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New proofs of some Dedekind η-function identities of level 6

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Abstract. Recently, Shaun Cooper proved several interesting η-function identities of level 6 while finding series and iterations for $1/\pi$. In this sequel, we present some new proofs of the η-function identities of level 6 discovered by Cooper. Here, in this article, we make use of the modular equation of degree 3 in two methods. We further give some interesting combinatorial interpretations of colored partitions. We also briefly describe a potential direction for further researches based upon some related recent developments involving the Jacobi's triple-product identity and the theta-function identities as well as on several other q-functions which emerged from the Rogers-Ramanujan continued fraction $R(q)$ and its such associates as $G(q)$ and $H(q)$. We point out the importance of the usage of the classical q-analysis and we also expose the current trend of falsely-claimed "generalization" by means of its trivial and inconsequential (p,q)-variation by inserting a forced-in redundant (or superfluous) parameter p.

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1. Introduction

The Dedekind η-function is defined as follows:

$$\eta(\tau) := \mathrm{e}^{\mathrm{i}\pi\tau/12} \prod_{j=1}^{\infty} \left(1 - e^{2\pi i\tau}\right) = q^{1/24} \prod_{j=1}^{\infty} \left(1 - q^j\right) \qquad \left(\Im(\tau) > 0\right).$$

Throughout this paper, we assume the $|q| < 1$ and employ the following standard notation:

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n).$$

Ramanujan's theta function $\mathfrak{f}(a,b)$ is defined by

$$\mathfrak{f}(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2} \qquad (|ab| < 1).$$

The function $\mathfrak{f}(a,b)$ satisfies the well-known triple-product identity of Jacobi [7, p. 35], which we state here as follows:

$$\mathfrak{f}(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Some important special cases of $\mathfrak{f}(a,b)$ may be recalled as follows (see [7, p. 36]):

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$\varphi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty$$

and

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

From this last definition, it is easily seen for the Dedekind η-function $\eta(\tau)$ that

$$f(-q) = q^{-1/24}\eta(\tau).$$

Following Ramanujan's work, we define

$$\chi(q) := (-q;q^2)_\infty.$$

Furthermore, in what follows, we find it to be convenient to write $f(-q^n) = f_n$.

A theta-function identity, which relates f_1, f_2, f_n and f_{2n} is called the theta-function identity of level $2n$. Ramanujan documented many modular equations which involve quotients of the function f_1 at different arguments. For example, we recall the following result [8, p. 204, Entry 51].

If

$$P := \frac{f_1^2}{q^{1/12} f_3^2} \quad \textit{and} \quad Q := \frac{f_2^2}{q^{1/6} f_6^2},$$

then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3. \tag{1}$$

The proof of the above result can be found in the monograph by Berndt [7], who also used similar types of identities in order to evaluate various continued fractions, Weber class invariants and many more. In fact, after the publication of [7], many researchers discovered similar identities (see, for example, [3–6, 13, 21, 24, 25, 27–29]).

Recently, Cooper [10, 11] established several interesting Dedekind η-function identities in his development of series expansions of π. In the present article, motivated by the above works, we prove some η-function identities of level 6 given by Cooper [10, 11] by using modular equations of degree 3. Moreover, as an application of the results derived here, we establish some interesting combinatorial interpretations of colored partitions.

Before concluding this section, we define a modular equation by Ramanujan's work. A modular equation of degree n is an equation relating α and β that is induced by

$$n\left(\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}\right) = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(\mathfrak{a},\mathfrak{b};\mathfrak{c};z) := \sum_{n=0}^{\infty} \frac{(\mathfrak{a})_n(\mathfrak{b})_n}{(\mathfrak{c})_n}\,\frac{z^n}{n!} \qquad (|z|<1)$$

denotes the Gauss hypergeometric function with

$$(\lambda)_0 := 1 \quad \text{and} \quad (\lambda)_n := \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) \qquad (n\in\{1,2,3,\cdots\}).$$

We say that β is of degree n over α and we refer to the following quotient:

$$m := \frac{z_1}{z_n}$$

as a multiplier for

$$z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right) \quad \text{and} \quad z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right).$$

2. Main Results

Our first main result as Theorem 1 below.

Theorem 1. *It is asserted for the Dedekind η-function that*

$$\frac{[\eta(\tau)]^5\,\eta(3\tau)}{\eta(2\tau)[\eta(6\tau)]^5} + 17 + 72\left(\frac{\eta(2\tau)[\eta(6\tau)]^5}{[\eta(\tau)]^5\,\eta(3\tau)}\right) = \left(\frac{\eta(2\tau)\eta(3\tau)}{\eta(\tau)\eta(6\tau)}\right)^{12}. \tag{2}$$

Proof. **[First Proof of Theorem** 1] Ramanujan [14, p. 238] and Berndt [7, pp. 230–238, Entry 13 (ix) and Entry 13 (xiv)] documented the following modular equations of degree 3. If β has degree 3 over α, we have

$$P := \left[16\alpha\beta(1-\alpha)(1-\beta)\right]^{1/8} \quad \text{and} \quad Q := \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4},$$

which yields

$$Q+\frac{1}{Q}+2\sqrt{2}\left(P-\frac{1}{P}\right)=0. \tag{3}$$

Also, from [7, pp. 122–124, Entry 10 (i) and Entry 12 (v)], we find for $q=e^{-y}$ that

$$\varphi(q)=\sqrt{z} \tag{4}$$

and

$$\chi(q)=2^{1/6}\left(\frac{x(1-x)}{q}\right)^{-1/24}. \tag{5}$$

If we now transform (3) by using (5), we obtain

$$\left(\frac{u}{v}\right)^{6}+\left(\frac{v}{u}\right)^{6}=(uv)^{3}-\frac{8}{(uv)^{3}}, \tag{6}$$

where

$$u:=u(q)=q^{-1/24}\,\chi(q)\quad\text{and}\quad v:=v(q)=q^{-1/8}\,\chi(q^{3}).$$

Upon multiplying both sides of (6) by

$$(uv)^{-18}\left(4u^{12}+48u^{3}v^{3}-3u^{9}v^{9}-4v^{12}\right),$$

we obtain

$$\frac{v^{3}}{u^{9}}+\frac{16}{u^{15}v^{3}}-\frac{4v^{6}}{u^{18}}+\frac{384}{u^{12}v^{12}}-\frac{72}{u^{6}v^{6}}+3+\frac{4u^{6}}{v^{18}}+\frac{80}{u^{3}v^{15}}-\frac{7u^{3}}{v^{9}}=0,$$

which is equivalent to

$$3\left(1-\frac{4u^{3}}{v^{9}}\right)^{2}-\frac{u^{8}}{v^{8}}\left(\frac{17}{u^{5}v}+\frac{v^{11}}{u^{17}}\right)\left(\frac{4v^{3}}{u^{9}}-1\right)\left(1-\frac{4u^{3}}{v^{9}}\right)+24\,\frac{u^{6}}{v^{18}}\left(\frac{4v^{3}}{u^{9}}-1\right)^{2}=0. \tag{7}$$

Also, from [7, pp. 230-238, Entry 13 (iẍ) and Entry (ẍiv̈)], if β has degree 3 over α, then we have

$$m=\frac{1-2\left(\dfrac{\beta^{3}(1-\beta)^{3}}{\alpha(1-\alpha)}\right)^{1/8}}{1-2(\alpha\beta)^{1/4}}\quad\text{and}\quad\frac{3}{m}=\frac{2\left(\dfrac{\alpha^{3}(1-\alpha)^{3}}{\beta(1-\beta)}\right)^{1/8}-1}{1-2(\alpha\beta)^{1/8}},$$

which would lead us to

$$\frac{m^{2}}{3}=\frac{1-2\left(\dfrac{\beta^{3}(1-\beta)^{3}}{\alpha(1-\alpha)}\right)^{1/8}}{2\left(\dfrac{\alpha^{3}(1-\alpha)^{3}}{\beta(1-\beta)}\right)^{1/8}-1}. \tag{8}$$

Next, on transforming (8) in terms of the theta function from (4) and (5), we have

$$\frac{[\varphi(q)]^{4}}{3[\varphi(q^{3})]^{4}}=\frac{1-\dfrac{4u^{3}}{v^{9}}}{\frac{4v^{3}}{u^{9}}-1}. \tag{9}$$

Employing (9) in (7), we find that

$$1-\left(\frac{17}{u^5v}+\frac{v^{11}}{u^{17}}\right)\left(\frac{v^2}{u^2}\ \frac{\varphi(q^3)}{\varphi(q)}\right)^4-\frac{72v^{14}}{u^{24}}\ \left(\frac{\varphi(q^3)}{\varphi(q)}\right)^8=0. \tag{10}$$

Furthermore, by using some known q-identities, we can easily see that

$$\varphi(q)=\frac{f_2^5}{(f_1f_4)^2},\quad \chi(q)=\frac{f_2^2}{f_1f_4}\quad\text{and}\quad \chi(-q)=\frac{f_1}{f_2}, \tag{11}$$

which readily yields

$$\frac{\varphi(q)}{\varphi(q^3)}=q^{1/6}\ \frac{u^2}{v^2}\ \frac{f_2}{f_6}. \tag{12}$$

Using (12) in (10), we deduce that

$$1-q^{2/3}\left(\frac{17}{u^5v}+\frac{v^{11}}{u^{17}}\right)\left(\frac{f_6}{f_2}\right)^4-\frac{72q^{4/3}}{u^{10}v^2}\left(\frac{f_6}{f_2}\right)^8=0. \tag{13}$$

Finally, if we replace q by $-q$ in the above equation (13), rewrite $u(-q)$ and $v(-q)$ in terms of f_n by using (11), multiply the resulting equation by $f_1^{17}f_3^2f_6^7$ and then simplify by using the Dedekind η-function, we complete the first proof of Theorem 1. □

Proof. [**Second Proof of Theorem** 1] Upon first rewriting the assertion (2) of Theorem 1 in terms of f_n and then dividing the resulting equation by $f_1^{17}f_2^{11}f_3^{13}f_6^7$, we obtain

$$1+17q\ \frac{f_2f_6^5}{f_1^5f_3}+72q^2\ \frac{f_2^2f_6^{10}}{f_1^{10}f_3^2}-\frac{f_2^{13}f_3^{11}}{f_1^{17}f_6^7}=0. \tag{14}$$

We also see from [29, Theorem 3.4 (i)] that, if

$$A=\frac{f_1}{q^{\frac{1}{24}}f_2}\quad\text{and}\quad B=\frac{f_3}{q^{\frac{1}{8}}f_6},$$

then

$$(AB)^3+\frac{8}{(AB)^3}=\left(\frac{B}{A}\right)^6-\left(\frac{A}{B}\right)^6. \tag{15}$$

Now, if we make use of P, Q, A and B as defined in (1) and (15), (14) reduces to

$$P^7Q^2(AB)^6+(17P^6-Q^6)Q(AB)^3+72P^5=0$$

or, equivalently, to

$$(AB)^3=\frac{(Q^6-17P^6)Q\pm\sqrt{(Q^6-17P^6)^2Q^2-288P^{12}Q^2}}{2P^7Q^2}. \tag{16}$$

Thus, by first using (16) in (15) and then factorizing the resulting equation, we obtain

$$L(P,Q)M(P,Q)=0,$$

where

$$L(P,Q)=P^6-9P^2Q^2-P^4Q^4+Q^6$$

and

$$M(P,Q)=32P^{14}Q^2(9P^{12}-8P^{10}Q^4-72P^8Q^2-26P^6Q^6+Q^{12})$$

Clearly, $L(P,Q)$ is the same as in (1), so we have completed our second proof of Theorem 1. □

We now state our second main result as Theorem 2 below.

Theorem 2. *For the Dedekind η-function $\eta(z)$, it is asserted that*

$$\frac{[\eta(2\tau)]^8[\eta(3\tau)]^4}{[\eta(\tau)]^4[\eta(6\tau)]^8} - 10 + 9\left(\frac{[\eta(\tau)]^4[\eta(6\tau)]^8}{[\eta(2\tau)]^8[\eta(3\tau)]^4}\right) = \left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6. \tag{17}$$

Proof. [**First Proof of Theorem** 2] On multiplying both sides of (6) by

$$u^{-10}v^{-26}\left(3u^{12} - u^9v^9 - 8u^3v^3 + 3v^{12}\right),$$

we obtain

$$\frac{3}{u^{10}v^2} + \frac{16}{u^7v^{11}} - \frac{64}{u^4v^{20}} + \frac{6u^2}{v^{14}} + \frac{16u^5}{v^{23}} - \frac{4}{uv^5} + \frac{u^8}{v^8} - \frac{4u^{11}}{v^{17}} + \frac{3u^{14}}{v^{26}} = 0,$$

which is equivalent to

$$\frac{3}{u^{10}v^2}\left(1 - \frac{4u^3}{v^9}\right)^2 + \left(\frac{10u^2}{v^{14}} - \frac{u^8}{v^8}\right)\left(1 - \frac{4u^3}{v^9}\right)\left(\frac{4v^3}{u^9} - 1\right) + \frac{3u^{14}}{v^{26}}\left(\frac{4v^3}{u^9} - 1\right)^2 = 0. \tag{18}$$

Now, if we use (9) in the equation (18), we find that

$$\frac{1}{u^{10}v^2} + \left(\frac{10}{(uv)^6} - 1\right)\frac{u^8}{v^8}\,\frac{[\varphi(q^3)]^4}{[\varphi(q)]^4} + \frac{9u^{14}}{v^{18}}\,\frac{[\varphi(q^3)]^8}{[\varphi(q)]^8} = 0,$$

which, in view of (12), yields

$$\frac{1}{u^{10}v^2} + q^{2/3}\left(\frac{10}{u^6v^6} - 1\right)\left(\frac{f_6}{f_2}\right)^4 + \frac{9q^{4/3}}{u^2v^{10}}\left(\frac{f_6}{f_2}\right)^8 = 0. \tag{19}$$

Finally, if we replace q by $-q$ in (19), rewrite $u(-q)$ and $v(-q)$ in terms of f_n by using (11), multiply both sides of the resulting equation by $f_2^{16}f_3^8$ and then simplify by using the Dedekind η-function, we complete our first proof of Theorem 2. □

Proof. [**Second Proof of Theorem** 2] We rewrite the assertion (17) of Theorem 2 in terms of f_n and divide the resulting equation by $f_1^2f_2^{16}f_3^{10}$. Then, by using P, Q, A and B as defined in (1) and (15), we obtain

$$(AB)^3 = \frac{\sqrt{9P^4 - 10P^2Q^4 + Q^8}}{PQ^2},$$

which, when used in (15) followed by factorization, leads us to

$$L(P,Q)M(P,Q) = 0, \tag{20}$$

where

$$L(P,Q) = P^6 - P^4Q^4 - 9P^2Q^2 + Q^6$$

and

$$M(P,Q) = 9P^{10} - P^8Q^4 - P^2Q^{10} + Q^{14}.$$

Since $L(P,Q)$ is the same as (1), we have completed our second proof of Theorem 2. □

Theorem 3. *The following Dedekind η-function identity holds trur*:

$$\frac{[\eta(2\tau)]^3[\eta(3\tau)]^9}{[\eta(\tau)]^3[\eta(6\tau)]^9} - 7 - 8\left(\frac{[\eta(\tau)]^3[\eta(6\tau)]^9}{[\eta(2\tau)]^3[\eta(3\tau)]^9}\right) = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^4. \tag{21}$$

Proof. [**First Proof of Theorem** 3] On multiplying both sides of (6) by

$$(uv)^{-18}\left(16u^3v^3 + 8u^{12} - 16v^{12} + u^9v^9\right),$$

we get

$$\frac{16v^6}{u^{18}} - \frac{17v^3}{u^9} + \frac{112}{u^{15}v^3} + \frac{16}{u^6v^6} + 1 + \frac{7u^3}{v^9} - \frac{128}{u^{12}v^{12}} - \frac{80}{u^3v^{15}} - \frac{8u^6}{v^{18}} = 0,$$

which is equivalent to

$$\left(1 + \frac{7u^3}{v^9} - \frac{8u^6}{v^{18}}\right)\left(\frac{4v^3}{u^9} - 1\right)^2 - \frac{9v^3}{u^9}\left(1 - \frac{4u^3}{v^9}\right)^2 = 0.$$

If we now use (9) in the above equation, we find that

$$1 + \frac{7u^3}{v^9} - \frac{8u^6}{v^{18}} - \frac{v^3}{u^9}\frac{\varphi^8(q)}{\varphi^8(q^3)} = 0,$$

which, in light of (12), yields

$$1 + \frac{7u^3}{v^9} - \frac{8u^6}{v^8} - \frac{q^{-4/3}u^7}{v^{13}}\left(\frac{f_2}{f_6}\right)^8 = 0. \tag{22}$$

Finally, if we replace q by $-q$ in the equation (22), rewrite $u(-q)$ and $v(-q)$ in terms of f_n by using (11), multiply the resulting equation by $f_2^6 f_3^{18}$, and then simplify by using the Dedekind η-function, we complete our first proof of Theorem 3. □

Proof. [**Second Proof of Theorem** 3] After rewriting the assertion (21) of Theorem 3 in terms of f_n, if we divide the resulting equation by $f_2^6 f_3^{18}$ and make use of the definitions of P, Q, A and B as in (1) and (15), we obtain

$$Q^6(AB)^6 - Q^3(7P^2 + P^5Q^2) - 8P^6 = 0$$

or, equivalently,

$$(AB)^3 = \frac{Q^3(7P^2 + P^5Q^2) \pm \sqrt{Q^6(7P^2 + P^5Q^2)^2 + 32P^6Q^6}}{2Q^6}. \tag{23}$$

Now, by using (23) in (15) and then factorizing, we have

$$L(P,Q)M(P,Q) = 0,$$

where

$$L(P,Q) == P^6 - P^4Q^4 - 9P^2Q^2 + Q^6$$

and

$$M(P,Q) = 32P^8Q^8(P^6 + 9P^2Q^2 + P^4Q^4)$$

Since $L(P,Q)$ is the same as in (1), our second proof of Theorem 3 is completed. □

Remark. Since the proof of Cooper's Dedekind η-function identities of level 6 are similar, we omit the proof of the following (presumably new) η-function identities.

$$1-9\left(\frac{[\eta(\tau)]^4[\eta(6\tau)]^8}{[\eta(2\tau)]^8[\eta(3\tau)]^4}\right)=\frac{[\eta(\tau)]^9[\eta(6\tau)]^3}{[\eta(2\tau)]^9[\eta(3\tau)]^3}, \tag{24}$$

$$1-\frac{[\eta(\tau)]^4[\eta(6\tau)]^8}{[\eta(2\tau)]^8[\eta(3\tau)]^4}=\frac{\eta(\tau)[\eta(3\tau)]^5}{[\eta(2\tau)]^5\ \eta(6\tau)}, \tag{25}$$

$$1-8\left(\frac{[\eta(\tau)]^3[\eta(6\tau)]^9}{[\eta(2\tau)]^3[\eta(3\tau)]^9}\right)=\frac{[\eta(\tau)]^8[\eta(6\tau)]^4}{[\eta(2\tau)]^4[\eta(3\tau)]^8} \tag{26}$$

and

$$1+\frac{[\eta(\tau)]^3[\eta(6\tau)]^9}{[\eta(2\tau)]^3[\eta(3\tau)]^9}=\frac{[\eta(2\tau)]^5\eta(6\tau)}{\eta(\tau)[\eta(3\tau)]^5}. \tag{27}$$

3. Applications to Colored Partitions

The Dedekind η-function identities, which we have proved in Section 2, have applications to the theory of partitions. In this section, we demonstrate the applications in colored partitions for Theorem 1 and 2. Similarly, we can identify other applications for the remaining identities derived in Section 2.

For convenience, we use the following standard notation:

$$(x_1,x_2,\cdots,x_m;q)_\infty:=\prod_{j=1}^{m}(x_j;q)_\infty. \tag{28}$$

Moreover, by definition, a positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called *colored partitions*. For example, if 1, 2 and 3 are assigned with two colors, then the possible partitions of 3 are given as follows:

3_i, 3_v, 2_v+1_v, 2_i+1_i, 2_i+1_v, 2_v+1_i, $1_i+1_i+1_i$, $1_v+1_v+1_v$, $1_i+1_i+1_v$ and $1_i+1_v+1_v$,

where we used the indices i (indigo) and v (violet) to differentiate two colors of 1, 2 and 3. Also, the generating function for the number of partitions of n is denoted by

$$\frac{1}{(q^a;q^b)_\infty^k},$$

with k colors and with all of the parts being congruent to $a \pmod b$.

Theorem 4. *Let $A(n)$ denote the number of partitions of n being split into parts congruent to ±1, ±2 or $+3$ modulo 6 with 5, 4 and 6 colors, respectively. Suppose also that $B(n)$ represents the number of partitions of n into several parts congruent to ±1, ±2 or $+3$ modulo 6 with 10, 8 and 12 colors, respectively. If $C(n)$ is the number of partitions of n being divided into parts congruent to ±1, ±2 or $+3$ modulo 6 with 17, 4 and 6 colors, respectively, then the following identity holds true*:

$$17A(n-1)+72B(n-2)-C(n)=0 \qquad (n\in\mathbb{N}\setminus\{1\}=\{2,3,4,\cdots\}). \tag{29}$$

Proof. Upon rewriting the assertion (2) of Theorem 1 by using Dedekind η-function, if we divide both sides of the resulting equation by $f_1^{17}f_2^{13}f_3^{13}f_6^{17}$ and rewrite it subject to the common base q^6, we have

$$1+\frac{17q}{\left(q_5^1,q_4^2,q_6^3,q_4^4,q_5^5;q^6\right)_\infty}+\frac{72q^2}{\left(q_{10}^1,q_8^2,q_{12}^3,q_8^4,q_{10}^5;q^6\right)_\infty}-\frac{1}{\left(q_{17}^1,q_4^2,q_6^3,q_4^4,q_{17}^5;q^6\right)_\infty}=0. \tag{30}$$

Since

$$\left(q^{a\pm};q^b\right)_\infty=\left(q^a,q^{b-a};q^b\right)_\infty \qquad (a,b\in\mathbb{N};\ a<b),$$

the equation (30) can be reduced to

$$1+\frac{17q}{\left(q_5^{1\pm},q_4^{2\pm},q_6^{3+};q^6\right)_\infty}+\frac{72q^2}{\left(q_{10}^{1\pm},q_8^{2\pm},q_{12}^{3+};q^6\right)_\infty}-\frac{1}{\left(q_{17}^{1\pm},q_4^{2\pm},q_6^{3+};q^6\right)_\infty}=0. \tag{31}$$

The above identity (31) generates $A(n)$, $B(n)$ and $C(n)$, and hence we have

$$1+17q\sum_{n=0}^{\infty}A(n)q^n+72q^2\sum_{n=0}^{\infty}B(n)q^n-\sum_{n=0}^{\infty}C(n)q^n=0. \tag{32}$$

Thus, upon extracting the powers of q^n in (32), we obtain the result (29) as asserted by Theorem 4. □

The following table verifies Theorem 4 for the partitions for $n = 2$.

Table 1. Verification of the Partitions for $n = 2$

$A(1) = 5$:	$1_r, 1_o, 1_y, 1_b, 1_v$
$B(0) = 1$:	
$C(2) = 157$:	$1_r + 1_r$, $1_y + 1_y$ and 15 more colors of the same type, $1_r + 1_y, 1_r + 1_b$ and 134 colors of similar type and $2_r, 2_0, 2_b, 2_y$

Theorem 5. *Let $A(n)$ represent the number of partitions of n into several parts congruent to ±1 or $+3$ modulo 6 with 10 and 12 colors, respectively. Suppose also that $B(n)$ denotes the number of partitions of n being split into parts congruent to ±1, ±2 or $+3$ modulo 6 with 6, 4 and 12 colors, respectively. Let $C(n)$ be the number of partitions of n being divided into parts congruent to ±1 or $+3$ modulo 6 with 2 and 12 colors, respectively. Also let $D(n)$ represent the number of partitions of n into several parts congruent to $2\pm$ modulo 6 with 4 colors. Then the following identity holds true*:

$$A(n)-10B(n-1)+9C(n-2)-D(n)=0 \qquad (n\in\mathbb{N}\setminus\{1\}). \tag{33}$$

Proof. We rewrite the assertion (17) of Theorem 2 using the Dedekind η-function, divide both sides of the resulting equation by $f_1^{10}f_2^{16}f_3^{10}f_6^{16}$ and rewrite it subject to the common base q^6. We thus find that

$$\frac{1}{\left(q_{10}^1,q_{12}^3,q_{10}^5;q^6\right)_\infty}-\frac{10q}{\left(q_6^1,q_4^2,q_{12}^3,q_4^4,q_6^5;q^6\right)_\infty}+\frac{9q^2}{\left(q_2^1,q_{12}^3,q_2^5;q^6\right)_\infty}-\frac{1}{\left(q_4^2,q_4^4;q^6\right)_\infty}=0,$$

which is equivalent to

$$\frac{1}{\left(q_{10}^{1\pm},q_{12}^{3+};q^6\right)_\infty}-\frac{10q}{\left(q_6^{1\pm},q_4^{2\pm},q_{12}^{3+};q^6\right)_\infty}+\frac{9q^2}{\left(q_2^{1\pm},q_{12}^{3+};q^6\right)_\infty}-\frac{1}{\left(q_4^{2\pm};q^6\right)_\infty}=0. \tag{34}$$

The identity (34) provides the generating functions of $A(n)$, $B(n)$, $C(n)$ and $D(n)$, and hence we have

$$\sum_{n=0}^{\infty} A(n)q^n - 10q \sum_{n=0}^{\infty} B(n)q^n + 9q^2 \sum_{n=0}^{\infty} C(n)q^n - \sum_{n=0}^{\infty} D(n)q^n = 0,$$

which, upon extracting the powers of q^n, yields the desired result as asserted by Theorem 5. □

The following table verifies Theorem 5 for the partitions for $n = 2$.

Table 2. Verification of the Partitions for $n = 2$

$A(2) = 55$:	$1_r + 1_r$, $1_y + 1_y$ and the remaining 8 colors of the same type, $1_r + 1_y$, $1_r + 1_b$ and the remaining 43 colors of similar type, $2_r, 2_y, 2_b$.
$B(1) = 6$:	$1_r, 1_y, 1_b, 1_o, 1_m, 1_v$.
$C(0) = 1$:	
$D(2) = 4$:	$2_r, 2_y, 2_b, 2_m$.

4. Jacobi's Triple-Product Identities, Theta-Function Identities and Associated q-Functions

In this section, we choose to present a brief description of some related recent developments involving the Jacobi's triple-product identities, theta-function identities and other associated q-functions.

First of all, we recall that the general theta function $\mathfrak{f}(a,b)$, which was introduced by Srinivasa Ramanujan (1887–1920) in Chapter 16 of his celebrated *Notebooks*, is given by (see also [15] and Section 1)

$$\begin{aligned}\mathfrak{f}(a,b) &= 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b,a) \qquad (|ab| < 1).\end{aligned} \tag{35}$$

Clearly, this last equation (35) shows that

$$\mathfrak{f}(a,b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \mathfrak{f}\Big(a(ab)^n, b(ab)^{-n}\Big) = \mathfrak{f}(b,a) \qquad (n \in \mathbb{Z}). \tag{36}$$

In fact, Ramanujan also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, can be written as follows (see [14, p. 35, Entry 19]):

$$\mathfrak{f}(a,b) = (-a;ab)_\infty \, (-b;ab)_\infty \, (ab;ab)_\infty \tag{37}$$

or, equivalently, by (see [12])

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n^2} z^n &= \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\left(1 + zq^{2n-1}\right)\left(1 + \frac{1}{z} q^{2n-1}\right) \\ &= \left(q^2;q^2\right)_\infty \left(-zq;q^2\right)_\infty \left(-\frac{q}{z};q^2\right)_\infty \qquad (|q| < 1;\ z \neq 0).\end{aligned} \tag{38}$$

The q-series identity (38) or its above-mentioned equivalent form was first proved by Carl Friedrich Gauss (1777–1855).

Next, we present the set of four theta functions $\vartheta_j(z,q)$ $(j = 1,2,3,4)$, which were introduced in 1829 by Carl Gustav Jacob Jacobi (1804–1851), defined by (see [12] and [30]; see also [16])

$$\begin{aligned}\vartheta_1(z,q) &= -i \sum_{n=-\infty}^{\infty} (-1)^n\, e^{(2n+1)iz} \\ &= 2\sum_{n=0}^{\infty} (-1)^n\, q^{\left(\frac{n+\frac{1}{2}}{2}\right)^2} \sin[(2n+1)z] \\ &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n\, q^{n(n+1)} \sin[(2n+1)z],\end{aligned} \tag{39}$$

$$\begin{aligned}\vartheta_2(z,q) &= \sum_{n=-\infty}^{\infty} q^{\left(\frac{n+\frac{1}{2}}{2}\right)^2} e^{(2n+1)iz} \\ &= 2\sum_{n=0}^{\infty} q^{\left(\frac{n+\frac{1}{2}}{2}\right)^2} \cos[(2n+1)z] \\ &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} q^{n(n+1)} \cos[(2n+1)z],\end{aligned} \tag{40}$$

$$\vartheta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2}\, e^{2niz} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz) \tag{41}$$

and

$$\vartheta_4(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{n^2}\, e^{2niz} = 1 + 2\sum_{n=1}^{\infty} (-1)^n\, q^{n^2} \cos(2nz), \tag{42}$$

where $z \in \mathbb{C}$ and $|q| < 1$. A set of three most interesting q-functions, which are related rather closely to such entities as Ramanujan's general theta function in (36), Jacobi's triple-product identity in (38) and Jacobi's theta functions in the equations (39) to (42), can now be introduced here as follows (see [1], [2], [20] and [17]):

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{n(3n-1)}{2}} = (q;q)_\infty = \frac{1}{\sqrt{3}}\, q^{-\frac{1}{24}}\, \vartheta_2\left(\frac{\pi}{6}, q^{\frac{1}{6}}\right), \tag{43}$$

$$\varphi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty\, (q^2;q^2)_\infty}{(q;q^2)_\infty\, (-q^2;q^2)_\infty} = \vartheta_3(0,q), \tag{44}$$

and

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{1}{2}\, q^{-\frac{1}{8}} \left[\vartheta_2(0, \sqrt{q}) - 1\right]. \tag{45}$$

The theory of the above-defined (Jacobi's) theta functions $\vartheta_j(z,q)$ $(j = 1,2,3,4)$ has a long history and many applications in a wide variety of research fields such as Number Theory (in, especially, Quadratic Forms and Elliptic Functions) and Quantum Physics. Besides, the subject of q-analysis, which is popularly known as the quantum analysis, has its roots in such important areas as (for example) Mathematical Physics, Analytic Number Theory and the Theory of Partitions. Motivated essentially by the potential for applications of q-series and q-products, in their recent investigation, Srivastava *et al.* [26] introduced and studied the three functions R_α, R_β and R_m, which emerged from the Rogers-Ramanujan continued fraction $R(q)$ and its such associates as $G(q)$ and $H(q)$ (see, for details, [26, Eqs. (12), (13) and (14)]) and for which some modular equations of degree 3 and higher can possibly be developed in a sequel to our present investigation. More recently, Srivastava *et al.* [22] developed several q-identities involving the theta functions $\varphi(q)$ and $\psi(q)$ defined by (44) and (45), respectively.

5. Conclusion

Motivated by a number of earlier works including especially the work of Cooper [10, 11], which we have cited herein, we have presented new and alternative proofs to the Dedekind η-function identities of level 6 documented in [11]. Here, in this sequel, we have used Ramanujan's modular equations of degree 3. Furthermore, as an application of the identities which we have presented here, we have established some interesting combinatorial interpretations of colored partitions.

The subject of the basic or quantum (or q-) analysis has found widespread applications which are based upon the extensive study of q-series and q-polynomials and, especially, q-hypergeometric functions and q-hypergeometric polynomials (see, for details, [23, pp. 350–351]). With a view to aiding and motivating the interested reader for further researches on the subject, therefore, we have chosen to cite some recent developments in addition to many of the above-cited works (see, for example, [9] and [19]) for potential usages of the basic or quantum (or q-) calculus.

In concluding our present work based upon the basic (or q-) analysis, we recall a recently-published review-cum-expository review article in which, in addition to employing the q-analysis in Geometric Function Theory of Complex Analysis, Srivastava [17] pointed out the fact that the results involving the q-analysis can easily (and possibly trivially) be translated into the corresponding results for the so-called (p,q)-analysis (with $0 < |q| < p \leqq 1$) by applying some parametric and argument variations, the additional parameter p being obviously redundant. Of course, this exposition and observation of Srivastava (see, for details, [17, p. 340] and [18, pp. 1511–1512]) would apply also to the results which we have considered in our present investigation for $|q| < 1$.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

[1] C. Adiga, N. A. S. Bulkhali, D. Ranganatha and H. M. Srivastava, Some new modular relations for the Rogers-Ramanujan type functions of order eleven with applications to partitions, *J. Number Theory* **158** (2016), 281–297.

[2] C. Adiga, N. A. S. Bulkhali, Y. Simsek and H. M. Srivastava, A continued fraction of Ramanujan and some Ramanujan-Weber class invariants, *Filomat* **31** (2017), 3975–3997.

[3] C. Adiga, T. Kim and M. S. M. Naika, Modular equations in the theory of signature 3 and *P-Q* identities, *Adv. Stud. Contemp. Math.* **7** (2003), 33–40. (2003)

[4] C. Adiga, K. R. Vasuki and B. R. Srivatsa Kumar, On modular relations for the functions analogous to Rogers-Ramanujan functions with applications to partitions, *South East Asian J. Math. Math. Soc.* **6** (2006), 131–144.

[5] N. S. Baruah, Modular equations for Ramanujan's cubic continued fraction, *J. Math. Anal. Appl.* **268** (2002), 244–255.

[6] N. D. Baruah, On some of Ramanujan's Schläfli-type "mixed" modular equations, *J. Number Theory* **100** (2003), 270–294.

[7] B. C. Berndt, *Ramanujan's Notebooks. Part* III, Springer-Verlag, Berlin, Heidelberg and New York, 1991.

[8] B. C. Berndt, *Ramanujan's Notebooks. Part* IV, Springer-Verlag, Berlin, Heidelberg and New York, 1994.

[9] J. Cao, H. M. Srivastava, H.-L. Zhou and S. Arjika, Generalized q-difference equations for q-hypergeometric polynomials with double q-binomial coefficients, *Mathematics* **10** (2022), Article ID 556, 1–17.

[10] S. Cooper, Series and iteration for $1/\pi$, *Acta Arith.* **141** (2010), 33–58.
[11] S. Cooper, *Ramanujan's Theta Functions*, Springer Nature Switzerland AG, Cham, Swizerland, 2017.
[12] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Regiomonti, Sumtibus Fratrum Bornträger, Königsberg, Germany, 1829; Reprinted in *Gesammelte Mathematische Werke* **1** (1829), 497–538, American Mathematical Society, Providence, Rhode Island, 1969, pp. 97–239.
[13] M. S. M. Naika, A note on cubic modular equations of degree two, *Tamsui Oxford J. Math. Sci.* **22** (2006), 1–8.
[14] S. Ramanujan, *Notebooks*, Vols. **1** and **2**, Tata Institute of Fundamental Research, Bombay, 1957.
[15] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa Publishing House, New Delhi, 1988.
[16] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, London and New York, 1966.
[17] H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. A*: *Sci.* **44** (2020), 327–344.
[18] H. M. Srivastava, Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations, *J. Nonlinear Convex Anal.* **22** (2021), 1501–1520.
[19] H. M. Srivastava, A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics, *Symmetry* **13** (2021), Article ID 2294, 1–22.
[20] H. M. Srivastava and M. P. Chaudhary, Some relationships between q-product identities, combinatorial partition identities and continued-fraction identities, *Adv. Stud. Contemp. Math.* **25** (2015), 265–272.
[21] H. M. Srivastava, M. P. Chaudhary and S. Chaudhary, Some theta-function identities related to Jacobi's triple-product identity, *European J. Pure Appl. Math.* **11** (2018), 1–9.
[22] H. M. Srivastava, M. P. Chaudhary, S. Chaudhary and G. A. Salilew, Jacobi's triple-product identity and an associated family of theta-function identities, *Math. Notes* **112** (2022), 756–773.
[23] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
[24] H. M. Srivastava and N. Saikia, Some congruences for overpartitions with restriction, *Math. Notes* **107** (2020), 488–498.
[25] H. M. Srivastava, N. V. Sayinath Udupa and B. R. Srivatsa Kumar, A class of mixed modular equations and their applications involving theta-function identities, *J. Nonlinear Convex Anal.* **23** (2022), 607–623.
[26] H. M. Srivastava, R. Srivastava, M. P. Chaudhary and S. Uddin, A family of theta-function identities based upon combinatorial partition identities and related to Jacobi's triple-product identity, *Mathematics* **8** (2020), Article ID 918, 1–14.
[27] H. M. Srivastava, B. R. Srivatsa Kumar and R. Narendra, Some modular equations analogous to Ramanujan's identities *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat.* (*RACSAM*) **115** (2021), Article ID 59, 1–12.
[28] K. R. Vasuki, On some of Ramanujan's *P*-*Q* modular equations, *J. Indian Math. Soc.* (*New Ser.*) **73** (2006), 131-143.
[29] K. R. Vasuki and T. G. Sreeramamurthy, A note on *P*-*Q* modular equations, *Tamsui Oxford J. Math. Sci.* **21** (2005), 109–120.
[30] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *with an Account of the Principal Transcendental Functions*, Fourth Edition, Cambridge University Press, Cambridge, London and New York, 1927.