Filomat 37:12 (2023), 3725–3735
https://doi.org/10.2298/FIL2312725B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized inequalities for nonuniform wavelet frames in linear canonical transform domain

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Abstract. A constructive algorithm based on the theory of spectral pairs for constructing nonuniform wavelet basis in $L^2(\mathbb{R})$ was considered by Gabardo and Nashed. In this setting, the associated translation set is a spectrum Λ which is not necessarily a group nor a uniform discrete set, given $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ (an integer) and r is an odd integer with $1 \leq r \leq 2N-1$ such that r and N are relatively prime and $\mathbb{Z}$ is the set of all integers. In this article, we continue this study based on non-standard setting and obtain some inequalities for the nonuniform wavelet system $\left\{\mathfrak{f}^{\mu}_{j,\lambda}(x) = (2N)^{j/2}\mathfrak{f}\Big((2N)^j x - \lambda\Big)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)}, j \in \mathbb{Z}, \lambda \in \Lambda\right\}$ to be a frame associated with linear canonical transform in $L^2(\mathbb{R})$. We use the concept of linear canonical transform so that our results generalise and sharpen some well-known wavelet inequalities.

1. Introduction

Frames were widely studied by Duffin and Schaeffer [9] in the light of non-harmonic Fourier series in year 1952. They were further investigated in 1986 by Daubechies et al.[7]. This process of frame study continued. The unique and interesting properties of frames and their duals make them able to play an important role in the characterization of signal and image processing, Image processing, signal spaces, sampling theory and many other fields. To be precise, we can say a frame is a collection of functions or signals in a separable Hilbert space which allows stable but not unique decomposition. Mathematically, we can say a family $\{f_k\}_{k=1}^{\infty}$ of functions of a Hilbert space $\mathbb{H}$ is called a *frame* for $\mathbb{H}$ if we can find constant $A, B > 0$ with the condition $f \in \mathbb{H}$,

$$A\big\|f\big\|_2^2 \leq \sum_{k=1}^{\infty} \Big|\Big\langle f, f_k\Big\rangle\Big|^2 \leq B\big\|f\big\|_2^2, \tag{1}$$

where A is lower frame bound while as B is the upper frame bound. When the bounds are equal, we have a *tight frame*. We have a *normalized tight frame* when $A = B = 1$. The frames which are born with the joint action of dilations and translations of finite number of signals will worth to study. To investigate these frames, for $a, b \in \mathbb{R}$ with $a > 1$, and $b > 0$, we define the wavelet systems as

$$\mathcal{F}(\mathfrak{f}, a, b) = \Big\{\mathfrak{f}_{j,k} =: a^{j/2}\mathfrak{f}\Big(a^j x - kb\Big) : j, k \in \mathbb{Z}\Big\}. \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 42C40; Secondary 42C15; 43A70; 11S85
Keywords. Nonuniform wavelets; Wavelet frame; Spectral pair; Linear canonical transform
Received: 13 June 2022; Accepted: 25 July 2022
Communicated by Dragan S. Djordjević
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One should note that the systems $\mathcal{F}(\mathfrak{f}, a, b)$ which will become frames for $L^2(\mathbb{R})$ have a wide range of applications [2, 6]. Thus it is important to impose the on $\mathfrak{f}$, a and b such that the system $\mathcal{F}(\mathfrak{f}, a, b)$ will become a frame. It was Daubechies, who in 1990 [7] obtained the first necessary conditions for wavelet frames, where as, Chui and Shi [6] obtained an improved result in 1993. Cassaza and Christensen [4] modified the results of Daubechies's sufficient condition for wavelet frames in $L^2(\mathbb{R})$. This process is continuing till date. For more details, we refer to [5, 8, 20, 23, 24]. Recently, Gabardo an Nashed [10, 11]developed the theory of nonuniform wavelets and wavelet sets in $L^2(\mathbb{R})$ for which the translation set is no longer a discrete subgroup of $\mathbb{R}$, but a union of two lattices. The nonuniform wavelet frames that are developed from spectral pairs were studied by Sharma and Manchanda [21] with the help of Fourier transforms. Actually they studied a necessary and sufficient conditions which makes the nonuniform wavelet system to be a frame for $L^2(\mathbb{R})$. More results in this subject can also be found in [15, 16, 18, 19] and the references therein.

The concept of novel multiresolution analysis in nonuniform settings was established by Shah and Lone [14]. They call it Nonuniform Multiresolution analysis associated with linear canonical transform (LCT-NUMRA). While as Bhat and Dar [1] introduced fractional vector-valued nonuniform MRA and associated wavelet packets (LCT-VNUMRA) where the associated subspace V_0^μ of $L^2\left(\mathbb{R}, \mathbb{C}^M\right)$ has an orthonormal basis of the form $\left\{\mathbf{\Phi}(x-\lambda)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)}\right\}_{\lambda\in\Lambda}$ where $\Lambda = \{0, r/N\} + 2\mathbb{Z}, N \geq 1$ is an integer and r is an odd integer such that r and N are relatively prime. These two authors have also constructed vector-valued nonuniform wavelet packets associated with the novel multiresolution analysis. Moreover they have developed a necessary condition and sufficient condition for nonuniform wavelet frames associated with linear canonical transform[2]. More results in this direction can be found in [3, 12, 13, 17, 22].

Motivated by the above described work, we present generalized inequalities for nonuniform wavelet frames in $L^2(\mathbb{R})$ via linear canonical transform transform. The inequalities we proposed are stated in terms of the linear canonical transform of the wavelet system's generating signals, and the inequalities are better than that obtained in [2] by Bhat and Dar.

The paper is structured as follows. In Section 2, we introduce some notations and preliminaries related to the nonuniform wavelets related to the one-dimensional spectral pairs. Then, we establish sufficient conditions for nonuniform wavelet frame associated with linear canonical transform.

2. Nonuniform Wavelet Frames in $L^2(\mathbb{R})$

For the sake of simplicity, we consider the second order matrix $\mu_{2\times 2} = (A, B, C, D)$. Let us first introduce the definition of Linear Canonical Transform.

Definition 2.1. *The linear canonical transform of any $f \in L^2(\mathbb{R})$ with respect to the unimodular matrix $\mu_{2\times 2} = (A, B, C, D)$is defined by*

$$\mathcal{L}[f](\zeta) = \begin{cases} \int_{\mathbb{R}} f(t)\mathcal{K}_\mu(t,\zeta)dt & B \neq 0 \\ \sqrt{D}\exp\frac{CD\zeta^2}{2} f(D\zeta) & B = 0. \end{cases} \tag{3}$$

where $\mathcal{K}_\mu(t,\zeta)$ is the kernel of linear canonical transform and is given by

$$\mathcal{K}_\mu(t,\zeta) = \frac{1}{\sqrt{2\pi\iota B}}\exp\left\{\frac{\iota(At^2 - 2t\zeta + D\zeta^2)}{2B}\right\}, \quad B \neq 0$$

The inversion formula corresponding to linear canonical transform (3) is defined by

$$f(t) = \int_{\mathbb{R}} \mathcal{L}[f](\zeta)\overline{\mathcal{K}_\mu(t,\zeta)}d\zeta.$$

Morever the well known Parsevel's formula of the linear canonical transform (3) may be stated as

$$\langle \mathcal{L}[f], \mathcal{L}[g]\rangle = \langle f, g\rangle, \quad \text{for all} \quad f, g, L^2(\mathbb{R}).$$

We first recall the definition of a nonuniform multiresolution analysis associated with linear canonical transform (as defined in Ref. [14]) and some of its properties.

Definition 2.2. *For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N-1$ such that r and N are relatively prime, a nonuniform multiresolution analysis associated with linear canonical transform is a sequence of closed subspaces $\left\{V_j^\mu : j \in \mathbb{Z}\right\}$ of $L^2(\mathbb{R})$ such that the following properties hold:*

(a) $V_j^\mu \subset V_{j+1}^\mu$ *for all* $j \in \mathbb{Z}$*;*

(b) $\bigcup_{j\in\mathbb{Z}} V_j^\mu$ *is dense in* $L^2(K)$*;*

(c) $\bigcap_{j\in\mathbb{Z}} V_j^\mu = \{0\}$*;*

(d) $f(t) \in V_j^\mu$ *if and only if* $f(2N\cdot)\, e^{-\iota\pi A\left(1-(2N)^2\right)t^2/B} \in V_{j+1}^\mu$ *for all* $j \in \mathbb{Z}$*;*

(e) There exists a signal φ *in* V_0^μ *such that* $\left\{\varphi(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\right\}$*, is a complete orthonormal basis for* V_0^μ*.*

Given a LCT- NUMRA $\left\{V_j^\mu : j \in \mathbb{Z}\right\}$, we define another sequence $\left\{W_j^\mu : j \in \mathbb{Z}\right\}$ of closed subspaces of $L^2(\mathbb{R})$ by $W_j^\mu := V_{j+1}^\mu \ominus V_j^\mu$, $j \in \mathbb{Z}$. These subspaces inherit the scaling property of V_j^μ, namely,

$$f(\cdot) \in W_j^\mu \quad \text{if and only if} \quad f(2N\cdot)\, e^{2\iota\pi\lambda\zeta/B} \in W_{j+1}^\mu. \tag{4}$$

Moreover, the subspaces $\left\{W_j^\mu : j \in \mathbb{Z}\right\}$ are mutually orthogonal, and we have the following orthogonal decomposition:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j^\mu = V_0^\mu \oplus \left(\bigoplus_{j\geq 0} W_j^\mu\right). \tag{5}$$

A set of signals $\left\{\mathfrak{f}_1^\mu, \mathfrak{f}_1^\mu, \ldots, \mathfrak{f}_{2N-1}^\mu\right\}$ in $L^2(\mathbb{R})$ is said to be a *set of basic wavelets* associated with the LCT-NUMRA $\left\{V_j^\mu : j \in \mathbb{Z}\right\}$ if the family of signals $\left\{\mathfrak{f}_\ell(t-\lambda)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda\right\}$ forms an orthonormal basis for W_0^μ.

In view of (4) and (5), it is clear that if $\{\mathfrak{f}_1, \mathfrak{f}_1, \ldots, \mathfrak{f}_{2N-1}\}$ is a set of basic wavelets, then $\left\{(2N)^{j/2}\mathfrak{f}_\ell\Big((2N)^j t - \lambda\Big)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda\right\}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$.

Given $N \geq 1$, where $N \in \mathbb{Z}$ with an odd integer r under the assumption $1 \leq r \leq 2N-1$ such that r and N are relatively prime, let us define

$$\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z} = \left\{\frac{rk}{N} + 2n : n \in \mathbb{Z}, k = 0, 1\right\}. \tag{6}$$

One can verify that Λ is neither a group nor a uniform discrete set. However, it is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Furthermore, Λ is the spectrum for the spectral set $\Upsilon = \left[0, \frac{1}{2}\right) \cup \left[\frac{N}{2}, \frac{N+1}{2}\right)$ and the pair (Λ, Υ) is called a *spectral pair*.

With $\mathfrak{f} \in L^2(\mathbb{R})$, let us consider the nonuniform wavelet system associated with linear canonical transform as

$$\mathcal{G}(\mathfrak{f}, j, \lambda, \mu) = \left\{\mathfrak{f}_{j,\lambda}^\mu =: (2N)^{j/2}\mathfrak{f}\Big((2N)^j t - \lambda\Big)e^{-\frac{\iota\pi A}{B}(t^2-\lambda^2)} : j \in \mathbb{Z}, \lambda \in \Lambda\right\}. \tag{7}$$

The above defied wavelet system $\mathcal{G}(\mathfrak{f}, j, \lambda, \mu)$ is a nonuniform wavelet frame for $L^2(\mathbb{R})$ associated with linear canonical transform, if we have the positive constants $0 < C \leq D < \infty$ such that

$$C\left\|f\right\|_2^2 \leq \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f, \mathfrak{f}_{j,\lambda}^\mu\right\rangle\right|^2 \leq D\left\|f\right\|_2^2, \tag{8}$$

is true for every signal $f \in L^2(\mathbb{R})$. To defend our main results, we require the help of the following well known lemma.

Lemma 2.3. *Suppose that $\{f_k\}_{k=1}^{\infty}$ is a family of elements in a Hilbert space $\mathbb{H}$ such that the inequalities (1) holds for all f in a dense subset $\mathfrak{D}$ of $\mathbb{H}$. Then, the same inequalities is true for all $f \in \mathbb{H}$.*

In the light of Lemma 2.3, let us consider the following set of signals:

$$\mathfrak{D} = \left\{ f \in L^2(\mathbb{R}) : \hat{f} \in L^\infty(\mathbb{R}) \text{ and } \hat{f} \text{ has compact support in } \mathbb{R} \right\}.$$

It is obvious that $\mathfrak{D}$ is a dense subspace of the Hilbert space. Hence, it suffices to show that the system $\mathcal{G}(\mathfrak{f}, j, \lambda, \mu)$ given by (7) is a frame for the Hilbert space if (8) hold for all $f \in \mathfrak{D}$. Further, we require the following lemma on nonuniform wavelet frames whose proof can be omitted for the sake of brevity.

Lemma 2.4. *Let $f \in \mathfrak{D}$ and $\mathfrak{f} \in L^2(\mathbb{R})$. If esssup$\left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\Big(((2N)^j B)^{-1}\zeta\Big)\right|^2 : \zeta \in [1, 2NB] \right\} < \infty$, then*

$$\sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left| \left\langle f, \mathfrak{f}^{\mu}_{j,\lambda} \right\rangle \right|^2 = \int_{\mathbb{R}} \left| \hat{f}\left(\frac{\zeta}{B}\right) \right|^2 \sum_{j\in\mathbb{Z}} \left| \hat{\mathfrak{f}}\Big(((2N)^j B)^{-1}\zeta\Big) \right|^2 d\zeta + \mathfrak{T}_{\mathfrak{f}}(f), \tag{9}$$

where $\mathfrak{T}_{\mathfrak{f}}(f) = \mathfrak{T}_0 + \mathfrak{T}_1 + \cdots + \mathfrak{T}_{2N-1}$ and for $0 \le p \le 2N-1$, $\mathfrak{T}_p$ is given by

$$\begin{aligned} \mathfrak{T}_p &= \frac{1}{4N} \sum_{j\in\mathbb{Z}} \sum_{\ell\neq p} \int_{\mathbb{R}} \left\{ \overline{\hat{f}\left(\frac{\zeta}{B} + (2N)^j \frac{p}{2}\right)} \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{p}{2}\right) \hat{f}\left(\frac{\zeta}{B} + (2N)^j \frac{\ell}{2}\right) \right. \\ &\quad \left. \times \overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{\ell}{2}\right)} \left(1 + e^{\pi i \frac{r}{N}(\ell - p)}\right) \right\} d\zeta. \end{aligned} \tag{10}$$

Let us now establish the first sufficient condition for the nonuniform wavelet system $\mathcal{G}(\mathfrak{f}, j, \lambda, \mu)$ associated with linear canonical transform given by (7) to be a frame for $L^2(\mathbb{R})$. Here we not only obtain various Inequalities for the nonuniform wavelet system but also modify the already obtained inequalities in [2]. So, we set

$$\Omega_{\mathfrak{f}}(m) = \operatorname{ess\,sup} \left\{ \sum_{j\in\mathbb{Z}} \left| \beta_{\mathfrak{f}}\left(m, \frac{\zeta}{(2N)^j B}\right) \right| : \zeta \in [1, 2NB] \right\}, \tag{11}$$

where

$$t_{\mathfrak{f}}(m, \zeta) = \sum_{k\in\mathbb{N}_0} \hat{\mathfrak{f}}\Big((2N)^k \zeta\Big) \overline{\hat{\mathfrak{f}}\left((2N)^k \Big(\frac{\zeta}{B} + \frac{m}{2}\Big)\right)}. \tag{12}$$

We also use the following set:

$$\Theta = \Big\{ (2N)Bk + \ell : k \in \mathbb{N}_0, 1 \le \ell \le 2N-1 \Big\}.$$

Analogously for the nonuniform case, we establish the first sufficient condition as follows.

Theorem 2.5. *Suppose $\mathfrak{f}^{\mu} \in L^2(\mathbb{R})$ such that*

$$\begin{aligned} \mathfrak{A}_{\mathfrak{f}} &= ess \inf_{\zeta\in[1,2NB]} \sum_{j\in\mathbb{Z}} \left| \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right) \right|^2 - \sum_{\nu\neq\nu'\in\Theta} \left[\Omega_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2}\right) \cdot \Omega_{\mathfrak{f}}\left(\frac{\beta-\nu}{2}\right) \right]^{1/2} > 0, \\ \mathfrak{B}_{\mathfrak{f}} &= ess \sup_{\zeta\in[1,2NB]} \sum_{j\in\mathbb{Z}} \left| \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right) \right|^2 + \sum_{\nu\neq\nu'\in\Theta} \left[\Omega_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2}\right) \cdot \Omega_{\mathfrak{f}}\left(\frac{\beta-\nu}{2}\right) \right]^{1/2} < \infty. \end{aligned}$$

Then $\left\{ \mathfrak{f}^{\mu}_{j,\lambda} : j \in \mathbb{Z}, \lambda \in \Lambda \right\}$ is a frame for $L^2(\mathbb{R})$ with bounds $\mathfrak{A}_{\mathfrak{f}}$ and $\mathfrak{B}_{\mathfrak{f}}$.

Proof. As the last series in (10) converges absolutely for every $f \in \mathfrak{D}$, we use Levi lemma to estimate $\mathfrak{T}_{\mathfrak{f}}(f)$ by rearranging the series as well as changing the orders of summation and integration . Therefore we have

$$
\begin{aligned}
\left|\mathfrak{T}_{\mathfrak{f}}(f)\right| &\leq \frac{1}{2NB}\sum_{p=0}^{2N-1}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\overline{\hat{f}\left(\frac{\zeta}{B}\right)}\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}\right)\right|\left\{\sum_{\ell\neq p}\left|\hat{f}\left(\frac{\zeta}{B}+(2N)^j\left(\frac{\ell-p}{2}\right)\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}+\frac{\ell-p}{2}\right)}\right|\right\}d\zeta \\
&= \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\overline{\hat{f}\left(\frac{\zeta}{B}\right)}\right|\left\{\sum_{k\in\mathbb{N}_0}\sum_{\nu\neq\nu'\in\Theta}\left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}\right)\right.\right. \\
&\qquad\left.\left.\times\hat{f}\left(\frac{\zeta}{B}+(2N)^{j+k}\left(\frac{\nu-\nu'}{2}\right)\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}+(2N)^k\left(\frac{\nu-\nu'}{2}\right)\right)}\right|\right\}d\zeta \\
&= \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\int_{\mathbb{R}}\left|\overline{\hat{f}\left(\frac{\zeta}{B}\right)}\right|\left\{\sum_{k\in\mathbb{N}_0}\sum_{\nu\neq\nu'\in\Theta}\sum_{j\in\mathbb{Z}}\left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^{j-k}}\right)\right.\right. \\
&\qquad\left.\left.\times\hat{f}\left(\frac{\zeta}{B}+(2N)^{j}\left(\frac{\nu-\nu'}{2}\right)\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^{j-k}}+(2N)^k\left(\frac{\nu-\nu'}{2}\right)\right)}\right|\right\}d\zeta \\
&= \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\int_{\mathbb{R}}\left|\overline{\hat{f}\left(\frac{\zeta}{B}\right)}\right|\left\{\sum_{j\in\mathbb{Z}}\sum_{\nu\neq\nu'\in\Theta}\left|\hat{f}\left(\frac{\zeta}{B}+(2N)^{j}\left(\frac{\nu-\nu'}{2}\right)\right)\right.\right. \\
&\qquad\left.\left.\times\sum_{k\in\mathbb{P}}\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^{j-k}}\right)\overline{\hat{\mathfrak{f}}\left((2N)^k\left(\frac{\zeta}{(2N)^jB}+\frac{\nu-\nu'}{2}\right)\right)}\right|\right\}d\zeta \\
&= \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\int_{\mathbb{R}}\left|\overline{\hat{f}\left(\frac{\zeta}{B}\right)}\right|\left\{\sum_{j\in\mathbb{Z}}\sum_{\nu\neq\nu'\in\Theta}\left|\hat{f}\left(\frac{\zeta}{B}+(2N)^{j}\left(\frac{\nu-\nu'}{2}\right)\right)\right.\right. \\
&\qquad\left.\left.\times\left|t_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^jB}\right)\right|\right\}d\zeta \\
&= \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\nu\neq\nu'\in\Theta}\int_{\mathbb{R}}\left\{\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|\left|t_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^jB}\right)\right|^{1/2}\right\} \\
&\qquad\times\left\{\left|\hat{f}\left(\frac{\zeta}{B}+(2N)^{j}\left(\frac{\nu-\nu'}{2}\right)\right)\right|\left|t_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^jB}\right)\right|^{1/2}\right\}d\zeta \\
&\leq \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\nu\neq\nu'\in\Theta}\left\{\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2\left|t_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^jB}\right)\right|d\zeta\right\}^{1/2} \\
&\qquad\times\left\{\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}+(2N)^{j}\left(\frac{\nu-\nu'}{2}\right)\right)\right|^2\left|t_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^jB}\right)\right|d\zeta\right\}^{1/2} \\
&\leq \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\sum_{\nu\neq\nu'\in\Theta}\left\{\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2\left|t_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^jB}\right)\right|d\zeta\right\}^{1/2} \\
&\qquad\times\left\{\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}+(2N)^{j}\left(\frac{\nu-\nu'}{2}\right)\right)\right|^2\left|t_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^jB}\right)\right|d\zeta\right\}^{1/2}
\end{aligned}
$$

$$= \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\sum_{\nu\neq\nu'\in\Theta}\left\{\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2\left|t_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^jB}\right)\right|d\zeta\right\}^{1/2}$$
$$\times\left\{\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat{f}(\zeta)\right|^2\left|t_{\mathfrak{f}}\left(-\frac{\nu-\nu'}{2},\frac{\zeta}{(2N)^j}\right)\right|d\zeta\right\}^{1/2}$$
$$\leq \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\sum_{\nu\neq\nu'\in\Theta}\left\{\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2\Omega_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2}\right)d\zeta\right\}^{1/2}$$
$$\times\left\{\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2\Omega_{\mathfrak{f}}\left(\frac{-(\nu-\nu')}{2}\right)d\zeta\right\}^{1/2}$$
$$= \frac{1}{2NB}\sum_{\nu=0}^{2N-1}\int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 d\zeta\sum_{\nu\neq\nu'\in\Theta}\left[\Omega_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2}\right)\cdot\Omega_{\mathfrak{f}}\left(\frac{-(\nu-\nu')}{2}\right)\right]^{1/2}.$$

It is clear from the expression (9) in Lemma 2.3 that

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f,\mathfrak{f}^{\mu}_{j,\lambda}\right\rangle\right|^2 \geq \int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}\right)\right|^{\cdot} - \sum_{\nu\neq\nu'\in\Theta}\left[\Omega_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2}\right)\cdot\Omega_{\mathfrak{f}}\left(\frac{-(\nu-\nu')}{2}\right)\right]^{1/2}\right\}d\zeta, \tag{13}$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f,\mathfrak{f}^{\mu}_{j,\lambda}\right\rangle\right|^2 \leq \int_{\mathbb{R}}\left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}\right)\right|^2 + \sum_{\nu\neq\nu'\in\Theta}\left[\Omega_{\mathfrak{f}}\left(\frac{\nu-\nu'}{2}\right)\cdot\Omega_{\mathfrak{f}}\left(\frac{-(\nu-\nu')}{2}\right)\right]^{1/2}\right\}d\zeta. \tag{14}$$

We take infimum in (13) and supremum in (14).Thus

$$\mathfrak{A}_{\mathfrak{f}}\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f,\mathfrak{f}^{\mu}_{j,\lambda}\right\rangle\right|^2 \leq \mathfrak{B}_{\mathfrak{f}}\|f\|_2^2,$$

hold for all $f\in\mathfrak{D}$. This completes the proof of Theorem 2.5. □

On the same platform as in [2], we modify the notations as:

$$\Xi = \left\{\nu\in\mathbb{R} : \text{there exists } (j,\lambda)\in\mathbb{Z}\times\Lambda \text{ such that } \nu = \frac{(2N)^jB(\lambda-\sigma)}{2}; \lambda\neq\sigma\right\}, \tag{15}$$

and for all $\nu\in\Xi$, we define

$$I(\nu) = \left\{(j,\lambda)\in\mathbb{Z}\times\Lambda : \nu = \frac{(2N)^jB(\lambda-\sigma)}{2}\right\}, \tag{16}$$

$$\Omega^{+}_{\nu}(\zeta) = \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)}\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}+\frac{\lambda-\sigma}{2}\right)}, \tag{17}$$

$$\Omega^{-}_{\nu}(\zeta) = \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)}\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^jB}-\frac{\lambda-\sigma}{2}\right)}. \tag{18}$$

Having defined above notations, we reintroduce the following result as.

Theorem 2.6. *Suppose $\mathfrak{f} \in L^2(\mathbb{R})$ such that*

$$\begin{aligned} \mathfrak{C}_{\mathfrak{f}} &= ess \inf_{\zeta\in[1,2NB]} \left\{ \Omega_0^+(\zeta) - \sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^+(\zeta)\right| \right\} > 0, \\ \mathcal{D}_{\mathfrak{f}} &= ess \sup_{\zeta\in[1,2NB]} \sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^+(\zeta)\right| < +\infty. \end{aligned}$$

Then $\left\{\mathfrak{f}^{\mu}_{j,\lambda} : j \in \mathbb{Z},\ \lambda \in \Lambda\right\}$ is a wavelet frame for $L^2(\mathbb{R})$ with bounds $\mathfrak{C}_{\mathfrak{f}}$ and $\mathcal{D}_{\mathfrak{f}}$.

Proof. We first note that $\Omega_0^+(\zeta) = \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(\zeta/(2N)^j\right)\right|^2$ by the definition of $\Omega_\nu^+(\zeta)$. We apply Lemma 2.3 to re-estimate $\mathfrak{T}_{\mathfrak{f}}(f)$ for $f \in \mathfrak{D}$ as

$$\begin{aligned} \left|\mathfrak{T}_{\mathfrak{f}}(f)\right| &= \Bigg| \frac{1}{4NB} \sum_{p=0}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\ell\neq p} \int_{\mathbb{R}} \Bigg\{ \overline{\hat{f}\left(\frac{\zeta}{B} + (2N)^j \frac{p}{2}\right)} \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{p}{2}\right) \hat{f}\left(\frac{\zeta}{B} + (2N)^j \frac{\ell}{2}\right) \\ &\qquad \times \overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{\ell}{2}\right)} \left(1 + e^{\pi i \frac{r}{N}(\ell - p)}\right) \Bigg\} d\zeta \Bigg| \\ &\le \frac{1}{2NB} \sum_{p=0}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\ell\neq p} \int_{\mathbb{R}} \left| \overline{\hat{f}\left(\frac{\zeta}{B} + (2N)^j \frac{p}{2}\right)} \hat{f}\left(\zeta + (2N)^j \frac{\ell}{2}\right) \right| \\ &\qquad \times \left| \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{p}{2}\right) \overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{\ell}{2}\right)} \right| d\zeta \\ &= \sum_{\nu\in\Xi\setminus\{0\}} \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \int_{\mathbb{R}} \left| \overline{\hat{f}\left(\frac{\zeta}{B} + (2N)^j \frac{\lambda}{2}\right)} \hat{f}\left(\zeta + (2N)^j \frac{\sigma}{2}\right) \right| \\ &\qquad \times \left| \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{\lambda}{2}\right) \overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{\sigma}{2}\right)} \right| d\zeta \\ &\le \sum_{\nu\in\Xi\setminus\{0\}} \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \int_{\mathbb{R}} \left| \overline{\hat{f}\left(\frac{\zeta}{B}\right)} \hat{f}\left(\frac{\zeta}{B} + (2N)^j \left(\frac{\lambda - \sigma}{2}\right)\right) \right| \\ &\qquad \times \left| \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right) \overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{\lambda - \sigma}{2}\right)} \right| d\zeta \\ &= \sum_{\nu\in\Xi\setminus\{0\}} \int_{\mathbb{R}} \left| \overline{\hat{f}\left(\frac{\zeta}{B}\right)} \hat{f}\left(\frac{\zeta}{B} + \frac{\nu}{2}\right) \right| \left\{ \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \left| \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right) \overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{\lambda - \sigma}{2}\right)} \right| \right\} d\zeta \\ &= \sum_{\nu\in\Xi\setminus\{0\}} \int_{\mathbb{R}} \left| \overline{\hat{f}\left(\frac{\zeta}{B}\right)} \hat{f}\left(\frac{\zeta}{B} + \frac{\nu}{2}\right) \right| \left|\Omega_\nu^+(\zeta)\right| d\zeta \qquad \big(\text{By Eq. (17)}\big) \\ &\le \sum_{\nu\in\Xi\setminus\{0\}} \left\{ \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left|\Omega_\nu^+(\zeta)\right| d\zeta \right\}^{1/2} \left\{ \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B} + \frac{\nu}{2}\right)\right|^2 \left|\Omega_\nu^+(\zeta)\right| d\zeta \right\}^{1/2} \\ &\le \left\{ \sum_{\nu\in\Xi\setminus\{0\}} \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left|\Omega_\nu^+(\zeta)\right| d\zeta \right\}^{1/2} \left\{ \sum_{\nu\in\Xi\setminus\{0\}} \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B} + \frac{\nu}{2}\right)\right|^2 \left|\Omega_\nu^+(\zeta)\right| d\zeta \right\}^{1/2}. \end{aligned} \tag{19}$$

Put $\eta = \dfrac{\zeta}{B} + \nu/2$. We have $\nu = (2N)^j B(\lambda - \sigma)$ for $(j, \lambda) \neq (j, \sigma) \in I(\nu)$ that

$$\begin{aligned}
\Omega_\nu^+(\zeta) &= \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B} + \frac{\lambda-\sigma}{2}\right)} \\
&= \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \hat{\mathfrak{f}}\left(((2N)^j B)^{-1}\left(\eta - \frac{\nu}{2}\right)\right)\overline{\hat{\mathfrak{f}}\left(((2N)^j B)^{-1}\left(\eta - \frac{\nu}{2}\right) + \frac{\lambda-\sigma}{2}\right)} \\
&= \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \hat{\mathfrak{f}}\left(\frac{\eta}{(2N)^j} - ((2N)^j B)^{-1}\frac{\nu}{2}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\eta}{(2N)^j} - ((2N)^j B)^{-1}\frac{\nu}{2} + \frac{\lambda-\sigma}{2}\right)} \\
&= \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \hat{\mathfrak{f}}\left(\frac{\eta}{(2N)^j} - \frac{\lambda-\sigma}{2}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\eta}{(2N)^j}\right)} \\
&= \overline{\Omega_\nu^-(\zeta)} \qquad \big(\text{By Eq. (18)}\big).
\end{aligned}$$

Therefore

$$\sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^+(\eta)\right| = \sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^-(\eta)\right|. \tag{20}$$

Again changing $\zeta + \nu/2$ to η in the last integration of (19), we obtain from (19) and (20) that

$$\begin{aligned}
\left|\mathfrak{T}_{\mathfrak{f}}(f)\right| &\leq \left\{\sum_{\nu\in\Xi\setminus\{0\}} \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left|\Omega_\nu^+(\zeta)\right| d\zeta\right\}^{1/2} \left\{\sum_{\nu\in\Xi\setminus\{0\}} \int_{\mathbb{R}} \left|\hat{f}(\zeta)\right|^2 \left|\overline{\Omega_\nu^-(\eta)}\right| d\eta\right\}^{1/2} \\
&= \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left\{\sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^+(\zeta)\right|\right\} d\zeta.
\end{aligned} \tag{21}$$

Hence from (21) and (9), we get

$$\int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left\{\Omega_0^+(\zeta) - \sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^+(\zeta)\right|\right\} d\zeta \leq \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \mathfrak{f}_{j,\lambda}^\mu\right\rangle\right|^2, \tag{22}$$

and

$$\sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \mathfrak{f}_{j,\lambda}^\mu\right\rangle\right|^2 \leq \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left\{\Omega_0^+(\zeta) + \sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^+(\zeta)\right|\right\} d\zeta,$$

or, equivalently

$$\sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \mathfrak{f}_{j,\lambda}^\mu\right\rangle\right|^2 \leq \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left\{\sum_{\nu\in\Xi} \left|\Omega_\nu^+(\zeta)\right|\right\} d\zeta. \tag{23}$$

We take infimum in (22) and supremum in (23), to get

$$\mathfrak{C}_{\mathfrak{f}}\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \mathfrak{f}_{j,\lambda}^\mu\right\rangle\right|^2 \leq \mathcal{D}_{\mathfrak{f}}\|f\|_2^2.$$

The proof of Theorem 2.6 is complete. □

Remark 2.7. *The crux of our results lies in the fact that the bounds obtained in Theorem 2.6 are far better than those of Bhat and Dar [2]. To be precise, we obtained:*

$$
\begin{aligned}
\mathfrak{A} &= \inf_{\zeta\in[1,2NB]} \left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(((2N)^j B)^{-1}\zeta\right)\right|^2 - \sum_{j\in\mathbb{Z}}\sum_{\ell\neq 0} \left|\hat{\mathfrak{f}}\left(((2N)^j B)^{-1}\zeta\right)\overline{\hat{\mathfrak{f}}\left(((2N)^j B)^{-1}\zeta+\ell/2\right)}\right| \right\} \\
&= \inf_{\zeta\in[1,2NB]} \left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\right|^2 - \sum_{\nu\in\Xi\setminus\{0\}} \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}+\frac{\lambda-\sigma}{2}\right)}\right| \right\} \\
&\leq \inf_{\zeta\in[1,2NB]} \left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\right|^2 - \sum_{\nu\in\Xi\setminus\{0\}} \left| \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}+\frac{\lambda-\sigma}{2}\right)}\right| \right\} \\
&= \inf_{\zeta\in[1,2NB]} \left\{ \Omega_0^+(\zeta) - \sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^+(\zeta)\right| \right\} \\
&= \mathfrak{C}_{\mathfrak{f}},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{B} &= \sup_{\zeta\in[1,2NB]} \left\{ \sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(((2N)^j B)^{-1}\zeta\right)\overline{\hat{\mathfrak{f}}\left(((2N)^j B)^{-1}\zeta+\ell/2\right)}\right| \right\} \\
&= \sup_{\zeta\in[1,2NB]} \left\{ \sum_{\nu\in\Xi\setminus\{0\}} \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}+\frac{\lambda-\sigma}{2}\right)}\right| \right\} \\
&\geq \sup_{\zeta\in[1,2NB]} \left\{ \sum_{\nu\in\Xi\setminus\{0\}} \left| \sum_{(j,\lambda)\neq(j,\sigma)\in I(\nu)} \hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\overline{\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}+\frac{\lambda-\sigma}{2}\right)}\right| \right\} \\
&= \sup_{\zeta\in[1,2NB]} \left\{ \sum_{\nu\in\Xi\setminus\{0\}} \left|\Omega_\nu^+(\zeta)\right| \right\} \\
&= \mathcal{D}_{\mathfrak{f}}.
\end{aligned}
$$

With the notations in (17) and (18), we define new sets as

$$
\Delta_\nu^+ = \operatorname{ess\,sup}\left\{ \left|\Omega_\nu^+(\zeta)\right| : \zeta\in[1,2NB] \right\}, \qquad \Delta_\nu^- = \operatorname{ess\,sup}\left\{ \left|\Omega_\nu^-(\zeta)\right| : \zeta\in[1,2NB] \right\}.
$$

Theorem 2.8. *Suppose* $\mathfrak{f}\in L^2(\mathbb{R})$ *such that*

$$
\begin{aligned}
\mathfrak{E}_{\mathfrak{f}} &= ess \inf_{\zeta\in[1,2NB]} \left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\right|^2 \right\} - \sum_{\nu\in\Xi\setminus\{0\}} \left[\Delta_\nu^+ \Delta_\nu^-\right]^{1/2} > 0 \\
\mathfrak{F}_{\mathfrak{f}} &= ess \sup_{\zeta\in[1,2NB]} \left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\right|^2 \right\} + \sum_{\nu\in\Xi\setminus\{0\}} \left[\Delta_\nu^+ \Delta_\nu^-\right]^{1/2} < \infty.
\end{aligned}
$$

Then $\left\{\mathfrak{f}^{\mu}_{j,\lambda} : j\in\mathbb{Z}, \lambda\in\Lambda\right\}$ *is a wavelet frame for* $L^2(\mathbb{R})$ *with bounds* $\mathfrak{E}_{\mathfrak{f}}$ *and* $\mathfrak{F}_{\mathfrak{f}}$.

Proof. By equation (19), we have

$$
\begin{aligned}
\left|\mathfrak{T}_{\mathfrak{f}}(f)\right| &\leq \sum_{\nu\in\Xi\setminus\{0\}} \left\{ \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left|\Omega_\nu^+(\zeta)\right| d\zeta \right\}^{1/2} \left\{ \int_{\mathbb{R}} \left|\hat{f}\left(\zeta+\frac{\nu}{2}\right)\right|^2 \left|\Omega_\nu^+(\zeta)\right| d\zeta \right\}^{1/2} \\
&= \sum_{\nu\in\Xi\setminus\{0\}} \left\{ \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left|\Omega_\nu^+(\zeta)\right| d\zeta \right\}^{1/2} \left\{ \int_{\mathbb{R}} \left|\hat{f}(\zeta)\right|^2 \left|\Omega_\nu^-(\zeta)\right| d\zeta \right\}^{1/2} \\
&\leq \sum_{\nu\in\Xi\setminus\{0\}} \left[\Delta_\nu^+ \Delta_\nu^-\right]^{1/2} \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 d\zeta.
\end{aligned}
$$

With the same lines on Theorem 2.5, we get

$$\int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\right|^2 - \sum_{\nu\in\Xi\setminus\{0\}} \left[\Delta_\nu^+ \,\Delta_\nu^-\right]^{1/2} \right\} d\zeta \le \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \mathfrak{f}_{j,\lambda}^{\mu}\right\rangle\right|^2,$$

and

$$\sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \mathfrak{f}_{j,\lambda}^{\mu}\right\rangle\right|^2 \le \int_{\mathbb{R}} \left|\hat{f}\left(\frac{\zeta}{B}\right)\right|^2 \left\{ \sum_{j\in\mathbb{Z}} \left|\hat{\mathfrak{f}}\left(\frac{\zeta}{(2N)^j B}\right)\right|^2 + \sum_{\nu\in\Xi\setminus\{0\}} \left[\Delta_\nu^+ \,\Delta_\nu^-\right]^{1/2} \right\} d\zeta$$

Hence

$$\mathfrak{E}_{\mathfrak{f}} \|f\|_2^2 \le \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\left\langle f, \mathfrak{f}_{j,\lambda}^{\mu}\right\rangle\right|^2 \le \mathfrak{F}_{\mathfrak{f}} \|f\|_2^2.$$

This completes the proof of Theorem 2.8. □

Conclusion

The paper deals with the synthesis problem for the nonuniform wavelets in the classic Hilbert space $L^2(\mathbb{R})$. We here proposed two types of the nonuniform frames and prove bounds for the appropriate wavelets using linear canonical transform. Several special transforms can be obtained from the linear canonical transform. For example, for $\mu = (1, B, 0, 1)$, gives the Fresnel transform, for $\mu = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$ the LCT yields us the fractional Fourier transform whereas for $\mu = (0, 1, -1, 0)$, we reach at the classical Fourier transform. Moreover, Bi-lateral Laplace, Gauss-Weierstrass, and Bargmann transform are also its special cases. Our results will therefore hold true for these transformations also.

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