}(m, n) &= \bigwedge \{\mathfrak{R}'(m, n), \mathfrak{R}_D(\nabla_p m, \nabla_p n)\}\\ &= \bigwedge \{\mathfrak{R}'(m, n), \top\}\\ &= \mathfrak{R}'(m, n)\\ &= \bigvee \{\mathfrak{R}(x, x) \mid \text{there exists } k \in \{1, 2\} \text{ such that } i_k(x) = x_1 = m \text{ and } i_k(x) = x_2 = n\}\end{aligned}$$

This implies that m and n have to be in the same component of $A \vee_p A$ which means $x = p$, i.e., $m = n$. So, the case $m = x_1$ and $n = x_2$ can not occur.

If $m = x_2$ and $n = x_1$, then similarly we have $m = n$. Thus, this case also can not occur.

Hence, by Lemma 3.8, $\overline{\mathfrak{R}}$ is discrete, and by Definition 4.2, $(A, \mathfrak{R})$ is T_0' at p. □

Theorem 4.6. *A $\mathcal{Q}$-reflexive space $(A, \mathfrak{R})$ is T_1 at p iff $\mathfrak{R}(x, p) = \perp = \mathfrak{R}(p, x)$ for all $x \in A$ with $x \neq p$.*

Proof. Suppose that $(A, \mathfrak{R})$ is T_1 at p and $x \in A$ with $x \neq p$. Let $m = x_1$, $n = x_2 \in A \vee_p A$. Note that

$$\begin{aligned} \mathfrak{R}(\rho_1 \mathcal{S}_p m, \rho_1 \mathcal{S}_p n) &= \mathfrak{R}(\rho_1(x,x), \rho_1(p,x)) = \mathfrak{R}(x,p) \\ \mathfrak{R}(\rho_2 \mathcal{S}_p m, \rho_2 \mathcal{S}_p n) &= \mathfrak{R}(\rho_2(x,x), \rho_2(p,x)) = \mathfrak{R}(x,x) = \top \\ \mathfrak{R}_D(\nabla_p m, \nabla_p n) &= \mathfrak{R}_D(x,x) = \top, \end{aligned}$$

where $\mathfrak{R}_D$ is the discrete $\mathcal{Q}$-reflexive relation on A and for each $i = 1, 2$, $\rho_i : A^2 \to A$ is the projection map. Since $m \neq n$ and $(A, \mathfrak{R})$ is T_1 at p, by Definition 4.2 and Lemmas 3.5, 3.8,

$$\begin{aligned} \bot &= \bigwedge \{\mathfrak{R}(\rho_i \mathcal{S}_p m, \rho_i \mathcal{S}_p n)_{(i=1,2)}, \mathfrak{R}_D(\nabla_p m, \nabla_p n)\} \\ &= \bigwedge \{\mathfrak{R}(x,p), \top\} = \mathfrak{R}(x,p) \end{aligned}$$

Similarly, if $m = x_2$, $n = x_1 \in A \vee_p A$, then

$$\bot = \bigwedge \{\mathfrak{R}(\rho_i \mathcal{S}_p m, \rho_i \mathcal{S}_p n)_{(i=1,2)}, \mathfrak{R}_D(\nabla_p m, \nabla_p n)\} = \mathfrak{R}(p,x)$$

Conversely, let $\mathfrak{R}^*$ be the initial $\mathcal{Q}$-reflexive relation on $A \vee_p A$ induced by $\mathcal{S}_p : A \vee_p A \to \mathcal{U}(A^2, \mathfrak{R}^2) = A^2$ and $\nabla_p : A \vee_p A \to \mathcal{U}(A, \mathfrak{R}_D) = A$, where $\mathfrak{R}^2$ is the product structure on A^2 induced by the projection maps ρ_i for $i = 1, 2$.

Suppose for all $x \in A$ with $x \neq p$, $\mathfrak{R}(x,p) = \bot = \mathfrak{R}(p,x)$. Let m and n be any points in $A \vee_p A$.

(i) If $m = n$, then $\mathfrak{R}^*(m,n) = \top$.

(ii) If $m \neq n$ and $\nabla_p m \neq \nabla_p n$, then $\mathfrak{R}_D(\nabla_p m, \nabla_p n) = \bot$ since $\mathfrak{R}_D$ is the discrete structure on A. By Lemma 3.5,

$$\mathfrak{R}^*(m,n) = \bigwedge \{\mathfrak{R}(\rho_i \mathcal{S}_p m, \rho_i \mathcal{S}_p n)_{(i=1,2)}, \mathfrak{R}_D(\nabla_p m, \nabla_p n)\} = \bot$$

(iii) Suppose $m \neq n$ and $\nabla_p m = \nabla_p n$. It follows that we must have $m = x_1$ and $n = x_2$ or $m = x_2$ and $n = x_1$. If $m = x_1$ and $n = x_2$, then by Lemma 3.5,

$$\begin{aligned} \mathfrak{R}^*(m,n) &= \bigwedge \{\mathfrak{R}(\rho_i \mathcal{S}_p x_1, \rho_i \mathcal{S}_p x_2)_{(i=1,2)}, \mathfrak{R}_D(\nabla_p x_1, \nabla_p x_2)\} \\ &= \bigwedge \{\mathfrak{R}(x,p), \top\} = \mathfrak{R}(x,p) \end{aligned}$$

By the assumption, $\mathfrak{R}(x,p) = \bot = \mathfrak{R}(p,x)$, we obtain $\mathfrak{R}^*(m,n) = \bot$.

Similarly, we get $\mathfrak{R}^*(m,n) = \bot$ for $m = x_2$ and $n = x_1$.

Hence, for all $m, n \in A \vee_p A$, we have

$$\mathfrak{R}^*(m,n) = \begin{cases} \top, & m = n \\ \bot, & m \neq n \end{cases}$$

By Lemma 3.8, it follows that $\mathfrak{R}^*$ is the discrete $\mathcal{Q}$-reflexive relation on $A \vee_p A$. Consequently, by Definition 4.2, $(A, \mathfrak{R})$ is T_1 at p. □

Remark 4.7. (i) In **Top** (the category of topological spaces and continuous mappings), $\overline{T_0}$ at p and T_0' at p (resp. T_1 at p) reduce to if for each $x \neq p$, there exists a neighborhood of x doesn't contain p or (resp. and) there exists a neighborhood of p doesn't contain x [2].

(ii) By Theorems 4.4, 4.5 and 4.6, if a $\mathcal{Q}$-reflexive space $(A, \mathfrak{R})$ is $\overline{T_0}$ at p or T_1 at p, then it is T_0' at p. But in general, the converse is not true. This is also a result of Remark 4.3 (ii).

Theorem 4.8. *A $\mathcal{Q}$-reflexive space $(A, \mathfrak{R})$ is $Pre\overline{T_2}$ at p iff the following conditions are satisfied.*

(I) For all $x \in A$ with $x \neq p$, $\mathfrak{R}(x,p) \wedge \mathfrak{R}(p,x) = \mathfrak{R}(x,p) = \mathfrak{R}(p,x)$.

(II) For any two distinct points $x, y \in A$ with $x \neq p \neq y$,
$\mathfrak{R}(x,p) \wedge \mathfrak{R}(p,y) = \mathfrak{R}(x,p) \wedge \mathfrak{R}(x,y) = \mathfrak{R}(p,y) \wedge \mathfrak{R}(x,y) = \mathfrak{R}(y,p) \wedge \mathfrak{R}(y,x) = \mathfrak{R}(p,x) \wedge \mathfrak{R}(y,x)$.

Proof. Suppose that $(A, \Re)$ is $Pre\overline{T_2}$ at p and $x \in A$ with $x \neq p$. Let $\rho_k : A^2 \to A$, $k = 1, 2$ be the projection maps and $m = x_1, n = x_2 \in A \vee_p A$. By Definition 4.2, we have

$$\begin{aligned}
\bigwedge\{\Re(\rho_k \mathcal{A}_p m, \rho_k \mathcal{A}_p n)_{(k=1,2)}\} &= \bigwedge\{\Re(\rho_k \mathcal{S}_p m, \rho_k \mathcal{S}_p n)_{(k=1,2)}\} \\
\bigwedge\{\Re(x, p), \Re(p, x)\} &= \bigwedge\{\Re(x, p), \top\} \\
\Re(x, p) \wedge \Re(p, x) &= \Re(x, p)
\end{aligned}$$

Similarly, if $m = x_2, n = x_1$, then we have $\Re(x, p) \wedge \Re(p, x) = \Re(p, x)$. Hence, $\Re(x, p) \wedge \Re(p, x) = \Re(x, p) = \Re(p, x)$.

Suppose x, y are any two distinct points of A and $x \neq p \neq y$. Let $m = x_i$, $n = y_j$ or $m = x_j$, $n = y_i$, and $i, j = 1, 2$ with $i \neq j$. Since $(A, \Re)$ is $Pre\overline{T_2}$ at p and by Definition 4.2, we have

$$\begin{aligned}
\bigwedge\{\Re(\rho_k \mathcal{A}_p m, \rho_k \mathcal{A}_p n)_{(k=1,2)}\} &= \bigwedge\{\Re(\rho_k \mathcal{S}_p m, \rho_k \mathcal{S}_p n)_{(k=1,2)}\} \\
\bigwedge\{\Re(x, p), \Re(p, y)\} &= \bigwedge\{\Re(x, p), \Re(x, y)\} \ \text{(for } m = x_1, n = y_2) \\
\bigwedge\{\Re(p, y), \Re(x, p)\} &= \bigwedge\{\Re(p, y), \Re(x, y)\} \ \text{(for } m = x_2, n = y_1) \\
\bigwedge\{\Re(y, p), \Re(p, x)\} &= \bigwedge\{\Re(y, p), \Re(y, x)\} \ \text{(for } m = y_1, n = x_2) \\
\bigwedge\{\Re(p, x), \Re(y, p)\} &= \bigwedge\{\Re(p, x), \Re(y, x)\} \ \text{(for } m = y_2, n = x_1)
\end{aligned}$$

and by the condition (I) ($\Re(x, p) = \Re(p, x), \Re(y, p) = \Re(p, y)$), it follows that $\Re(x, p) \wedge \Re(p, y) = \Re(x, p) \wedge \Re(x, y) = \Re(p, y) \wedge \Re(x, y) = \Re(y, p) \wedge \Re(y, x) = \Re(p, x) \wedge \Re(y, x)$.

Conversely, assume that the conditions are true. We prove that $(A, \Re)$ is $Pre\overline{T_2}$ at p. Let $\Re_{\mathcal{A}_p}$ and $\Re_{\mathcal{S}_p}$ be two initial structures on $A \vee_p A$ induced by $\mathcal{A}_p : A \vee_p A \to \mathcal{U}(A^2, \Re^2) = A^2$ and $\mathcal{S}_p : A \vee_p A \to \mathcal{U}(A^2, \Re^2) = A^2$ respectively, and $\Re^2$ be the product structure on A^2 induced by the projection maps $\rho_k : A^2 \to A$ for $k = 1, 2$. We need to show that $\Re_{\mathcal{A}_p} = \Re_{\mathcal{S}_p}$.

First, note that $\Re_{\mathcal{A}_p}$ and $\Re_{\mathcal{S}_p}$ are symmetric at p by the assumption (I).

Suppose m and n are any two points in $A \vee_p A$.

If $m = n$, then $\Re_{\mathcal{A}_p}(m, n) = \top = \Re_{\mathcal{S}_p}(m, n)$.

If $m \neq n$ and they are in the same component of the wedge $A \vee_p A$, i.e., $m = x_i$ and $n = y_i$ for $i = 1, 2$, then

$$\begin{aligned}
\Re_{\mathcal{A}_p}(m, n) &= \bigwedge\{\Re(\rho_k \mathcal{A}_p m, \rho_k \mathcal{A}_p n)_{(k=1,2)}\} \\
&= \bigwedge\{\Re(x, y), \Re(p, p) = \top\} \\
&= \bigwedge\{\Re(\rho_k \mathcal{S}_p m, \rho_k \mathcal{S}_p n)_{(k=1,2)}\} \\
&= \Re_{\mathcal{S}_p}(m, n)
\end{aligned}$$

Suppose $m \neq n$ and they are in the different factor of the wedge $A \vee_p A$. We have the following cases for m and n:

Case I: $m = x_1$ and $n = x_2$ or $m = x_2$ and $n = x_1$ for all $x \in A$ with $x \neq p$.

If $m = x_1$ and $n = x_2$, then for $k = 1, 2$,

$$\Re_{\mathcal{A}_p}(m, n) = \bigwedge\{\Re(\rho_k \mathcal{A}_p x_1, \rho_k \mathcal{A}_p x_2)\} = \Re(x, p) \wedge \Re(p, x),$$

$$\Re_{\mathcal{S}_p}(m, n) = \bigwedge\{\Re(\rho_k \mathcal{S}_p x_1, \rho_k \mathcal{S}_p x_2)\} = \Re(x, p).$$

By the assumption (I), it follows that $\Re_{\mathcal{A}_p}(m, n) = \Re_{\mathcal{S}_p}(m, n)$.

Similarly, if $m = x_2$ and $n = x_1$, then by the assumption (I), we get $\Re_{\mathcal{A}_p}(m, n) = \Re_{\mathcal{S}_p}(m, n)$.

Case II: $m = x_i$, $n = y_j$ or $m = x_j$, $n = y_i$, where x, y are any two distinct points of A with $x \neq p \neq y$, and $i, j = 1, 2$ with $i \neq j$.

If $m = x_1$ and $n = y_2$ (resp. $m = x_2$ and $n = y_1$), then for $k = 1, 2$,

$$\mathfrak{R}_{\mathcal{A}_p}(m,n) = \bigwedge\{\mathfrak{R}(\rho_k\mathcal{A}_p x_1, \rho_k\mathcal{A}_p y_2)\} = \mathfrak{R}(x,p) \wedge \mathfrak{R}(p,y) \ \text{(resp. } \mathfrak{R}(p,y) \wedge \mathfrak{R}(x,p)),$$

$$\mathfrak{R}_{\mathcal{S}_p}(m,n) = \bigwedge\{\mathfrak{R}(\rho_k\mathcal{S}_p x_1, \rho_k\mathcal{S}_p y_2)\} = \mathfrak{R}(x,p) \wedge \mathfrak{R}(x,y) \ \text{(resp. } \mathfrak{R}(p,y) \wedge \mathfrak{R}(x,y)).$$

By the assumption (II), it follows that $\mathfrak{R}_{\mathcal{A}_p}(m,n) = \mathfrak{R}_{\mathcal{S}_p}(m,n)$.

Similarly, if $m = y_1$ and $n = x_2$ (resp. $m = y_2$ and $n = x_1$), then for $k = 1, 2$,

$$\mathfrak{R}_{\mathcal{A}_p}(m,n) = \bigwedge\{\mathfrak{R}(\rho_k\mathcal{A}_p y_1, \rho_k\mathcal{A}_p x_2)\} = \mathfrak{R}(y,p) \wedge \mathfrak{R}(p,x) \ \text{(resp. } \mathfrak{R}(p,x) \wedge \mathfrak{R}(y,p)),$$

$$\mathfrak{R}_{\mathcal{S}_p}(m,n) = \bigwedge\{\mathfrak{R}(\rho_k\mathcal{S}_p y_1, \rho_k\mathcal{S}_p x_2)\} = \mathfrak{R}(y,p) \wedge \mathfrak{R}(y,x) \ \text{(resp. } \mathfrak{R}(p,x) \wedge \mathfrak{R}(y,x)).$$

By the assumption (II), it follows that $\mathfrak{R}_{\mathcal{A}_p}(m,n) = \mathfrak{R}_{\mathcal{S}_p}(m,n)$.

Hence, we obtain $\mathfrak{R}_{\mathcal{A}_p}(m,n) = \mathfrak{R}_{\mathcal{S}_p}(m,n)$ for any points $m, n \in A \vee_p A$, and by Lemma 3.5 and Definition 4.2, $(A, \mathfrak{R})$ is $Pre\overline{T_2}$ at p. □

Theorem 4.9. *A Q-reflexive space $(A, \mathfrak{R})$ is $PreT_2'$ at p iff for all $x \in A$ with $x \neq p$, $\mathfrak{R}(x,p) = \bot = \mathfrak{R}(p,x)$.*

Proof. Assume that $(A, \mathfrak{R})$ is $PreT_2'$ at p and $x \in A$ with $x \neq p$. Let $\rho_k : A^2 \to A$, $k = 1, 2$ be the projection maps and $\mathfrak{R}'$ be the final Q-reflexive relation on $A \vee_p A$ induced by $i_1, i_2 : \mathcal{U}(A, \mathfrak{R}) = A \to A \vee_p A$, where i_1 and i_2 are the canonical injection maps. For $m = x_1$, $n = x_2 \in A \vee_p A$, by Definition 4.2, note that

$$\begin{aligned} \mathfrak{R}'(m,n) &= \bigwedge \{\mathfrak{R}(\rho_1\mathcal{S}_p m, \rho_1\mathcal{S}_p n), \mathfrak{R}(\rho_2\mathcal{S}_p m, \rho_2\mathcal{S}_p n)\} \\ &= \bigwedge \{\mathfrak{R}(x,p), \mathfrak{R}(x,x) = \top\} = \mathfrak{R}(x,p) \end{aligned}$$

and since m and n are in the different factor of the wedge $A \vee_p A$, it follows from Lemmas 3.6 and 3.9 that $\mathfrak{R}'(m,n) = \mathfrak{R}(x,p) = \bot$.

Similarly, for $m = x_2$, $n = x_1 \in A \vee_p A$, then

$$\mathfrak{R}'(m,n) = \bigwedge \{\mathfrak{R}(\rho_k\mathcal{S}_p x_2, \rho_k\mathcal{S}_p x_1)_{(k=1,2)}\} = \mathfrak{R}(p,x)$$

and it follows that $\mathfrak{R}'(m,n) = \mathfrak{R}(p,x) = \bot$ by Lemmas 3.6 and 3.9.

Conversely, let $\mathfrak{R}_{\mathcal{S}_p}$ be the initial Q-reflexive relation on $A \vee_p A$ induced by $\mathcal{S}_p : A \vee_p A \to \mathcal{U}(A^2, \mathfrak{R}^2) = A^2$, where $\mathfrak{R}^2$ is the product structure on A^2 induced by the projection maps ρ_k for $k = 1, 2$.

Suppose that $\mathfrak{R}(x,p) = \bot = \mathfrak{R}(p,x)$ for all $x \in A$ with $x \neq p$. We prove that $(A, \mathfrak{R})$ is $PreT_2'$ at p, i.e., $\mathfrak{R}' = \mathfrak{R}_{\mathcal{S}_p}$. Let m and n be any points in $A \vee_p A$.

If $m = n$, then $\mathfrak{R}'(m,n) = \top = \mathfrak{R}_{\mathcal{S}_p}(m,n)$.

Suppose that $m \neq n$ and they are in the same component of the wedge $A \vee_p A$. If $m = x_i$ and $n = y_i$ for $x, y \in A$ and $i = 1, 2$, then by Lemmas 3.5 and 3.6,

$$\begin{aligned} \mathfrak{R}_{\mathcal{S}_p}(m,n) &= \bigwedge\{\mathfrak{R}(\rho_k\mathcal{S}_p m, \rho_k\mathcal{S}_p n)_{(k=1,2)}\} \\ &= \bigwedge\{\mathfrak{R}(x,y), \mathfrak{R}(p,p) = \top\} = \mathfrak{R}(x,y) \\ \mathfrak{R}'(m,n) &= \bigvee \{\mathfrak{R}(x,y) \mid i_k(x) = x_k,\ i_k(y) = y_k : k = 1, 2\} = \mathfrak{R}(x,y) \end{aligned}$$

Hence, we get $\mathfrak{R}'(m,n) = \mathfrak{R}(x,y) = \mathfrak{R}_{\mathcal{S}_p}(m,n)$.

Suppose $m \neq n$ and they are in the different factor of the wedge $A \vee_p A$. We have the following cases for m and n:

Case I: $m = x_1$ and $n = x_2$ or $m = x_2$ and $n = x_1$ for all $x \in A$ with $x \neq p$.

If $m = x_1$ and $n = x_2$ (resp. $m = x_2$ and $n = x_1$), then by the assumption,

$$\mathfrak{R}_{\mathcal{S}_p}(m,n) = \bigwedge\{\mathfrak{R}(\rho_k\mathcal{S}_p x_1, \rho_k\mathcal{S}_p x_2)_{(k=1,2)}\} = \mathfrak{R}(x,p) \ (\textit{resp.}\ \mathfrak{R}(p,x)) = \bot$$

and by Lemma 3.9, $\Re'(m,n) = \perp$ since m and n are in the different factor of the wedge. It follows that $\Re'(m,n) = \Re_{\mathcal{S}_p}(m,n)$.

Case II: $m = x_i$, $n = y_j$ or $m = x_j$, $n = y_i$, where x, y are any two distinct points of A with $x \neq p \neq y$, and $i, j = 1, 2$ with $i \neq j$.

If $m = x_1$ and $n = y_2$, then by the assumption,

$$\begin{aligned}\Re_{\mathcal{S}_p}(m,n) &= \bigwedge\{\Re(\rho_k\mathcal{S}_p m, \rho_k\mathcal{S}_p n)_{(k=1,2)}\}\\ &= \bigwedge\{\Re(x,p) = \perp, \Re(x,y)\} = \perp\end{aligned}$$

and by Lemma 3.9, $\Re'(m,n) = \perp$ since m and n are in the different factor of the wedge. Thus, $\Re'(m,n) = \Re_{\mathcal{S}_p}(m,n)$.

Similarly, if $m = y_1$ and $n = x_2$ or $m = x_2$ and $n = y_1$ or $m = y_2$ and $n = x_1$, then by the assumption and Lemma 3.9, we have $\Re'(m,n) = \Re_{\mathcal{S}_p}(m,n)$.

Hence, for any points $m, n \in A \vee_p A$ we obtain $\Re'(m,n) = \Re_{\mathcal{S}_p}(m,n)$, and by Lemmas 3.5, 3.6 and Definition 4.2, $(A, \Re)$ is $PreT_2'$ at p. □

Theorem 4.10. *A $\mathcal{Q}$-reflexive space $(A,\Re)$ is $\overline{T_2}$ (resp. T_2') at p iff for all $x \in A$ with $x \neq p$, $\Re(x,p) = \perp = \Re(p,x)$.*

Proof. It follows from Definition 4.2 and Theorems 4.4, 4.5, 4.8 and 4.9. □

Theorem 4.11. *Let $(A,\Re)$ be a $\mathcal{Q}$-reflexive space and $p \in A$. Then the following are equivalent.*

(i) $(A,\Re)$ *is* $\overline{T_2}$ *at* p *for all* $p \in A$.
(ii) $(A,\Re)$ *is* T_2' *at* p *for all* $p \in A$.
(iii) $\Re(x,p) = \perp = \Re(p,x)$ *for all* $x, p \in A$ *with* $x \neq p$.
(iv) $\Re$ *is the discrete* $\mathcal{Q}$*-reflexive relation, i.e.,* $\Re = \Re_D$.

Proof. It follows from Lemma 3.8 and Theorem 4.10. □

Remark 4.12. (i) In **Top**, $Pre\overline{T_2}$ at p is equivalent to $PreT_2'$ at p and they both reduce to for each point x with $x \neq p$, there exist disjoint neighborhoods of x and p, if the set $\{x,p\}$ is not indiscrete [2]. Moreover, $\overline{T_2}$ at p is equivalent to T_2' at p and they both reduce to classical Hausdorff condition at p [2].

(ii) For an arbitrary topological category $\mathcal{E}$ with B an object in $\mathcal{E}$, the constant map at p, $p\colon X \to X$ is called a retract map if there exists a map $r\colon X \to X$ such that the composition $rp = id$, the identity map on X [4]. If $p\colon X \to X$ is a retract map, then by Theorem 2.6 of [4] and Theorem 3.1 of [5], $PreT_2'$ at p implies $Pre\overline{T_2}$ at p but the reverse implication is not true, in general [11].

(iii) In **Q-RRel**, by Theorems 4.6, 4.9 and 4.10, we have T_1 at p, $PreT_2'$ at p, $\overline{T_2}$ at p and T_2' at p are equivalent.

(iv) Local separation axioms for the category ∞**pqsMet** given in [11, 12] are the special forms of our results. For example, if we take quantale $([0,\infty], \geq, +)$, then Theorems 4.4, 4.5 (resp. Theorem 4.10) reduce to Theorem 4 (resp. Theorem 6) of [12] and Theorem 4.8 (resp. Theorem 4.9) reduces to Theorem 4 (resp. Theorem 5) of [11].

Corollary 4.13. *Let $(A,\Re)$ be a $Pre\overline{T_2}$ $\mathcal{Q}$-reflexive space at a point $p \in A$. Then the following are equivalent.*

(i) $(A,\Re)$ *is* $\overline{T_0}$ *at* p.
(ii) $(A,\Re)$ *is* T_1 *at* p.
(iii) $(A,\Re)$ *is* $PreT_2'$ *at* p.
(iv) $(A,\Re)$ *is* $\overline{T_2}$ *at* p.
(v) $(A,\Re)$ *is* T_2' *at* p.

5. Closedness and strong closedness

Let X be a set, p be a point in X and $\vee_p^\infty X$ be the *infinite wedge product* of X at p, that is formed by taking countably separate copies of X and identifying them at p.

In the infinite wedge $\vee_p^\infty X$, a point x is represented as x_i if it lies in the i-th component.

Definition 5.1. ([3]) Let $\vee_p^\infty X$ be the infinite wedge product at p and $X^\infty = X \times X \times ...$ be the countable cartesian product of X.

(i) The *infinite principle axis map at p*, $A_p^\infty : \vee_p^\infty X \longrightarrow X^\infty$ is stated by $A_p^\infty(x_i) = (p, p, ..., p, x, p, ...)$.

(ii) The *infinite fold map at p*, $\nabla_p^\infty : \vee_p^\infty X \longrightarrow X^\infty$ is stated by $\nabla_p^\infty(x_i) = x$ for all $i \in I$.

Note that the map $\mathcal{A}_p^\infty$ is the unique map arising from the multiple pushout of $p : 1 \to X$ for which $\mathcal{A}_p^\infty i_j = (p, p, ..., p, id, p, ...) : X \to X^\infty$, where the identity map, id, is in the j-th place [7].

Definition 5.2. ([2, 3]) Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be a topological functor, $X \in Ob(\mathcal{E})$ with $\mathcal{U}(X) = B$ and $p \in B$. Let F be a subset of B. We denote the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/F = (B\backslash F) \cup \{*\}$ by X/F, where q is the epi map that is the identity on $B\backslash F$ and identifying F with a point $\{*\}$.

(i) $\{p\}$ is *closed* provided that the initial lift of the $\mathcal{U}$-source $\{A_p^\infty : \vee_p^\infty B \to \mathcal{U}(X^\infty) = B^\infty$ and $\nabla_p^\infty : \vee_p^\infty B \to \mathcal{U}\mathcal{D}(B^\infty) = B^\infty\}$ is discrete, where $\mathcal{D}$ is the discrete functor.

(ii) $F \subset X$ is closed provided that $\{*\}$, the image of F, is closed in X/F or $F = \emptyset$.

(iii) $F \subset X$ is strongly closed provided that X/F is T_1 at $\{*\}$ or $F = \emptyset$.

Remark 5.3. In ***Top***, the notion of closedness coincides with the usual one [2] and F is strongly closed provided that F is closed and for each $x \notin F$ there exists a neighbourhood of F missing x. For T_1 topological spaces, the notions of closedness and strong closedness coincide [2].

Theorem 5.4. *Let $(A, \mathfrak{R})$ be a $\mathcal{Q}$-reflexive relation space and $p \in A$. p is closed in A iff for all $x \in A$ with $x \neq p$, $\mathfrak{R}(x, p) \wedge \mathfrak{R}(p, x) = \bot$.*

Proof. Let $(A, \mathfrak{R})$ is a $\mathcal{Q}$-reflexive space, $p \in A$ and $\overline{\mathfrak{R}}$ be the initial $\mathcal{Q}$-reflexive relation on $\vee_p^\infty A$ induced by $\mathcal{A}_p^\infty : \vee_p^\infty A \to \mathcal{U}(A^\infty, \mathfrak{R}^*) = A^\infty$ and $\nabla_p^\infty : \vee_p^\infty A \to \mathcal{U}(A, \mathfrak{R}_d) = A$ where $\mathfrak{R}_d$ is the discrete $\mathcal{Q}$-reflexive relation on A, and $\mathfrak{R}^*$ be the product $\mathcal{Q}$-reflexive relation on A^∞ induced by $\rho_i : A^\infty \to A$ $(i \in I)$ projection maps.

Suppose that $\{p\}$ is closed in A. We prove that for all $x \in X$ with $x \neq p$, $\mathfrak{R}(x, y) \wedge \mathfrak{R}(y, x) = \bot$. Note that for $i, j, k \in I$ with $i \neq j$ and $i \neq k \neq j$,

$$\begin{aligned}
\mathfrak{R}(\rho_i \mathcal{A}_p^\infty(x_i), \rho_i \mathcal{A}_p^\infty(x_j)) &= \mathfrak{R}(x, p) \\
\mathfrak{R}(\rho_j \mathcal{A}_p^\infty(x_i), \rho_j \mathcal{A}_p^\infty(x_j)) &= \mathfrak{R}(p, x) \\
\mathfrak{R}(\rho_k \mathcal{A}_p^\infty(x_i), \rho_k \mathcal{A}_p^\infty(x_j)) &= \mathfrak{R}(p, p) = \top \\
\mathfrak{R}_d(\nabla_p^\infty(x_i), \nabla_p^\infty(x_j)) &= \mathfrak{R}_d(x, x) = \top
\end{aligned}$$

Since p is closed in A and $x_i \neq x_j$ $(i \neq j)$, by Definition 5.2 and Lemma 3.5, we have

$$\begin{aligned}
\bot = \overline{\mathfrak{R}}(x_i, x_j) &= \bigwedge \{\mathfrak{R}(\rho_h \mathcal{A}_p^\infty(x_i), \rho_h \mathcal{A}_p^\infty(x_j))_{(h=i,j,k)}, \mathfrak{R}_d(\nabla_p^\infty(x_i), \nabla_p^\infty(x_j))\} \\
&= \bigwedge \{\mathfrak{R}(x, p), \mathfrak{R}(p, x), \top\} \\
&= \mathfrak{R}(x, p) \wedge \mathfrak{R}(p, x)
\end{aligned}$$

and hence, $\mathfrak{R}(x, p) \wedge \mathfrak{R}(p, x) = \bot$.

Conversely, assume that the condition is true, i.e., $\mathfrak{R}(x, p) \wedge \mathfrak{R}(p, x) = \bot$ for all $x \in A$ with $x \neq p$. We prove that p is closed. Let $m, n \in \vee_p^\infty A$ be any points.

(i) If $m = n$, then $\overline{\mathfrak{R}}(m, n) = \top$.

(ii) If $m \neq n$ and $\nabla_p^\infty m \neq \nabla_p^\infty n$, then $\mathfrak{R}_d(\nabla_p^\infty m, \nabla_p^\infty n) = \bot$ since $\mathfrak{R}_d$ is the discrete structure. By Lemma 3.5, we have $\overline{\mathfrak{R}}(m, n) = \bot$.

(iii) Suppose $m \neq n$ and $\nabla_p^\infty m = x = \nabla_p^\infty n$ for some $x \in X$ with $x \neq p$. It follows that $m = x_i$ and $n = x_j$ for $i, j \in I$ with $i \neq j$. Let $m = x_i$, $n = x_j$ and $i, j, k \in I$ with $i \neq j$ and $i \neq k \neq j$. Then,

$$\begin{array}{rcl} \Re(\rho_h \mathcal{A}_p^\infty m, \rho_h \mathcal{A}_p^\infty n)_{(h=i,j,k)} & = & \{\Re(x,p), \Re(p,x), \Re(p,p) = \top\} \\ \Re_d(\nabla_p^\infty m, \nabla_p^\infty n) & = & \Re_d(x,x) = \top \end{array}$$

and it follows that

$$\begin{array}{rcl} \overline{\Re}(m,n) & = & \bigwedge \{\Re(\rho_h \mathcal{A}_p^\infty m, \rho_h \mathcal{A}_p^\infty n)_{(h=i,j,k)}, \Re_d(\nabla_p^\infty m, \nabla_p^\infty n))\} \\ & = & \bigwedge \{\Re(x,p), \Re(p,x), \top\} \\ & = & \Re(x,p) \wedge \Re(p,x) \end{array}$$

By the assumption, $\Re(x,p) \wedge \Re(p,x) = \perp$ and we obtain $\overline{\Re}(m,n) = \perp$.

Consequently, for all $m, n \in \vee_p^\infty X$, we get

$$\overline{\Re}(m,n) = \begin{cases} \top, & m = n \\ \perp, & m \neq n \end{cases}$$

and by Lemma 3.8, $\overline{\Re}$ is the discrete $\mathcal{Q}$-reflexive relation on $\vee_p^\infty A$. Hence, by Definition 5.2, $\{p\}$ is closed in A. □

Theorem 5.5. *Let $(A, \Re)$ be a $\mathcal{Q}$-reflexive space that $\mathcal{Q}$ has a prime bottom element and $p \in A$. $\{p\}$ is closed in A iff $\Re(x,p) = \perp$ or $\Re(p,x) = \perp$ for all $x \in A$ with $x \neq p$.*

Proof. It follows from the definition of the prime bottom element and Theorem 5.4. □

Theorem 5.6. *Let $(A, \Re)$ be a $\mathcal{Q}$-reflexive space that $\mathcal{Q}$ has a prime bottom element and F be a nonempty subset of A. F is closed iff for all $y \in F$ and $x \in A$ with $x \notin F$, $\Re(x,y) = \perp$ or $\Re(y,x) = \perp$.*

Proof. Let $(A, \Re)$ be a $\mathcal{Q}$-reflexive space, $\emptyset \neq F \subset A$ and $\Re'$ be the quotient $\mathcal{Q}$-reflexive relation on A/F induced from the epi map $q : A \to A/F$. Suppose F is closed and $x \in A$ with $x \notin F$. Since $q(x) = x \neq * = q(F)$ and F is closed, by Definition 5.2, $*$ is closed in A/F. By Theorem 5.5, $\Re'(x,*) = \perp$ or $\Re'(*,x) = \perp$. If $\Re'(x,*) = \perp$, then by Lemma 3.6,

$$\perp = \Re'(x,*) = \bigvee \{\Re(x,y) \mid \text{there exists } y \in A \text{ such that } q(y) = *\},$$

and it follows that $\Re(x,y) = \perp$ for all $y \in F$. Similarly, if $\Re'(*,x) = \perp$, then $\Re(y,x) = \perp$ for all $y \in F$. Hence, for all $y \in F$ and $x \in A$ with $x \notin F$, we have $\Re(x,y) = \perp$ or $\Re(y,x) = \perp$.

Conversely, assume that the condition is true. We prove that F is closed. Let $a \neq *$ be a point in A/F and $\Re'$ is the quotient structure on A/F. By assumption and Lemma 3.6, if $\Re(a,y) = \perp$ for all $y \in F$, then $\Re'(a,*) = \perp$, or if $\Re(y,a) = \perp$ for all $y \in F$, then $\Re'(*,a) = \perp$. Consequently, by Theorem 5.5 and Definition 5.2, $*$ is closed in A/F and F is closed. □

Theorem 5.7. *Let $(A, \Re)$ be a $\mathcal{Q}$-reflexive space and F be a nonempty subset of A. F is strongly closed iff for all $y \in F$ and $x \in A$ with $x \notin F$, $\Re(x,y) = \perp = \Re(y,x)$.*

Proof. Let $(A, \Re)$ be a $\mathcal{Q}$-reflexive space and $\emptyset \neq F \subset A$. Suppose F is strongly closed and $x \in A$ with $x \notin F$. By Definition 5.2, $(A/F, \Re')$ is T_1 at $*$ since F is strongly closed and $q(x) = x \neq * = q(F)$, where $\Re'$ is the quotient structure on A/F induced from the epi map $q : A \to A/F$. By Theorem 4.6, $\Re'(x,*) = \perp = \Re'(*,x)$. By Lemma 3.6,

$$\perp = \Re'(x,*) = \bigvee \{\Re(x,y) \mid \text{there exists } y \in A \text{ such that } q(y) = *\},$$

and this means $\Re(x, y) = \bot$ for all $y \in F$. Similarly,

$$\bot = \Re'(*, x) = \bigvee \{\Re(y, x) \mid \text{there exists } y \in A \text{ such that } q(y) = *\}$$

and it follows that for all $y \in F$, $\Re(y, x) = \bot$. Consequently, we get for all $y \in F$ and $x \in A$ with $x \notin F$, $\Re(x, y) = \bot = \Re(y, x)$.

Conversely, assume that the condition is true. We prove that F is strongly closed. Let $a \neq *$ be a point in A/F and $\Re'$ is the quotient structure on A/F. Note that, by Lemma 3.6,

$$\Re'(a, *) = \bigvee \{\Re(a, y) \mid \text{there exists } y \in A \text{ such that } q(y) = *\},$$

$$\Re'(*, a) = \bigvee \{\Re(y, a) \mid \text{there exists } y \in A \text{ such that } q(y) = *\}$$

and by assumption, we have $\Re'(a, *) = \bot = \Re'(*, a)$. Hence, by Theorem 4.6 and Definition 5.2, $(A/F, \Re')$ is T_1 at $*$ and F is strongly closed. □

Remark 5.8. (i) Let $(A, \Re)$ be a $\mathcal{Q}$-reflexive space that $\mathcal{Q}$ has a prime bottom element and F be a nonempty subset of A. By Theorems 5.6 and 5.7, if F is strongly closed, then F is closed, i.e., in **Q-RRel**, strong closedness implies closedness. But in general, the notions of closedness and strong closedness are independent of each other for an arbitrary topological category (cf. [24] Remark 4.4).

(ii) The closed subsets for the categories ∞**pqsMet** and **RRel** given in [10, 13] are the special forms of our results. For example, if we take quantale $([0, \infty], \geq, +)$ (resp. $(\{0, 1\}, \leq, \wedge)$), then Theorems 5.4-5.7 reduce to Theorems 3.2, 3.4 of [13] (resp. Theorem 3.8 of [10]) for the category ∞**pqsMet** (resp. **RRel**).

6. Zero-dimensional quantale-valued reflexive spaces

Recall that zero-dimensionality for a topological space (X, τ) is defined as X has a basis comprising of clopen sets. In [29], Stine showed that a topological space (X, τ) is zero-dimensional iff for $i \in I$, there exists a family of functions $f_i : (X, \tau) \to (X_i, \tau_{i_d})$ such that τ is the topology induced by (X_i, τ_{i_d}) via f_i, where (X_i, τ_{i_d}) is the family of discrete topological spaces. In view of the categorical counterparts, we have the following definition given by Stine.

Definition 6.1. ([29]) Let $\mathcal{U} : \mathcal{C} \to \mathcal{E}$ be a topological and $\mathcal{D} : \mathcal{E} \to \mathcal{C}$ be the discrete functor. An object X in $\mathcal{C}$ is called a zero-dimensional object if and only if there exists $A_i \in Ob(\mathcal{E})$ and morphisms $f_i : \mathcal{U}(X) \to A_i$ for $i \in I$ such that $\{\overline{f}_i : X \to \mathcal{D}(A_i)\}_{i\in I}$ is the initial lift of $\{f_i : \mathcal{U}(X) \to \mathcal{U}(\mathcal{D}(A_i)) = A_i\}_{i\in I}$.

Remark 6.2. For the forgetful functor $\mathcal{U}$: **Top** $\to$ **Set**, Definition 6.1 reduces to usual definition of zero-dimensional topological space.

Theorem 6.3. *Let $(A, \Re)$ be a $\mathcal{Q}$-reflexive space and $(A_i, \Re_{i_D})$ be the discrete $\mathcal{Q}$-reflexive spaces for $i \in I$. $(A, \Re)$ is zero-dimensional provided that there exists $f_i\colon (A, \Re) \to (A_i, \Re_{i_D})$ such that $\forall x, y \in X$,*

$$\Re(x, y) = \begin{cases} \top, & f_i(x) = f_i(y),\ \forall i \in I \\ \bot, & f_i(x) \neq f_i(y),\ \exists i \in I \end{cases}$$

Proof. Assume that $(A, \Re)$ is zero-dimensional. Let $(A_i, \Re_{i_D})$ be the discrete $\mathcal{Q}$-reflexive spaces for $i \in I$ and $f_i\colon A \to A_i$ be a family of functions. By Definition 6.1, $f_i\colon (A, \Re) \to (A_i, \Re_{i_D})$ is the initial lift of $f_i\colon A \to A_i$. Note that, for $x, y \in A$ and by Lemma 3.8,

$$\Re(x, y) = \bigwedge_{i\in I} \{\Re_{i_D}(f_i(x), f_i(y))\} = \bigwedge_{i\in I} \begin{cases} \top, & f_i(x) = f_i(y) \\ \bot, & f_i(x) \neq f_i(y) \end{cases}$$

(i) If $f_i(x) = f_i(y)$ for all $i \in I$, then $\Re(x, y) = \top$.

(ii) Similarly, if for at least one $i \in I$, $f_i(x) \neq f_i(y)$, then $\Re(x,y) = \perp$ by definition of the initial structure.

Conversely, let A be a nonempty set, $(A_i, \Re_{i_D})$ be the discrete Q-reflexive spaces for $i \in I$ and $f_i \colon A \to A_i$ be a family of functions. Assume that the condition is true, i.e., there exists $f_i \colon (A, \Re) \to (A_i, \Re_{i_D})$ such that $\forall x, y \in X$,

$$\Re(x,y) = \begin{cases} \top, & f_i(x) = f_i(y),\ \forall i \in I \\ \perp, & f_i(x) \neq f_i(y),\ \exists i \in I \end{cases}$$

We prove that $(A, \Re)$ is zero-dimensional, i.e., by Definition 6.1, $f_i \colon (A, \Re) \to (A_i, \Re_{i_D})$ is the initial lift of $f_i \colon A \to A_i$. It is obvious that for each $i \in I$, f_i is a Q-monotone mapping.

Let $g \colon (B, \Re_B) \to (A, \Re)$ is a mapping. We prove that g is Q-monotone if and only if $f_i \circ g$ is Q-monotone for all $i \in I$. The necessity is obvious since compositions of Q-monotone mappings are Q-monotone. Suppose for each $i \in I$, $f_i \circ g \colon (B, \Re_B) \to (A_i, \Re_{i_D})$ is a Q-monotone mapping. It follows that, for $x, y \in B$

$$\Re_B(x,y) \quad \leq \quad \bigwedge_{i \in I} \{\Re_{i_D}(f_i(g(x)), f_i(g(y)))\}$$

and by assumption we have

$$\Re(g(x), g(y)) = \begin{cases} \top, & f_i(g(x)) = f_i(g(y)),\ \forall i \in I \\ \perp, & f_i(g(x)) \neq f_i(g(y)),\ \exists i \in I \end{cases}$$

If for all $i \in I$, $f_i(g(x)) = f_i(g(y))$, then $\Re_B(x,y) \leq \mathcal{R}(g(x), g(y)) = \top$.

Let $f_i(g(x)) \neq f_i(g(y))$ for at least one $i \in I$. It follows that $\Re(g(x), g(y)) = \perp$, and $\Re_B(x,y) = \perp$ since $f_i \circ g$ is Q-monotone for all $i \in I$, i.e., $\Re_B(x,y) \leq \bigwedge_{i \in I} \{\Re_{i_D}(f_i(g(x)), f_i(g(y)))\} = \perp$. Hence, $\Re_B(x,y) \leq \Re(g(x), g(y))$.

Consequently, $g \colon (B, \Re_B) \to (A, \Re)$ is Q-monotone and therefore, $(A, \Re)$ is zero-dimensional. □

Example 6.4. Suppose $Q = ([0,1], \leq, *)$ is a triangular norm with a binary operation $*$ stated by $\alpha * \beta = (\alpha - 1 + \beta) \vee 0$ for all $\alpha, \beta \in [0,1]$ (Lukasiewicz t-norm), where the bottom and top elements are $\perp = 0$ and $\top = 1$. Let $A = \{a, b, c, d\}$, $B_i = \{x_i, y_i\}$ for $i = 1, 2, 3$, $\Re_{i_D}$ be the discrete Q-reflexive relation on B_i for $i = 1, 2, 3$ with the Lukasiewicz t-norm $Q = ([0,1], \leq, *)$, and the map $f_i \colon (A, \Re) \to (B_i, \Re_{i_D})$, $i = 1, 2, 3$, be defined as

$$f_i(t) = \begin{cases} x_i, & t = a, c \\ y_i, & t = b, d. \end{cases}$$

Define a Q-reflexive relation $\Re : A \times A \to Q$ by

$$\Re(m,n) = \Re(n,m) = \begin{cases} 1, & m = n \text{ or } (m,n) = (a,c), (b,d) \\ 0, & (m,n) \in \{a,c\} \times \{b,d\}. \end{cases}$$

Then $(A, \Re)$ is zero-dimensional.

Corollary 6.5. *(i) All indiscrete Q-reflexive spaces are zero-dimensional.*

(ii) If $|A| = 1$ (cardinality), then every Q-reflexive space $(A, \Re)$ is zero-dimensional.

(iii) Let $(A, \Re)$ be a Q-reflexive space with $|A| = 2$ and $(A_i, \Re_{i_D})$ be the discrete Q-reflexive space for $i \in I$. $(A, \Re)$ is zero-dimensional provided that there exists $f_i \colon (A, \Re) \to (A_i, \Re_{i_D})$ such that

$$\Re = \begin{cases} \Re_I, & f_i \text{ is constant},\ \forall i \in I \\ \Re_D, & \text{otherwise}. \end{cases}$$

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