Filomat 37:12 (2023), 3715–3724
https://doi.org/10.2298/FIL2312715H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on fractional Simpson-like type inequalities for functions whose third derivatives are convex

Fatih Hezenci[a], Hüseyin Budak[a]

[a]*Department of Mathematics, Faculty of Science and Arts, Duzce University, Turkiye*

Abstract. In this paper, equality is established for Riemann–Liouville fractional integral. With the aid of this equality, it is proved some fractional Simpson-like type inequalities for functions whose third derivatives in absolute value are convex. By using special cases of the main results, previously obtained Simpson type inequalities are found for the Riemann–Liouville fractional integral. Furthermore, the mathematical example is presented to verify the newly established inequality.

1. Introduction

The inequality theory is a considerable topic and remains an interesting research area with numerous number of applications in many mathematical fields. In addition, convex functions have also a significant place in the theory of inequality. Many inequalities have been investigated for convex functions but the most prominent is the Simpson type inequality, because of its rich geometrical importance and applications. The following inequality is one of the well-known outcome in the literature as the classical Simpson type inequality for four times continuously differentiable functions.

Theorem 1.1. *Let $\mathfrak{F} : [\eta, \mu] \to \mathbb{R}$ denote a four times continuously differentiable function on (η, μ), and let $\left\|\mathfrak{F}^{(4)}\right\|_\infty = \sup_{x\in(\eta,\mu)} \left|\mathfrak{F}^{(4)}(x)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{6}\left[\mathfrak{F}(\eta) + 4\mathfrak{F}\left(\frac{\eta+\mu}{2}\right) + \mathfrak{F}(\mu)\right] - \frac{1}{\mu-\eta}\int_\eta^\mu \mathfrak{F}(x)dx\right| \le \frac{1}{2880}\left\|\mathfrak{F}^{(4)}\right\|_\infty (\mu-\eta)^4 .$$

The convex theory is an impressive method to solve a large number of problems from varied branches of mathematics. Hence, many papers are established the Simpson type inequalities for convex function. For instance, Sarikaya et al. proved the new variants of Simpson type inequalities with the aid of differentiable convex function in the papers [33]. For results with respect to these types of inequalities one can see Refs. [12, 25] and the references therein. In addition to these, Simpson type inequalities for various convex classes have been studied extensively by many authors (see, [17, 23, 27, 32] and the references therein).

Twice differentiable convex functions have been established by many authors to get significant inequalities. For example, some Simpson type inequalities were presented for functions whose absolute values of

2020 *Mathematics Subject Classification*. Primary 26D15, 26D10; Secondary 26D07, 41A55
Keywords. Simpson type inequalities, Convex function, Fractional integrals, Third derivative
Received: 10 June 2022; Accepted: 23 July 2022
Communicated by Dragan S. Djordjević
Email addresses: fatihezenci@gmail.com (Fatih Hezenci), hsyn.budak@gmail.com (Hüseyin Budak)

derivatives are convex in [31]. Moreover, J. Park proved new estimates on the generalization of Hadamard, Ostrowski and Simpson type inequalities for the case of functions whose second derivatives in absolute value at certain powers are convex and quasi–convex functions in [28]. Furthermore, it was proved some fractional Simpson type inequalities for functions whose second derivatives in absolute value are convex in [8]. It can be referred to [3, 9, 15, 35] for further information about twice differentiable functions.

Some inequalities of Simpson type for functions whose three derivatives in absolute value are the class of (α, m)-geometric-arithmetically-convex functions established in the paper [20]. In addition to this, some applications to special means of positive real numbers were given in this paper. In [1], some inequalities of Simpson type for quasi-convex functions in terms of third derivatives are presented and applications to Simpson numerical quadrature rule is also established. Furthermore, Ozdemir et. al. presented some inequalities by s-convex and s-concave functions in [26]. In the paper [11], the authors proved new inequalities of Simpson type for functions whose third derivatives are extended s-convex functions, and apply these inequalities to provide some inequalities of special means. In the paper [29], J. Park established some new integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex and $s-$convex in the second sense. For further information related to these subjects, we refer to reader to [10, 36] and the references therein.

Mathematical preliminaries of fractional calculus theory, which will be used throughout this paper, will be presented as follows:

The well-known *Gamma function* and *Beta function* are defined by

$$\Gamma(x) := \int\limits_0^\infty \xi^{x-1} e^{-\xi} d\xi$$

and

$$\beta(x, y) := \int\limits_0^1 \xi^{x-1} (1-\xi)^{y-1} d\xi = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively for $0 < x, y < \infty$ and $x, y \in \mathbb{R}$.

Let us consider $\mathfrak{F} \in L_1[\eta, \mu]$. The *Riemann–Liouville integrals* $J_{\eta+}^{\alpha}\mathfrak{F}$ and $J_{\mu-}^{\alpha}\mathfrak{F}$ of order $\alpha > 0$ with $\eta \geq 0$ are defined by

$$J_{\eta+}^{\alpha}\mathfrak{F}(x) = \frac{1}{\Gamma(\alpha)} \int_\eta^x (x-\xi)^{\alpha-1} \mathfrak{F}(\xi) d\xi, \quad x > \eta$$

and

$$J_{\mu-}^{\alpha}\mathfrak{F}(x) = \frac{1}{\Gamma(\alpha)} \int_x^\mu (\xi - x)^{\alpha-1} \mathfrak{F}(\xi) d\xi, \quad x < \mu,$$

respectively. Let us note that $J_{\eta+}^{0}\mathfrak{F}(x) = J_{\mu-}^{0}\mathfrak{F}(x) = \mathfrak{F}(x)$. Let us also note that $\alpha = 1$ in above. Then, the fractional integral becomes to the classical integral.

The fractional integral inequalities and applications have been established with the aid of the Riemann–Liouville fractional integral. For instance, Sarikaya et al. established some Simpson type inequalities for the case of functions whose second derivatives are convex [34]. Moreover, Iqbal et. al. generalized the Simpson type inequalities based on differentiable functions to Riemann-Liouville fractional integrals in the paper [16]. Some Simpson type inequalities using s-$(\alpha, m)-$convex function by Riemann–Liouville fractional integrals were given in [21]. The reader is referred to [5–7, 13, 14, 24, 37] and the references therein for more information and unexplained subjects about several properties of Riemann–Liouville fractional integrals and various fractional integral operators. Whereas Simpson type inequalities for Riemann–Liouville fractional integrals have been considered by the authors, some mathematicians have also established the

Simpson type inequalities for other types of fractional integrals such as k-fractional integral Conformable fractional integrals, Katugampola fractional integrals, etc. Concerning some papers related to these subjects see [2, 18, 19, 22, 30], and references therein.

The aim of this paper is to establish Simpson-like type inequalities involving Riemann-Liouville fractional integrals for a function whose 3rd derivatives are convex. The whole design of the present paper takes the form of three sections including introduction. In Sect. 2, an identity is investigated for function whose 3rd derivatives are convex. By utilizing this equality, it is established several Simpson-like type inequalities to the case of function whose 3rd derivatives in absolute value are convex. Moreover, some remarks and corollaries are presented in this section. Furthermore, we give mathematical example to support the main results. In Sect. 3, some conclusions and further directions of research are discussed.

2. Main results

Lemma 2.1. *Let us note that $\mathfrak{F} : [\eta, \mu] \to \mathbb{R}$ is a three times differentiable function (η, μ) such that $\mathfrak{F}''' \in L_1([\eta, \mu])$. Then, the following equality holds:*

$$\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^{\alpha}}\left[J^{\alpha}_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^{\alpha}_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right] \tag{1}$$

$$=\frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\int_0^1\left(\xi^{\alpha+2}-\xi^2\right)\left[\mathfrak{F}'''\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)-\mathfrak{F}'''\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]d\xi.$$

Proof. With the aid of the integration by parts, we obtain

$$\begin{aligned}
I &= \int_0^1\left(\xi^{\alpha+2}-\xi^2\right)\left[\mathfrak{F}'''\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)-\mathfrak{F}'''\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]d\xi \\
&= \frac{2}{\mu-\eta}\left(\xi^{\alpha+2}-\xi^2\right)\left[\mathfrak{F}''\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)+\mathfrak{F}''\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]\Bigg|_0^1 \\
&\quad -\frac{2}{\mu-\eta}\int_0^1\left((\alpha+2)\xi^{\alpha+1}-2\xi\right)\left[\mathfrak{F}''\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)+\mathfrak{F}''\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]d\xi \\
&= -\frac{2}{\mu-\eta}\Bigg[\frac{2}{\mu-\eta}\left((\alpha+2)\xi^{\alpha+1}-2\xi\right)\left[\mathfrak{F}'\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)-\mathfrak{F}'\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]\Bigg|_0^1 \\
&\quad -\frac{2}{\mu-\eta}\int_0^1\left((\alpha+1)(\alpha+2)\xi^{\alpha}-2\right)\left[\mathfrak{F}'\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)-\mathfrak{F}'\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]d\xi\Bigg] \\
&= \frac{4}{(\mu-\eta)^2}\Bigg[\frac{2}{\mu-\eta}\left((\alpha+1)(\alpha+2)\xi^{\alpha}-2\right)\left[\mathfrak{F}\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)+\mathfrak{F}\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]\Bigg|_0^1
\end{aligned} \tag{2}$$

$$-\frac{2\alpha(\alpha+1)(\alpha+2)}{\mu-\eta}\int_0^1 \xi^{\alpha-1}\left[\mathfrak{F}\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)+\mathfrak{F}\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]d\xi\Bigg]$$

$$=\frac{16}{(\mu-\eta)^3}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]$$

$$-\frac{8\alpha(\alpha+1)(\alpha+2)}{(\mu-\eta)^3}\int_0^1 \xi^{\alpha-1}\left[\mathfrak{F}\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)+\mathfrak{F}\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right]d\xi.$$

With the change of the variable $x=\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta$ and $x=\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\eta$ for $\xi\in[0,1]$, equality (2) can be rewritten as follows

$$I=\frac{16}{(\mu-\eta)^3}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right] \tag{3}$$

$$-\frac{2^{\alpha+3}(\alpha+1)(\alpha+2)\Gamma(\alpha+1)}{(\mu-\eta)^{\alpha+3}}\left[J^{\alpha}_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^{\alpha}_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right].$$

If we multiply the both sides of (3) by $\frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}$, then (1) will be obtained. □

Theorem 2.2. *Assume that the assumptions of Lemma 2.1 are valid. Assume also that the function $|\mathfrak{F}'''|$ is convex on $[\eta,\mu]$. Then, the following inequality holds:*

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^{\alpha}}\left[J^{\alpha}_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^{\alpha}_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right| \tag{4}$$

$$\leq\frac{(\mu-\eta)^3\alpha}{48(\alpha+1)(\alpha+2)(\alpha+3)}\left[\left|\mathfrak{F}'''(\eta)\right|+\left|\mathfrak{F}'''(\mu)\right|\right].$$

Proof. By taking modulus in Lemma 2.1, we have

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^{\alpha}}\left[J^{\alpha}_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^{\alpha}_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right| \tag{5}$$

$$\leq\frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\int_0^1\left|\xi^{\alpha+2}-\xi^2\right|\left|\mathfrak{F}'''\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)-\mathfrak{F}'''\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right|d\xi.$$

From the fact that $|\mathfrak{F}'''|$ is convex, it follows

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^{\alpha}}\left[J^{\alpha}_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^{\alpha}_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right|$$

$$\leq \frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 \left|\xi^{\alpha+2}-\xi^2\right|\left[\frac{\xi}{2}\left|\mathfrak{F}'''(\mu)\right|+\frac{2-\xi}{2}\left|\mathfrak{F}'''(\eta)\right|+\frac{\xi}{2}\left|\mathfrak{F}'''(\eta)\right|+\frac{2-\xi}{2}\left|\mathfrak{F}'''(\mu)\right|\right]d\xi$$

$$=\frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\int_0^1 \left|\xi^{\alpha+2}-\xi^2\right|d\xi\left[\left|\mathfrak{F}'''(\eta)\right|+\left|\mathfrak{F}'''(\mu)\right|\right]=\frac{(\mu-\eta)^3\alpha\left[\left|\mathfrak{F}'''(\eta)\right|+\left|\mathfrak{F}'''(\mu)\right|\right]}{48(\alpha+1)(\alpha+2)(\alpha+3)}.$$

The proof of Theorem 2.2 is completed. □

Example 2.3. *Consider a function $\mathfrak{F}:[\eta,\mu]=[0,1]\to\mathbb{R}$ by $\mathfrak{F}(x)=x^5$. Then, the left hand side of (4) becomes to*

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(0)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{1}{2}\right)+\mathfrak{F}(1)\right]-2^{\alpha-1}\Gamma(\alpha+1)\left[J^{\alpha}_{\frac{1}{2}-}\mathfrak{F}(0)+J^{\alpha}_{\frac{1}{2}+}\mathfrak{F}(1)\right]\right|$$

$$=\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{\alpha^2+3\alpha}{32}+1\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\Gamma(\alpha)}\left[\int_0^{\frac{1}{2}}\xi^{\alpha-1}\xi^5d\xi+\int_{\frac{1}{2}}^1(1-\xi)^{\alpha-1}\xi^5d\xi\right]\right|$$

$$=\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\frac{\alpha^2+3\alpha}{32}+1\right]-\frac{1}{64(\alpha+5)}\left[\alpha+\frac{(\alpha+6)\left(\alpha^4+14\alpha^3+91\alpha^2+334\alpha+640\right)}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\right]\right|.$$

The right hand side of the inequality (4) reduces to $\frac{5\alpha}{4(\alpha+1)(\alpha+2)(\alpha+3)}$. *Hence, we have the following inequality*

$$\frac{1}{32}\left|\frac{\alpha^2+3\alpha+32}{(\alpha+1)(\alpha+2)}-\frac{1}{2(\alpha+5)}\left[\alpha+\frac{(\alpha+6)\left(\alpha^4+14\alpha^3+91\alpha^2+334\alpha+640\right)}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\right]\right|\leq\frac{5\alpha}{4(\alpha+1)(\alpha+2)(\alpha+3)}. \tag{6}$$

As one can see in Figure 1, the result of our appropriate choices in Example 2.3 is provided in the inequality (4).

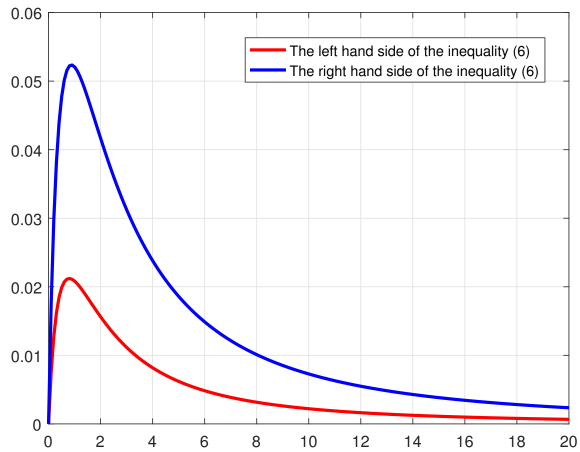

Figure 1: Curves for the result of Example 2.3 calculated and drawn with MATLAB.

Remark 2.4. *Let us consider $\alpha = 1$ in Theorem 2.2. Then, the following Simpson type inequality holds:*

$$\left|\frac{1}{6}\left[\mathfrak{F}(\eta)+4\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{1}{\mu-\eta}\int_{\eta}^{\mu}\mathfrak{F}(\xi)\,d\xi\right| \leq \frac{(\mu-\eta)^3}{1152}\left[\left|\mathfrak{F}'''(\eta)\right|+\left|\mathfrak{F}'''(\mu)\right|\right],$$

which is given by [20, (3) of Remark 3.1].

Theorem 2.5. *Suppose that the assumptions of Lemma 2.1 hold. Suppose also that the function $|\mathfrak{F}'''|^q$, $q > 1$ is convex on $[\eta,\mu]$. Then, the following inequalities hold:*

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^{\alpha}}\left[J^{\alpha}_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^{\alpha}_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right|$$

$$\leq \frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\varphi(\alpha,p)\left[\left(\frac{3\left|\mathfrak{F}'''(\eta)\right|^q+\left|\mathfrak{F}'''(\mu)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{F}'''(\eta)\right|^q+3\left|\mathfrak{F}'''(\mu)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\mu-\eta)^3}{2^{2+\frac{2}{q}}(\alpha+1)(\alpha+2)}\varphi(\alpha,p)\left[\left|\mathfrak{F}'''(\eta)\right|+\left|\mathfrak{F}'''(\mu)\right|\right]$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\varphi(\alpha,p)=\int_0^1\left(\left|\xi^{\alpha+2}-\xi^2\right|^p d\xi\right)^{\frac{1}{p}}.$$

Proof. By applying Hölder inequality in the inequality (5), it follows

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^{\alpha}}\left[J^{\alpha}_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^{\alpha}_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right|$$

$$\leq \frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\left[\int_0^1\left(\left|\xi^{\alpha+2}-\xi^2\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathfrak{F}'''\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)\right|^q d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\int_0^1\left(\left|\xi^{\alpha+2}-\xi^2\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathfrak{F}'''\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right|^q d\xi\right)^{\frac{1}{q}}\right].$$

By using convexity of $|\mathfrak{F}'''|^q$, we obtain

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^{\alpha}}\left[J^{\alpha}_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^{\alpha}_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right|$$

$$\leq \frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\int_0^1\left(\left|\xi^{\alpha+2}-\xi^2\right|^p d\xi\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left[\frac{\xi}{2}\left|\mathfrak{F}'''(\mu)\right|^q+\frac{2-\xi}{2}\left|\mathfrak{F}'''(\eta)\right|^q\right]d\xi\right)^{\frac{1}{q}}+\left(\int_0^1\left[\frac{\xi}{2}\left|\mathfrak{F}'''(\eta)\right|^q+\frac{2-\xi}{2}\left|\mathfrak{F}'''(\mu)\right|^q\right]d\xi\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\int_0^1\left(\left|\xi^{\alpha+2}-\xi^2\right|^p d\xi\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathfrak{F}'''(\eta)\right|^q+\left|\mathfrak{F}'''(\mu)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{F}'''(\eta)\right|^q+3\left|\mathfrak{F}'''(\mu)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

Let us consider $\eta_1=3\left|\mathfrak{F}'''(\eta)\right|^q$, $\mu_1=\left|\mathfrak{F}'''(\mu)\right|^q$, $\eta_2=\left|\mathfrak{F}'''(\eta)\right|^q$ and $\mu_2=3\left|\mathfrak{F}'''(\mu)\right|^q$. Using the facts that,

$$\sum_{k=1}^n(\eta_k+\mu_k)^s\le\sum_{k=1}^n\eta_k^s+\sum_{k=1}^n\mu_k^s,\ 0\le s<1$$

and $1+3^{\frac{1}{q}}\le 4$. The desired result can be obtained straightforwardly. This finalizes the proof of Theorem 2.5. □

Corollary 2.6. *Consider $\alpha=1$ in Theorem 2.5. Then, we can obtain*

$$\left|\frac{1}{6}\left[\mathfrak{F}(\eta)+4\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{1}{\mu-\eta}\int_\eta^\mu\mathfrak{F}(\xi)\,d\xi\right|$$

$$\le\frac{(\mu-\eta)^3}{96}\beta(2p+1,p+1)\left[\left(\frac{3\left|\mathfrak{F}'''(\eta)\right|^q+\left|\mathfrak{F}'''(\mu)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{F}'''(\eta)\right|^q+3\left|\mathfrak{F}'''(\mu)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\le\frac{(\mu-\eta)^3}{3\cdot 2^{3+\frac{2}{q}}}\beta(2p+1,p+1)\left[\left|\mathfrak{F}'''(\eta)\right|+\left|\mathfrak{F}'''(\mu)\right|\right].$$

Theorem 2.7. *Suppose that the assumptions of Lemma 2.1 hold. Suppose also that the function $|\mathfrak{F}'''|^q$, $q\ge 1$ is convex on $[\eta,\mu]$. Then, the following inequality*

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^\alpha}\left[J^\alpha_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^\alpha_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right|$$

$$\le\frac{(\mu-\eta)^3\alpha}{48(\alpha+1)(\alpha+2)(\alpha+3)}\left[\left(\frac{3(\alpha+3)\left|\mathfrak{F}'''(\mu)\right|^q+(5\alpha+23)\left|\mathfrak{F}'''(\eta)\right|^q}{8(\alpha+4)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{3(\alpha+3)\left|\mathfrak{F}'''(\eta)\right|^q+(5\alpha+23)\left|\mathfrak{F}'''(\mu)\right|^q}{8(\alpha+4)}\right)^{\frac{1}{q}}\right]$$

is valid.

Proof. With the aid of the power-mean inequality in (5), we get

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^\alpha}\left[J^\alpha_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^\alpha_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right|$$

$$\leq \frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\left[\left(\int_0^1\left|\xi^{\alpha+2}-\xi^2\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\xi^{\alpha+2}-\xi^2\right|\left|\mathfrak{F}'''\left(\frac{\xi}{2}\mu+\left(\frac{2-\xi}{2}\right)\eta\right)\right|^q d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\xi^{\alpha+2}-\xi^2\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\xi^{\alpha+2}-\xi^2\right|\left|\mathfrak{F}'''\left(\frac{\xi}{2}\eta+\left(\frac{2-\xi}{2}\right)\mu\right)\right|^q d\xi\right)^{\frac{1}{q}}\right].$$

From the fact that $|\mathfrak{F}'''|^q$ is convex, it follows

$$\left|\frac{1}{(\alpha+1)(\alpha+2)}\left[\mathfrak{F}(\eta)+\left(\alpha^2+3\alpha\right)\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu-\eta)^\alpha}\left[J^\alpha_{\frac{\eta+\mu}{2}-}\mathfrak{F}(\eta)+J^\alpha_{\frac{\eta+\mu}{2}+}\mathfrak{F}(\mu)\right]\right|$$

$$\leq \frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\left(\int_0^1\left|\xi^{\alpha+2}-\xi^2\right|d\xi\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\xi^{\alpha+2}-\xi^2\right|\left[\frac{\xi}{2}\left|\mathfrak{F}'''(\mu)\right|^q+\frac{2-\xi}{2}\left|\mathfrak{F}'''(\eta)\right|^q\right]d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\xi^{\alpha+2}-\xi^2\right|\left[\frac{\xi}{2}\left|\mathfrak{F}'''(\eta)\right|^q+\frac{2-\xi}{2}\left|\mathfrak{F}'''(\mu)\right|^q\right]d\xi\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\mu-\eta)^3}{16(\alpha+1)(\alpha+2)}\left(\frac{\alpha}{3(\alpha+3)}\right)^{1-\frac{1}{q}}\left[\left(\frac{\alpha}{8(\alpha+4)}\left|\mathfrak{F}'''(\mu)\right|^q+\frac{\alpha(5\alpha+23)}{24(\alpha+3)(\alpha+4)}\left|\mathfrak{F}'''(\eta)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\alpha}{8(\alpha+4)}\left|\mathfrak{F}'''(\eta)\right|^q+\frac{\alpha(5\alpha+23)}{24(\alpha+3)(\alpha+4)}\left|\mathfrak{F}'''(\mu)\right|^q\right)^{\frac{1}{q}}\right],$$

which completes the proof of Theorem 2.7. □

Remark 2.8. *If we choose $\alpha=1$ in Theorem 2.7, then the following Simpson type inequality holds:*

$$\left|\frac{1}{6}\left[\mathfrak{F}(\eta)+4\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{1}{\mu-\eta}\int_\eta^\mu\mathfrak{F}(\xi)\,d\xi\right|$$

$$\leq\frac{(\mu-\eta)^3}{1152}\left[\left(\frac{3\left|\mathfrak{F}'''(\mu)\right|^q+7\left|\mathfrak{F}'''(\eta)\right|^q}{10}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathfrak{F}'''(\eta)\right|^q+7\left|\mathfrak{F}'''(\mu)\right|^q}{10}\right)^{\frac{1}{q}}\right],$$

which is given by [20, (2) of Remark 3.1].

3. Conclusion

An identity is investigated Riemann–Liouville fractional integral. By using this equality, it is established fractional Simpson-like type inequalities for three times differentiable function. With the help of special choices in our results, previously obtained Simpson type inequalities are found. Moreover, we present mathematical example to provide the main results. In future studies of the mathematicians, improvement or generalization of our results can be researched by using different kind of convex function classes or other types fractional integral operators.

References

[1] M. Alomari and S. Hussainz, Two inequalities of Simpson type for Quasi-Convex functions and applications, Applied Mathematics E-Notes 11 (2011) 110–117.
[2] P. Agarwal, J. Tariboon, and S.K. Ntouyas, Some generalized Riemann–Liouville k-fractional integral inequalities, Journal of Inequalities and Applications 2016(122) (2016).
[3] M. A. Ali, H. Kara, J. Tariboon, S. Asawasamrit, H. Budak, and F. Hezenci, Some new Simpson's-Formula-Type inequalities for twice–differentiable convex functions via generalized fractional operators, Symmetry 13(12) (2021) 2249.
[4] M. Alomari, M. Darus, and S.S. Dragomir, New inequalities of Simpson's type for s-convex functions with applications, RGMIA Res. Rep. Coll. 12(4) (2009).
[5] H. Budak, H. Kara, M. Z. Sarikaya, and M. E. Kiris, New extensions of the Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, Miskolc Mathematical Notes 21(2) (2020) 665–678.
[6] H. Budak, F. Hezenci, and H. Kara, On parameterized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integrals, Mathematical Methods in the Applied Sciences 44(17) (2021) 12522–12536.
[7] H. Budak, F. Hezenci, and H. Kara, On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co–ordinated convex functions via generalized fractional integrals, Advances in Difference Equations 2021 (2021) 1–32.
[8] F. Hezenci, H. Budak, and H. Kara, New version of Fractional Simpson type inequalities for twice differentiable functions, Advances in Difference Equations 2021(460) (2021).
[9] X. You, F. Hezenci, H. Budak, and H. Kara, New Simpson type inequalities for twice differentiable functions via generalized fractional integrals, AIMS Mathematics 7(3) (2021) 3959–3971.
[10] L. Chun and F. Qi, Integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex, Journal of Inequalities and Applications 2013(451) (2013).
[11] L. Chun and F. Qi, Inequalities of Simpson Type for Functions Whose Third Derivatives Are Extended s-Convex Functions and Applications to Means, Journal of Computational Analysis and Applications 19(3) (2015) 555–569.
[12] S. S. Dragomir, R. P. Agarwal, P. Cerone, On Simpson's inequality and applications, Journal of Inequalities and Applications 5 (2000) 533–579.
[13] F. Ertuğral and M. Z. Sarikaya, Simpson type integral inequalities for generalized fractional integral, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 113(4)(2019) 3115–3124.
[14] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, Wien: Springer-Verlag, 1997.
[15] S. Hussain and S. Qaisar, More results on Simpson's type inequality through convexity for twice differentiable continuous mappings, SpringerPlus 5(1) (2016) 1–9.
[16] M. Iqbal, S. Qaisar, S. Hussain, On Simpson's type inequalities utilizing fractional integrals, Journal of Computational Analysis and Applications 23(6) (2017) 1137–1145.
[17] İ. İşcan, Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions, Journal of Mathematics 2014 (2014).
[18] S. Kermausuor, Simpson's type inequalities via the Katugampola fractional integrals for s-convex functions, Kragujevac Journal of Mathematics 45(5) (2021) 709–720.
[19] H. Lei, G. Hu, J. Nie and T. Du, Generalized Simpson-type inequalities considering first derivatives through the k-Fractional Integrals, IAENG International Journal of Applied Mathematics 50(3) (2020) 1–8.
[20] Y. Li and T. Du, Some Simpson type integral inequalities for functions whose third derivatives are (α, m)-GA-convex functions, Journal of the Egyptian Mathematical Society 24(2) (2016) 175–180.
[21] X. Li, S. Qaisar, J. Nasir et al. Some results on integral inequalities via Riemann–Liouville fractional integrals, Journal of Inequalities and Applications 2019(214) (2019).
[22] B. Z. Liu, An inequality of Simpson type, Proc. R. Soc. A 461 (2005) 2155–2158.
[23] W. Liu, Some Simpson type inequalities for h-convex and (α, m)-convex functions, Journal of Computational Analysis and Applications 16(5) (2014) 1005–1012.
[24] S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, NewYork:Wiley, 1993.
[25] M. E. Ozdemir, A. O. Akdemir, and H. Kavurmacı, On the Simpson's inequality for convex functions on the coordinates, Turkish Journal of Analysis and Number Theory 2(5) (2014) 165–169.
[26] M. E. Ozdemir, M. Avcı, and H. Kavurmaci, Simpson type inequalities for functions whose third derivatives in the absolute value are s-convex and s-concave functions, arXiv:1112.3865v1, https://doi.org/10.48550/arXiv.1112.3865.
[27] J. Park, On Simpson-like type integral inequalities for differentiable preinvex functions, Applied Mathematical Sciences 7(121) (2013) 6009–6021.

[28] J. Park, On Some Integral Inequalities for Twice Differentiable Quasi–Convex and Convex Functions via Fractional Integrals, Applied Mathematical Sciences 9(62) (2015) 3057–3069.

[29] J. Park, Hermite-Hadamard type inequalities for functions whose third derivatives are convex and s-convex, Applied Mathematical Sciences (8)1 (2014) 13–31.

[30] S. Rashid, A. O. Akdemir, F. Jarad, M. A. Noor, and K.I. Noor, Simpson's type integral inequalities for k-fractional integrals and their applications, AIMS Mathematics 4(4) (2019) 1087–1100.

[31] M. Z. Sarikaya, N. Aktan, On the generalization of some integral inequalities and their applications, Mathematical and Computer Modelling 54(9-10) (2011) 2175–2182.

[32] Sarikaya, M. Z. Sarikaya, E. Set, and M. E. Özdemir, On new inequalities of Simpson's type for s-convex functions. Computers & Mathematics with Applications 60(8) (2010) 2191–2199.

[33] M. Z. Sarikaya, E. Set, and M. E. Özdemir, On new inequalities of Simpson's type for convex functions, RGMIA Res. Rep. Coll. 13(2) (2010), Article2.

[34] M. Z. Sarikaya, E. Set, and M. E. Özdemir, On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex, Journal of Applied Mathematics, Statistics and Informatics 9(1) (2013) 37–45.

[35] M. Vivas-Cortez, T. Abdeljawad, P. O. Mohammed, and Y. Rangel-Oliveros, Simpson's integral inequalities for twice differentiable convex functions, Mathematical Problems in Engineering, 2020.

[36] Bo-Yan Xi, Shu-Hong Wang, and F. Qi, Some inequalities of Hermite-Hadamard type for functions whose 3rd derivatives are $P-$convex, Applied Mathematics 3 (2012) 1898–1902.

[37] X. You, M. A. Ali, H. Budak, H. Kara, and D. Zhao, Some parameterized Simpson's type inequalities for differentiable convex functions involving generalized fractional integrals, Advances in Continuous and Discrete Models 2022(1) (2022) 1–22.