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Common fixed points for set-valued generalized contractions on a metric space with graphical structure

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Abstract. This manuscript deals with existence of common fixed points and coincidence points for set-valued generalized f-contraction in the setting of metric space having graphical structure. This result enables us to derive fixed points for set-valued generalized contractions on a metric space having directed graph. The main theorem generalizes and improves several results in the literature. An invariant approximation result on a normed linear space is derived from our main result. As an implementation of our main result, we deduce the sufficient criteria for occurrence of solution for the Caputo fractional differential equation.

1. Introduction

In the year 1989, Mizoguchi and Takahashi [14] extended the well known Nadler's [15] theorem for nonlinear set-valued map on a complete metric space. They [14] established that, a set-valued map F from a metric space (Q, d) into $(CB(Q), H)$ (where $CB(Q)$ contains all non-void bounded closed subsets of Q and H acts as a Hausdorff metric on $CB(Q)$) possess a fixed point if for each p, q lies in Q fulfills $H(Fp, Fq) \leq k(d(p, q))d(p, q)$ where $k \in W = \{h : [0, \infty) \to [0, 1) | \limsup_{r \to t^+} h(r) < 1, \forall\, 0 \leq t < \infty\}$. Later, there are various elegant extensions of this result, that can be found in [4, 10, 12, 17]. Subsequently, in the year of 2007, Berinde and Berinde [4] generalized the Mizoguchi-Takahashi's [14] result for set-valued map in the below stated manner.

Theorem 1.1. *[4] Let us assume (Q, d) is complete and $F : Q \to CB(Q)$ in order that for every p and q in Q,*

$$H(Fp, Fq) \leq k(d(p, q))d(p, q) + Md(q, Fp), \tag{1}$$

where $k \in W$ and M is non-negative real number. Then the map F possess a fixed point.

On another side, many researchers are interested for seeking the coincidence points as well as common fixed points for set-valued mappings. A several significant results towards this direction can be found in [10, 12, 16, 18]. In 2007, Kamran [10] improved the above mentioned Theorem 1.1 through the concept of set-valued common fixed point theory. The author [10] derived the succeeding result.

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Theorem 1.2. *[10] Suppose the maps $f : Q \to Q$ and $F : Q \to CB(Q)$ in order that for every p, q lies in Q,*

$$H(Fp, Fq) \leq k(d(fp, fq))d(fp, fq) + Md(fq, Fp), \tag{2}$$

where $k \in W$ and $M \geq 0$. Then there is an element q^* having $fq^* \in Fq^*$ if fQ is complete and for every $q \in Q$, $Fq \subset fQ$. Moreover, $q^* = fq^* \in Fq^*$, whenever $ffq^* \in Ffq^*$ and $ffq^* = fq^*$.

We note that, the above mentioned theorem deduces Theorem 1.1 due to Berinde-Berinde [4] by considering f to be an identity map.

Jachymski [7] first studied the fixed points through the language of graph in the year 2008. He extended the renowned Banach contraction principle via G-contraction on a complete metric space consists with a directed graph G. This result enables the author [7] to extend and unify some fixed point theorems proved in metric spaces as well as metric spaces having partial order. Followed by Jachymski [7], an extension of Mizoguchi-Takahashi's [14] result was derived by Sultana and Vetrivel [17] in the year 2014, on a metric space consisting with a directed graph. The authors [17] demonstrated the below stated theorem.

Theorem 1.3. *[17] Assume a complete metric space (Q, d) incorporates with a directed graph G and a map $F : Q \to CB(Q)$ in order that for every $p, q \in Q$ having $(p, q) \in E(G)$ fulfills*

(i) $H(Fp, Fq) \leq k(d(p, q))d(p, q)$ for $k \in W$,
(ii) if $v_1 \in Fp$ and $v_2 \in Fq$ having $d(v_1, v_2) \leq d(p, q)$ implies $(v_1, v_2) \in E(G)$.

Then F possess a fixed point if there is $q_0 \in Q$ and $N \in \mathbb{N}$ in order that,

(a) $[q_0]_G^N \cap Fq_0 \neq \emptyset$,
(b) for any sequence $\{z_n\}_n$ in Q with $z_n \in [z_{n-1}]_G^N \cap Fz_{n-1}$ for every positive integer n and $z_n \to z$, then there is $\{z_{n_k}\}_k$ in order that for each $k \in \mathbb{N}$, $(z_{n_k}, z) \in E(G)$.

This result enables the authors [17] to derive the fixed point for set-valued uniformly local contraction. As an application of the Theorem 1.3, the authors [17] also improved the Kelisky and Rivlin [11, Theorem 1] result for Bernstein type nonlinear operator on a complete normed linear space. For more results on fixed point using the perception of graph theory can be found in [5, 12].

In this present article, we improve the Theorem 1.2 due to Kamran [10] for set-valued mappings on metric space having graphical structure. Our result enables us to extend the aforementioned Theorem 1.1 for mappings on a complete metric space consisting with graph. Moreover, the above Theorem 1.3 for $N = 1$, follows from our main result. As an application of our result, we scrutinize the the occurrence of a solution for a Caputo fractional differential equation. Furthermore, an invariant approximation result is also established through the idea of common fixed point theory.

2. Preliminaries

In this part, we utilize some essential symbols and definition, which is needful all through this article. Up to the end of this work we denote $CB(Q) = \{P : P \subseteq Q,$ and P is non-void bounded closed$\}$ and a below sated function H which becomes a Hausdorff metric on $CB(Q)$. Consider U and V lies in $CB(Q)$,

$$H(U, V) = \max\left\{\sup_{u \in U} d(u, V),\ \sup_{v \in V} d(v, U)\right\},$$

whereas the notation $d(u, V)$ represents the $\inf_{v \in V} d(u, v)$ and similarly $d(v, U)$. One immediate result is followed from this definition.

Lemma 2.1. *[10] Consider U, $V \in CB(Q)$, then for $s > 1$, the occurrence of u in U having $d(u, v) \leq sH(U, V)$ for every $v \in V$ is guaranteed.*

On the other side, an element q^* is described as a coincidence point for a couple of mappings $f : Q \to Q$ and $F : Q \to CB(Q)$, if $\{fq^*\} \subset Fq^*$. For the set-valued map F, fixed point of F is identified by $Fix(F) = \{q \in Q : q \in Fq\}$ and an element q^* is noted as a common fixed point for the maps f and F if $q^* = fq^* \in Fq^*$. Additionally, map f is described as a F-weakly commuting [10] at an element y in Q if $ffy \in Ffy$. Up to the end, the collection of common fixed points and coincidence points for the maps f and F are expressed by $Fix(f, F)$ and $Coin(f, F)$ respectively.

Now we introduce some basic terminology of graph theory, which is essential for this article. For the metric space (Q, d), let us assume a directed graph $G(V(G), E(G))$, where the notation $V(G)$ represents the collection of all vertices, which is nothing but whole set Q and the collection of edges $E(G)$ includes the set $\Delta = \{(s, s) : (s, s) \in Q \times Q\}$. Also this reflexive graph G does not possess any parallel edges. The character G^{-1} represents the graph, where the edges are in opposite direction of the edges of G, in other word $E(G^{-1}) = \{(v, u) \in Q \times Q : (u, v) \in E(G)\}$. Again we treat $\tilde{G}$ by means of, $E(\tilde{G})$ is symmetric. By this ideology, $E(\tilde{G}) = E(G) \cup E(G^{-1})$.

For $p, q \in Q$, a path [7] between p and q in G of length $J \in \mathbb{N} \cup \{0\}$ if there is a alternating sequence $(z_i)_{i=0}^{J}$ in Q in order that $z_0 = p$, $z_J = q$ and $(z_{i-1}, z_i) \in E(G)$ for each $i \in \{1, 2, \cdots, J\}$. We represent

$$[p]_G^1 = \{q \in Q : \text{ there is directed path from } p \text{ to } q\}.$$

3. Main results

All through this part we consider the metric space (Q, d) incorporates with a directed graph $G(V(G), E(G))$, whereas $V(G) = Q$, $\Delta \subseteq E(G)$ and G has no parallel edges. Now we present our main theorem for ensuring the common fixed point for the generalized set-valued contraction, which is described below.

Definition 3.1. *For a given map $f : Q \to Q$, a set-valued map $F : Q \to CB(Q)$ is defined as a generalized G_f contraction if for every p, q in Q having $(fp, fq) \in E(G)$:*

(a) $H(Fp, Fq) \leq k(d(fp, fq))d(fp, fq) + Md(fq, Fp)$, where $k \in W$ and $M \geq 0$,
(b) if $fu \in Fp$ and $fv \in Fq$ with $d(fu, fv) \leq d(fp, fq)$, then $(fu, fv) \in E(G)$.

Now our main theorem for generalized G_f-contraction is stated below.

Theorem 3.2. *Assume $f : Q \to Q$ in order that (fQ, d) is complete. Suppose that $F : Q \to CB(Q)$ is generalized G_f contraction with $Fq \subset fQ$, for every $q \in Q$. Then the presence of $q^* \in Q$ having $fq^* \in Fq^*$ can be ensured if the following hold:*

(i) $F(q_0) \cap [fq_0]_G^1 \neq \emptyset$ for some $q_0 \in Q$;

(ii) if $\{z_n\}_n \in Q$ with $(z_n, z_{n+1}) \in E(G)$ for every positive integer n and z_n converge to z, then there is $\{z_{n_t}\}_t$ in order that for every $t \in \mathbb{N}$, $(z_{n_t}, z) \in E(G)$.

Moreover, F and f possess a common fixed point whenever the map f is weakly commuting at q^* along with $ffq^* = fq^*$.

Proof. Due to the fact that $F(q_0) \cap [fq_0]_G^1 \neq \emptyset$ and $Fq_0 \subset fQ$ we obtain $q_1 \in Q$ with $(fq_0, fq_1) \in E(G)$. Now using the Lemma 2.1, for $\varepsilon_1 = \frac{1}{\sqrt{k(d(fq_0, fq_1))}} > 1$, there is $q_2 \in Q$ with $fq_2 \in F(q_1)$ satisfying

$$\begin{aligned} d(fq_1, fq_2) &\leq \varepsilon_1 H(Fq_0, Fq_1) \\ &\leq \varepsilon_1[k(d(fq_0, fq_1))d(fq_0, fq_1) + Md(fq_1, Fq_0)] \\ &\leq \sqrt{k(d(fq_0, fq_1))}d(fq_0, fq_1) \leq d(fq_0, fq_1). \end{aligned}$$

As F is generalized G_f-contraction and $(fx_0, fx_1) \in E(G)$ with $fx_1 \in Fx_0$, $fx_2 \in Fx_1$ then the last inequation leads to $(fq_1, fq_2) \in E(G)$. Again for $\varepsilon_2 = \frac{1}{\sqrt{k(d(fq_1, fq_2))}}$ we get $fq_3 \in F(q_2)$ fulfilling

$$\begin{aligned} d(fq_2, fq_3) &\leq \varepsilon_2 H(Fq_1, Fq_2) \\ &\leq \varepsilon_2[k(d(fq_1, fq_2))d(fq_1, fq_2) + Md(fq_2, Fq_1)] \\ &\leq \sqrt{k(d(fq_1, fq_2))}d(fq_1, fq_2) \leq d(fq_1, fq_2). \end{aligned}$$

The previous inequality approaches to $(fq_2, fq_3) \in E(G)$. In this manner for each $\varepsilon_n = \frac{1}{\sqrt{k(d(fq_{n-1}, fq_n))}} > 1$ we acquire $f(q_{n+1}) \in F(q_n)$, which leads to

$$\begin{aligned} d(fq_n, fq_{n+1}) &\leq \varepsilon_n H(Fq_{n-1}, Fq_n) \\ &\leq \varepsilon_n[k(d(fq_{n-1}, fq_n))d(fq_{n-1}, fq_n) + Md(fq_n, Fq_{n-1})] \\ &\leq \sqrt{k(d(fq_{n-1}, fq_n))}d(fq_{n-1}, fq_n) \qquad (3) \\ &\leq d(fq_{n-1}, fq_n). \end{aligned}$$

Thus we can develop two sequences $\{f(q_n)\}_n$ and $\{F(q_n)\}_n$, where $f(q_n) \in F(q_{n-1})$ and for each $n \geq 1$, $(fq_{n-1}, fq_n) \in E(G)$. Again it is clearly observed that the sequence $\{d_n\}_n$, where $d_n = d(fp_n, fp_{n+1})$ is monotone decreasing and bounded below. Therefore it converges to some $a \geq 0$. If $a > 0$, then from the equation (3),

$$a \leq \sqrt{\limsup_{d_{n-1} \to a} k(d_{n-1})}\, a < a;$$

this is absurd. Hence $d_n \to 0$, while $n \to \infty$. Again for each n, form (3) it yields

$$d(fq_n, fq_{n+1}) \leq \prod_{i=0}^{n} \sqrt{k(d(fq_{i-1}, fq_i))}d(fq_0, fq_1). \qquad (4)$$

Since $k : [0, \infty) \to [0, 1)$ with $\limsup_{t \to s^+} k(t) < 1$, then we take $\delta > 0$ and $\beta \in (0, 1)$ in order that $k(t) < \beta^2$ for every $0 < t < \delta$. Again we obtain $N \in \mathbb{N}$ in order that $d(fp_{n-1}, fp_n) < \delta$, for each $n \geq N$ on account of $\{d_n\}_n$ converges to 0. Consequently (4) implies,

$$\begin{aligned} d(fq_n, fq_{n+1}) &\leq \beta^{n-(N-1)} \prod_{j=1}^{N-1} \sqrt{k(d(fq_{j-1}, fq_j))}d(fq_0, fq_1) \\ &< \beta^{n-(N-1)} d(fq_0, fq_1). \end{aligned}$$

Now for any arbitrary $m \in \mathbb{N}$, we have

$$\begin{aligned} d(fq_n, fq_{n+m}) &\leq \sum_{i=1}^{m} d(fq_{n+i-1}, fq_{n+i}) \\ &\leq \beta^{n-(N-1)} \left[1 + \beta + \cdots + \beta^{m-1}\right] d(fq_0, fq_1) \\ &\leq \beta^{n-(N-1)} \frac{1 - \beta^m}{1 - \beta} d(fq_0, fq_1) \leq \frac{\beta^{n-(N-1)}}{1 - \beta} d(fq_0, fq_1). \end{aligned}$$

As a consequence $\{fq_n\}_n$ becomes Cauchy in the complete metric space fQ. Thus $\{fq_n\}_n$ is converges to fq^*, for some $q^* \in Q$.

As for each n, $(fq_{n-1}, fq_n) \in E(G)$ and $fq_n \to fq^*$, then by the hypothesis there is $\{fq_{n_r}\}_r$ having $(fq_{n_r}, fq^*) \in E(G)$, for each $r \geq 1$. Now for each r,

$$\begin{aligned} d(fq^*, Fq^*) &\leq d(fq^*, fq_{n_r+1}) + d(fq_{n_r+1}, Fq^*) \\ &\leq d(fq^*, fq_{n_r+1}) + H(Fq_{n_r}, Fq^*) \\ &\leq d(fq^*, fq_{n_r+1}) + k(d(fq_{n_r}, fq^*))d(fq_{n_r}, fq^*) + Md(fq^*, Fq_{n_r}). \end{aligned}$$

As, $fq_{n_r-1} \in Fq_{n_r}$, then taking $r \to \infty$, the last inequation leads to $fq^* \in Fq^*$.

Now let $p = fq^*$, then $ffq^* = fq^*$ follows that $fp = p$. Again f and F are weakly commuting at q^*, hence $ffq^* \in Ffq^*$ leads to $p = fp \in Fp$. □

Remark 3.3. *Theorem 1.2 due to Kamran [10] is followed from above mentioned Theorem 3.2 by choosing a graph G in order that $V(G) = Q$, $E(G) = \{(p, q) : p,\, q \in Q\}$.*

The below-stated example illustrates the above Theorem 3.2. Further, it indicates that Theorem 3.2 is indeed an extension of the Theorem 1.2 due to Kamran [10].

Example 3.4. *Let $Q = \left\{\frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^n}, \cdots\right\} \cup \{0,1\}$ be incorporated with the usual metric. Suppose that a set-valued map $F : Q \to CB(Q)$ is defined by*

$$F(q) = \begin{cases} \left\{0, \frac{1}{2}\right\} & \text{if} \quad q = 0, \\ \left\{\frac{1}{2}, \frac{1}{2^{n+2}}\right\} & \text{if} \quad q = \frac{1}{2^n} \text{ where } n \in \mathbb{N}, \\ \left\{\frac{1}{2}\right\} & \text{if} \quad q = 1, \end{cases}$$

and $f : Q \to Q$ is defined as

$$f(q) = \begin{cases} \frac{1}{2^{n+1}} & \text{if} \quad q = \frac{1}{2^n},\ n \in \mathbb{N} \cup \{0\} \\ 0 & \text{if} \quad q = 0. \end{cases}$$

Choose a graph G in order that $V(G) = Q$ and $E(G) = \{(p, q) \in Q \times Q : d(p, q) < \frac{1}{4}\}$. Consequently, $E(G)$ includes the set Δ and G does not possess any parallel edges. Consider $p = 0$ and $q = \frac{1}{2^n}$, where $n \geq 2$. Then it occurs $(fp, fq) \in E(G)$ and

$$H(Fp, Fq) = H\left(\left\{0, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2^{n+2}}\right\}\right) = \frac{1}{2^{n+2}} \leq \frac{1}{2} d(fp, fq).$$

Now, consider $p = \frac{1}{2^n}$ and $q = \frac{1}{2^m}$, where $m \geq n \geq 1$. Then we see that $(fp, fq) \in E(G)$ and

$$H(Fp, Fq) = H\left(\left\{\frac{1}{2}, \frac{1}{2^{n+2}}\right\}, \left\{\frac{1}{2}, \frac{1}{2^{m+2}}\right\}\right) = \frac{1}{2^{n+2}} - \frac{1}{2^{m+2}} \leq \frac{1}{2}(d(fp, fq)).$$

Hence for every p, $q \in Q$ with $(fp, fq) \in E(G)$, $H(Fp, Fq) \leq k(d(fp, fq))d(fp, fq) + Md(fq, Fp)$ where $k(t) = \frac{1}{2}$ for $0 \leq t < \infty$ and $M \geq 0$. Moreover, if $fu \in Fp$ and $fv \in Fq$ with $d(fu, fv) \leq d(fp, fq)$, then $(fu, fv) \in E(G)$. Hence F is a generalized G_f contraction for the chosen map f. It is easy to see that $Fq_0 \cap [fq_0]^1_G \neq \emptyset$ for $q_0 = \frac{1}{2}$. Again, let $\{z_n\}_n \in Q$ in order that $(z_n, z_{n+1}) \in E(G)$ for $n \geq 1$ and $z_n \to z$. Subsequently, we achieve a natural number L so that $d(z_n, z) < \frac{1}{4}$ for each $n \geq L$. Therefore we obtain a subsequence $\{z_{n_t}\}_t$ in order that $(z_{n_t}, z) \in E(G)$ for each $t \geq 1$. Further, we observe that (fQ, d) is complete and for each $q \in Q$, $Fq \subset fQ$. Thus all the criteria of the Theorem 3.2 are fulfilled and it ensures the existence of a point $q^* \in Q$ with $fq^* \in Fq^*$. We note that 0 is a coincidence point of f and F, that is, $f0 \in F0$. Moreover, $ff0 = f0$ and $ff0 \in Ff0$. It is worth to mention that 0 is a common fixed point of f and F.

However, for $p = 0$ and $q = 1$, we see that $H(F0, F1) = \frac{1}{2}$, $d(f0, f1) = \frac{1}{2}$ and $d(f1, F0) = 0$. Therefore for all $k \in W$ and $M \geq 0$, $H(F0, F1) = d(f0, f1) > k(d(f0, f1))d(f0, f1) + Md(f1, F0)$. This indicates that F does not meet the condition (2) of Theorem 1.2 due to Kamran.

The upcoming corollary is an extension of the Theorem 1.1 for the mappings on metric space having graphical structure by considering f to be an identity map in the above mentioned Theorem 3.2.

Corollary 3.5. *Assume that (Q, d) is complete and for each (p, q) lies in $E(G)$, a map $F : Q \to CB(Q)$ fulfills,*

(i) *$H(Fp, Fq) \leq k(d(p, q))d(p, q) + Md(q, Fp)$ where $k \in W$ and M is non-negative real number,*
(ii) *if $w_1 \in Fp$ and $w_2 \in Fq$ having $d(w_1, w_2) \leq d(p, q)$, then $(w_1, w_2) \in E(G)$.*

Then F possess a fixed point if there is $q_0 \in Q$ in order that

(a) *$F(q_0) \cap [q_0]^1_G \neq \emptyset$,*
(b) *if $\{z_n\}_n \in Q$ with $(z_n, z_{n+1}) \in E(G)$ for every positive integer n and z_n converge to z, then there is $\{z_{n_t}\}_t$ such that for each $t \in \mathbb{N}$, $(z_{n_t}, z) \in E(G)$.*

Remark 3.6. *The Theorem 1.3 for $N = 1$ due to Sultana and Vetrivel [17] follows from our Theorem 3.2 by choosing f to be an identity map and $M = 0$.*

Edelstein [6, Theorem 5.2] improved the well known Banach contraction principle for single valued uniformly local contraction [6] in the year 1961. Later, Nadler [15, Theorem 6] derived the fixed points for set-valued uniformly local contraction [15] on a metric space. Subsequently, the Nadler's [15] result was generalized by Sultana and Vetrivel [17, Theorem 5] for the mappings satisfied the Mizoguchi-Takahashi's [14] contraction locally. In the succeeding corollary, we present the occurrence of fixed points for the mappings that satisfies the equation (1) locally.

Corollary 3.7. *Let us assume (Q, d) is complete and $F : Q \to CB(Q)$ in order that for $r > 0$ and for every p, q lies in Q having $d(p, q) < r$ meets the contractive condition* (1). *Then occurrence of q^* having $q^* \in Fq^*$ can be assured if there is $q_1 \in Fq_0$ and $d(q_0, q_1) < r$.*

Proof. Take a graph G in Q having $V(G) = Q$ and the collection of edges $E(G) = \{(p, q) : p, q \in Q \text{ and } d(p, q) < r\}$. Subsequently, $E(G)$ does not possess any parallel edges and $\Delta \subseteq E(G)$. Assume that f is an identity map on the set Q. Let p, q lies in Q with $(fp, fq) = (p, q) \in E(G)$. Therefore $d(p, q) = d(fp, fq) < r$ and hence the equation (1) hold. If $fu \in Fp$, $fv \in Fq$ with $d(fu, fv) \le d(fp, fq)$, consequently we obtain $d(fu, fv) < r$, which yields that $(fu, fv) \in E(G)$. Thus F turns out to be a G_f contraction.

Further, there is $q_0 \in Q$ and $q_1 \in Fq_0$ with $d(q_0, q_1) < r$, hence it is simple to visualize that $Fq_0 \cap [q_0]^1_G \neq \emptyset$. Let $\{z_n\}_n \in Q$ converge z with $(z_n, z_{n+1}) \in E(G)$ for every n, in consequence we achieve a natural number L in order that $d(z_n, z) < r$ for every $n \ge L$. Consequently, we get a sub-sequence $\{z_{n_t}\}_t$ such that $d(z_{n_t}, z) < r$, for every $t \in \mathbb{N}$, hence $(z_{n_t}, z) \in E(G)\ \forall t \ge 1$. Thus every criteria of the Theorem 3.2 are fulfilled. Subsequently there is q^* having $q^* \in Fq^*$. □

In the year 2008, Jachymski [7, Theorem 4.1] derived the convergence of iterates for certain linear operator on a complete normed linear space through fixed points on Banach spaces having graphical structure. In the succeeding theorem, Sultana and Vetrivel [17] generalized this Jachymski's [7, Theorem 4.1] result for some nonlinear operator. In fact, the below sated result due to Sultana and Vetrivel [17] follows from our main Theorem 3.2 by choosing some suitable graph.

Theorem 3.8. *[17] Assume Q_0 is a closed subspace of a complete normed linear space Q and $g : Q \to Q$ in order that for each p, q lies in Q having $p - q \in Q_0$ follows*

$$||gp - gq|| \le k(||p - q||)||p - q||, \text{ where } k \in W. \tag{5}$$

Therefore for each q in Q, $\{\lim_{n\to\infty} g^n q\} = (q + Q_0) \cap \{q^* \in Q : g(q^*) = q^*\}$ if $(I - g)(Q) \subseteq Q_0$.

Proof. Consider the space Q having a graph G in order that $V(G) = Q$ and $E(G) = \{(p, q) \in Q \times Q : p - q \in Q_0\}$. Let us choose f to be an identity map on Q and $F : Q \to CB(Q)$ in order that for each q in Q, $Fq = \{gq\}$. Evidently, F fulfills all the criteria of Theorem 3.2 and hence for each $q \in Q$, $\lim_{n\to\infty} g^n q \in \{q^* \in Q : g(q^*) = q^*\}$.

Since $(I - g)Q \subseteq Q_0$ and Q_0 is closed, it appears that $\lim_{n\to\infty} g^n q \in (q + Q_0)$. Thus, $\lim_{n\to\infty} g^n q \in (q + Q_0) \cap \{q^* \in Q : g(q^*) = q^*\}$. Let $r_1, r_2 \in (q + Q_0) \cap \{q^* \in Q : g(q^*) = q^*\}$. Then $r_1 - r_2 \in Q_0$. Hence and by equation (5), we have $||gr_1 - gr_2|| \le k(||r_1 - r_2||)|r_1 - r_2| < ||r_1 - r_2||$. Therefore we obtain $r_1 = r_2$. Hence proved. □

Several problems related to invariant approximations for single valued maps are established through the common fixed point theorems (see, [1, 16]). Also invariant approximation theorems for the hybrid couple of single valued and multivalued mappings satisfied strict contraction are found in [9, 18]. In the following result, we scrutinize invariant approximation for single valued and multivalued map meeting the equation (2) on a normed linear space. Let us consider a non-void subset D of a normed linear space $(Q, ||.||)$. Now for $z \in Q$, a set $B_D(z) = \{y \in D : d(z, D) = ||y - z||\}$ is defined as a collection of best D-approximates of z over D.

Theorem 3.9. *Assume a non-void subset D of a normed linear space $(Q, \|.\|)$. Suppose that $f : D \to D$ and $F : D \to CB(D)$ in order that*

(i) *for every $p, q \in B_D(z)$, $H(Fp, Fq) \leq k(\|fp - fq\|)\|fp - fq\| + Md(fq, Fp)$ where $k \in W$ and M is non-negative real number,*

(ii) $f(B_D(z)) = B_D(z)$,

(iii) *$f(B_D(z))$ is a complete subspace of $B_D(z)$,*

(iv) *$\sup_{a \in Fq} \|a - z\| \leq \|fq - z\|$ for each $q \in B_D(z)$.*

Then $Fix(f, F) \cap B_D(z) \neq \emptyset$ if $w \in Coin(f, F) \cap B_D(z)$ having $ffw = fw$ and $ffw \in Ffw$.

Proof. Let $q \in B_D(z)$ and $w \in Fq$. As $f(B_D(z)) = B_D(z)$, therefore $fq \in B_D(z)$ for every $q \in B_D(z)$. Hence the notion of $B_D(z)$ yields that $\|fq - z\| = d(z, D)$. Further,

$$\|w - z\| \leq \sup_{a \in Fq} \|a - z\| \leq \|fq - z\| = d(z, D).$$

Hence $w \in B_D(z)$. Therefore $Fq \subseteq B_D(z)$ for each $q \in B_D(z)$. Since for all $q \in D$, the set Fq are closed, hence Fq is closed for any $q \in B_D(z)$ also. Subsequently, $F|_{B_D(z)} : B_D(z) \to CB(B_D(z))$ and $f|_{B_D(z)} : B_D(z) \to B_D(z)$. Then eventually

$$Fix(f|_{B_D(z)}, F|_{B_D(z)}) = Fix(f, F) \cap B_D(z).$$

Hence the theorem follows by applying the Theorem 3.2 under the consideration $Q = B_D(z)$ along with a graph G having $V(G) = B_D(z)$ and $E(G) = B_D(z) \times B_D(z)$. □

4. Applications

This segment contains with an application of our main Theorem 3.2. We derive the occurrence of solution for Caputo fractional differential equation by applying the Theorem 3.2.

4.1. Fractional differential equation

A fractional differential equation consists with fractional derivatives of the form $D^\lambda (\lambda > 0)$, where λ need not be a natural number. This class of differential equations actually generalizes the ordinary differential equations. The occurrence of solution for several fractional differential equations were derived through the notion of fixed point theory, which can be found in [3, 12, 13]. We here consider a generalized fractional differential equation

$$^{c}D^\lambda q(t) = g(t, f(q(t))) \text{ whenever } q \in C[0, 1],\ 0 < t < 1 \text{ and } 1 < \lambda \leq 2, \tag{6}$$

having boundary criteria $q(0) = 0$ and $q(1) = \int_0^\eta q(s)ds$, for some $\eta \in (0, 1)$, where $f : C[0, 1] \to C[0, 1]$, $g : [0, 1] \times C[0, 1] \to \mathbb{R}$ are both continuous. The notation $^cD^\lambda$ represents the Caputo fractional derivative [3] with order λ. Also the collection of all continuous maps from $[0, 1]$ into $\mathbb{R}$ are represented as $C([0, 1], \mathbb{R})$, which incorporates with $\|.\| = \max_{t \in [0,1]} |w(t)|$.

In the coming result, we look into the sufficient criteria for ensuring a solution of the aforesaid fractional differential equation (6), through our main Theorem 3.2.

Theorem 4.1. *Choose the fractional differential equation as mentioned in* (6). *Suppose that the below criteria hold:*

(a) *there is $k \in W$ in order that for $0 \leq t \leq 1$,*

$$\left|g(t, f(p(t))) - g(t, f(q(t)))\right| \leq \frac{\Gamma(\lambda + 1)}{5} k\left(\left\|fp - fq\right\|\right)\left|fp(t) - fq(t)\right| \text{ where } \Gamma \text{ denotes the gamma function,}$$

(b) for every $q \in C[0,1]$, $Fq \subseteq f(C[0,1])$, where $F : C[0,1] \to C[0,1]$ in order that

$$\begin{aligned}F(q(t)) =&\frac{1}{\Gamma(\lambda)}\Big\{\int_0^t (t-r)^{\lambda-1} g(r, f(q(r)))dr\\&- \frac{2t}{(2-\eta^2)}\int_0^1 (1-r)^{\lambda-1} g(r, f(q(r)))dr\\&+ \frac{2t}{(2-\eta^2)}\int_0^\eta \int_0^r (r-s)^{\lambda-1} g(r, f(q(r)))ds\, dr\Big\}.\end{aligned}$$

Then the above mentioned fractional differential equation (6) possess a solution if the set $f(C[0,1])$ is closed in $C[0,1]$.

Proof. We assume that $Q = C[0,1]$ incorporate with a graph G having $V(G) = Q$ and $E(G) = \{(p,q) : (p,q) \in Q \times Q\}$. Now it is simple to visualize that $q \in Q$ is the solution of (6), if and only if it fulfills the succeeding equation,

$$\begin{aligned}q(t) =&\frac{1}{\Gamma(\lambda)}\Big\{\int_0^t (t-r)^{\lambda-1} g(r, f(q(r)))dr\\&- \frac{2t}{(2-\eta^2)}\int_0^1 (1-r)^{\lambda-1} g(r, f(q(r)))dr\\&+ \frac{2t}{(2-\eta^2)}\int_0^\eta \int_0^r (r-s)^{\lambda-1} g(r, f(q(r)))ds\, dr\Big\}.\end{aligned} \tag{7}$$

For every p and q lies in Q it yields,

$$\begin{aligned}\Big|F(p(t)) - F(q(t))\Big| =&\Big|\frac{1}{\Gamma(\lambda)}\Big\{\int_0^t (t-r)^{\lambda-1}(g(r, f(p(r))) - g(r, f(q(r))))dr\\&-\frac{2t}{(2-\eta^2)}\Big(\int_0^1 (1-r)^{\lambda-1}(g(r, f(p(r))) - g(r, f(q(r))))dr\\&-\int_0^\eta \int_0^r (r-s)^{\lambda-1}(g(r, f(p(r))) - g(r, f(q(r))))ds\, dr\Big)\Big\}\Big|\\&\le\frac{\Gamma(\lambda+1)}{5\Gamma(\lambda)}k(\|fp - fq\|)\Big\{\int_0^t |(t-r)|^{\lambda-1}|fp(r) - fq(r)|dr + \frac{2t}{(2-\eta^2)}\int_0^1 (1-r)^{\lambda-1}|fp(r) - fq(r)|dr\\&+\frac{2t}{(2-\eta^2)}\int_0^\eta \int_0^r |(r-s)|^{\lambda-1}|fp(r) - fq(r)|ds\, dr\Big\} \qquad [\text{from } (a)]\\&\le\frac{\Gamma(\lambda+1)}{5\Gamma(\lambda)}k(\|fp - fq\|)\sup_{t\in[0,1]}\Big\{\int_0^t |(t-r)|^{\lambda-1}dr + \frac{2t}{(2-\eta^2)}\int_0^1 (1-r)^{\lambda-1}dr\\&+\frac{2t}{(2-\eta^2)}\int_0^\eta \int_0^r |(r-s)|^{\lambda-1}ds\, dr\Big\}\|fp - fq\|\\&\le k(\|fp - fq\|)\|fp - fq\|.\end{aligned}$$

Therefore $\|F(p(t)) - F(q(t))\| \le k(\|fp - fq\|)\|fp - fq\|$. Thus considering $M = 0$ we can conclude that F and f has a function $q^*(t) \in Q$ in order that $Fq^*(t) = f(q^*(t))$. Hence $f(q^*(t))$ fulfills the equation (7), therefore the given equation (6) has a solution. □

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