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Integral inequalities in fractional calculus with general analytic kernels

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Abstract. Many different definitions of fractional calculus have been proposed in the literature, especially in recent years, and these can be classified into groups with similar properties. Many recent papers have studied inequalities for fractional integrals of particular types of functions, such as Hermite–Hadamard inequalities and related results. Here we provide theorems valid for a whole general class of fractional operators (anything defined using an integral with an analytic kernel function), so that it is no longer necessary to prove such results for each model one by one. We consider several types of fractional integral inequalities, which apply to functions of convex and synchronous type, and extend them to the full generality of fractional calculus with analytic kernels.

1. Introduction

Fractional calculus was theorised and developed as a generalisation of the classical calculus, in which integrals and derivatives can be taken not only to integer orders but to any real or complex number order. The main fundamental definitions for fractional integrals and fractional derivatives were formulated in the 19th century [32], but in the 21st century many other definitions have been proposed, some more useful than others. Nowadays, research in fractional calculus falls into two main categories: pure, establishing the mathematical theory of fractional operators, and applied, using these operators to model various real-world systems [37].

Following the theme of generalisation, which is always at the core of pure mathematics, it is clear that pure mathematical results should always be proved in the most general possible framework. Applications require specific formulae which can be fitted to specific real data, but there is no reason to apply the same mathematical proof many times to different results if they are all special cases of one general result.

For this reason, recently [8] it was proposed to consider the operators of fractional calculus as falling within various broad classes. Within each class, many results can be proven in a general setting, instead of wasting time repeating the same proofs in many specific settings. One of these classes, which we shall focus on in the current work, is the class of fractional integral operators with analytic kernel functions, introduced in [14].

One topic of recent interest within mathematical analysis and specifically fractional calculus is the study of integral inequalities. These have been used in fields such as ordinary and partial differential equations and

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integral equations [23, 39] and special functions [29]. In particular, fractional integral inequalities have been vital in providing bounds to solve some boundary value problems in fractional calculus, and in establishing the existence and uniqueness of solutions for certain fractional differential equations [5, 13, 33, 40]. But integral inequalities have also been studied for their own sake: many papers have been published on this topic, especially the Hermite–Hadamard inequality for convex functions and its variants [9, 24, 25, 28, 34–36, 38].

We seek to stem the flow of very similar papers by proving the corresponding results in general classes of fractional operators, obviating the need for proving them in each specific case. Some papers [4, 28] have already proceeded in this direction, by proving integral inequalities in the class of fractional operators with respect to functions. We continue that work by proving the Hermite–Hadamard and other inequalities in the class of fractional operators with general analytic kernels.

The structure of our paper is as follows. Section 2 is for preliminary definitions, results, and discussion. Section 3 is for proving the main results concerning the Hermite–Hadamard inequality for fractional integrals with general analytic kernels. Section 4 is for further fractional integral inequalities that apply to synchronous functions. Section 5 is for the conclusions.

2. Preliminaries

2.1. Fractional calculus

We start by defining the Riemann–Liouville fractional integral, the most fundamental starting point of fractional calculus.

Definition 2.1 ([27, 30, 32]). *For $f \in L^1[a,b]$ and $\mathrm{Re}(\alpha) > 0$, the left Riemann–Liouville fractional integral to order α of $f(x)$ is defined as*

$$ {}^{RL}_{a}I^{\alpha}_{x} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \qquad x \in [a,b], \tag{1}$$

where Γ denotes the standard and well-known gamma function, $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t$. For $f \in C^n[a,b]$ and $n-1 \le \mathrm{Re}(\alpha) < n$, the left Riemann–Liouville fractional derivative to order α of $f(x)$ is defined as

$$ {}^{RL}_{a}D^{\alpha}_{x} f(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\, {}^{RL}_{a}I^{n-\alpha}_{x} f(x), \qquad x \in [a,b]. \tag{2}$$

*When we interpret ${}^{RL}D^{-\alpha}_{a+} f(x) = {}^{RL}I^{\alpha}_{a+} f(x)$, we find that the formula (2) is the analytic continuation in α of the formula (1); thus, it makes sense to consider both differentiation and integration as cases of a single operator which we call **differintegration**.*

Definition 2.2 ([32]). *For $f \in L^1[a,b]$ and $\mathrm{Re}(\alpha) > 0$, the right Riemann–Liouville fractional integral to order α of $f(x)$ is defined as*

$$ {}^{RL}_{x}I^{\alpha}_{b} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \qquad x \in [a,b]. \tag{3}$$

For $f \in C^n[a,b]$ and $n-1 \le \mathrm{Re}(\alpha) < n$, the right Riemann–Liouville fractional derivative to order α of $f(x)$ is defined as

$$ {}^{RL}_{x}D^{\alpha}_{b} f(x) := (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\, {}^{RL}_{x}I^{n-\alpha}_{b} f(x), \qquad x \in [a,b]. \tag{4}$$

Other well-established definitions of fractional derivatives involve the Caputo derivative [12] and the Hilfer derivative [19, Chapter II] which interpolates between Riemann–Liouville and Caputo. Many of the alternative definitions of fractional integrals and derivatives arising in recent years involve replacing the power function kernel of (1) and (3) with other kernel functions. We mention in particular the Atangana–Baleanu [6, 7] and Prabhakar [22, 31] definitions, which involve Mittag-Leffler kernel functions, although one can obtain a viable model of fractional calculus by many other possible kernel functions. These can be considered as special cases of a general class of fractional operators which we define as follows.

Definition 2.3 ([14]). *Let $[a,b]$ be an interval and $x \in [a,b]$. For $f \in L^1[a,b]$ and $\alpha,\beta > 0$, the left fractional integral with analytic kernel function A and parameters α,β of $f(x)$ is defined as*

$$ {}^{A}_{a}I^{\alpha,\beta}_{x} f(x) := \int_a^x (x-\xi)^{\alpha-1} A\left((x-\xi)^{\beta}\right) f(\xi)\,\mathrm{d}\xi, \tag{5}$$

where $A : D(0,R) \to \mathbb{C}$ is a complex analytic function with power series

$$ A(x) = \sum_{n=0}^{\infty} a_n x^n \tag{6}$$

and the coefficients $a_n = a_n(\alpha,\beta)$ are permitted to depend on the parameters α and β if desired, and $R > (b-a)^{\beta}$.

(Note that in general the parameters α and β may be complex, as originally written in [14]. However, for the purposes of this paper we restrict them to be real, since we cannot do inequalities in the complex plane.)

Alternatively, the generalised integral operator (5) can be written as an infinite series of Riemann–Liouville fractional integrals, thus confirming its status as part of fractional calculus:

$$ {}^{A}_{a}I^{\alpha,\beta}_{x} f(x) = \sum_{n=0}^{\infty} a_n \Gamma(\alpha+n\beta)\ {}^{RL}_{a}I^{\alpha+n\beta}_{x} f(x) \tag{7}$$

$$ = A_{\Gamma}\left({}^{RL}_{a}I^{\beta}_{x}\right) {}^{RL}_{a}I^{\alpha}_{x} f(x), \tag{8}$$

the series (7) being locally uniformly convergent according to [14], where the transformed function A_{Γ} is defined by

$$ A_{\Gamma}(x) := \sum_{n=0}^{\infty} a_n \Gamma(\alpha+n\beta) x^n. \tag{9}$$

Similarly, the right fractional integral with analytic kernel function A and parameters α,β of $f(x)$ is defined as

$$ {}^{A}_{x}I^{\alpha,\beta}_{b} f(x) := \int_x^b (\xi-x)^{\alpha-1} A\left((\xi-x)^{\beta}\right) f(\xi)\,\mathrm{d}\xi \tag{10}$$

$$ = \sum_{n=0}^{\infty} a_n \Gamma(\alpha+n\beta)\ {}^{RL}_{x}I^{\alpha+n\beta}_{b} f(x) \tag{11}$$

$$ = A_{\Gamma}\left({}^{RL}_{x}I^{\beta}_{b}\right) {}^{RL}_{x}I^{\alpha}_{b} f(x), \tag{12}$$

where A and A_{Γ} are defined by (6) and (9) as above.

Remark 2.4. *The Riemann–Liouville fractional integral, as well as the Prabhakar fractional integral and Atangana–Baleanu fractional derivatives, all mentioned above, are special cases of Definition 2.3 given by the following choices of A, α, and β:*

(i) When $A(x) = \frac{1}{\Gamma(\alpha)}$ is constant, we have

$$ {}^{RL}_{a}I^{\alpha}_{x} f(x) = {}^{A}_{a}I^{\alpha,0}_{x} f(x), $$
$$ {}^{RL}_{x}I^{\alpha}_{b} f(x) = {}^{A}_{x}I^{\alpha,0}_{b} f(x). $$

(ii) When $A(x) = E^{\gamma}_{\beta,\alpha}(\omega x)$ is the 3-parameter Mittag-Leffler function, we have

$$ {}^{P}_{a}I^{\alpha,\beta,\gamma,\omega}_{x} f(x) = {}^{A}_{a}I^{\alpha,\beta}_{x} f(x). \tag{13}$$

(iii) When $A(x) = \frac{B(\alpha)}{1-\alpha} E_{\alpha}\left(\frac{-\alpha}{1-\alpha}\right)$ is the 1-parameter Mittag-Leffler function with constant factor, we have

$$ {}^{ABR}_{a}D^{\alpha}_{x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\, {}^{A}_{a}I^{1,\alpha}_{x} f(x), \tag{14}$$

$$ {}^{ABC}_{a}D^{\alpha}_{x} f(x) = {}^{A}_{a}I^{1,\alpha}_{x} f'(x). \tag{15}$$

For more details about the Mittag-Leffler functions and their connections with fractional calculus, we refer the reader to the classical texts [17, 18, 26].

2.2. Hermite–Hadamard inequalities

The classical Hermite–Hadamard inequality, from which a great deal of further literature has been extended, is stated as follows.

Proposition 2.5. *If $f : [a,b] \to \mathbb{R}$ is an L^1 convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{16}$$

Two fractional versions of this classical result were proved in 2013 [35] and 2017 [36]. The Hermite–Hadamard inequality for Riemann–Liouville fractional integrals can be stated in either of the two following ways.

Proposition 2.6 ([35]). *If $f : [a,b] \to \mathbb{R}$ is convex and L^1, and $\alpha > 0$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[{}^{RL}_{\,a}I^\alpha_x f(b) + {}^{RL}_{\,x}I^\alpha_b f(a)\Big] \leq \frac{f(a)+f(b)}{2}, \tag{17}$$

Proposition 2.7 ([36]). *If $f : [a,b] \to \mathbb{R}$ is convex and L^1, and $\alpha > 0$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[{}^{RL}_{\frac{a+b}{2}}I^\alpha_x f(b) + {}^{RL}_{\,x}I^\alpha_{\frac{a+b}{2}} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{18}$$

The difference between the inequalities (17) and (18) is that, in the interval $[a,b]$, the former uses integration forwards from the beginning and backwards from the end, while the latter uses integration both ways from the centre.

A version of the Hermite–Hadamard inequality for h-convex functions has also been proved in 2008 [34], and this too has been extended to fractional integrals of such functions [38]. This generalisation of the notion of convexity also includes other proposed types of convexity, such as exp-convexity [21]; therefore, Hermite–Hadamard inequalities for fractional integrals of exp-convex functions are also known, as special cases of existing results in the literature.

Many other extensions of the Hermite–Hadamard theorem have been proposed, with proofs at varying levels of difficulty compared with the original result, and the results investigated at varying levels of detail. These include results in various types of fractional calculus, e.g. for fractional integrals of a function with respect to another function [25, 28], those of Hilfer type [9] and for other models of fractional calculus involving Mittag-Leffler kernels [15]. We do not state all the results in detail here, but they are usually similar in form and function, although not identical, to the original Hermite–Hadamard result (16) and its fractional version (17).

3. Hermite–Hadamard inequalities for general integral operators

In this section, we shall prove analogues of the fractional Hermite–Hadamard inequalities (17)–(18) for fractional integrals with general analytic kernels. The main results here are Theorem 3.1 (a generalisation of Proposition 2.6) and Theorem 3.5 (a generalisation of Proposition 2.7).

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex L^1 function, and α, β, A be as in Definition 2.3 for fractional integrals with general analytic kernels.*

If all coefficients a_n of the analytic function A are real positive, then the Hermite–Hadamard inequality is as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)^\alpha B\left((b-a)^\beta\right)}\Big[{}^{A}_{a}I^{\alpha,\beta}_x f(b) + {}^{A}_{x}I^{\alpha,\beta}_b f(a)\Big] \leq \frac{f(a)+f(b)}{2}, \tag{19}$$

where

$$B(x) := \sum_{n=0}^{\infty} \frac{a_n x^n}{\alpha + n\beta}. \tag{20}$$

In general, if the coefficients a_n are real but may be either positive or negative, then the Hermite–Hadamard inequality is as follows:

$$f\left(\frac{a+b}{2}\right)B_+\left((b-a)^\beta\right) - \frac{f(a)+f(b)}{2}B_-\left((b-a)^\beta\right) \\ \leq \frac{1}{2(b-a)^\alpha}\left[{}^{A}_{a}I^{\alpha,\beta}_{x} f(b) + {}^{A}_{x}I^{\alpha,\beta}_{b} f(a)\right] \leq \frac{f(a)+f(b)}{2}B_+\left((b-a)^\beta\right) - f\left(\frac{a+b}{2}\right)B_-\left((b-a)^\beta\right), \tag{21}$$

where

$$B_+(x) := \sum_{n\,:\,a_n>0} \frac{|a_n|x^n}{\alpha + n\beta}, \qquad B_-(x) := \sum_{n\,:\,a_n<0} \frac{|a_n|x^n}{\alpha + n\beta}, \tag{22}$$

noting that both $B_+(x)$ and $B_-(x)$ are positive for $x > 0$ and that $B_+(x) - B_-(x) = B(x)$.

Proof. From the series definition (7), we have:

$${}^{A}_{a}I^{\alpha,\beta}_{x} f(b) + {}^{AB}_{x}I^{\alpha,\beta}_{b} f(a) = \sum_{n=0}^{\infty} a_n \Gamma(\alpha + n\beta)\left({}^{RL}_{a}I^{\alpha+n\beta}_{x} f(b) + {}^{RL}_{x}I^{\alpha+n\beta}_{b} f(a)\right).$$

Now we wish to use the inequality (17), but we must think carefully about sign. Since the gamma function is positive on $\mathbb{R}^+$, and $2(b-a)^{\alpha+n\beta}$ is also positive, we can deduce:

$$\frac{2(b-a)^{\alpha+n\beta}}{\alpha+n\beta} f\left(\frac{a+b}{2}\right) \leq \Gamma(\alpha+n\beta)\left({}^{RL}_{a}I^{\alpha+n\beta}_{x} f(b) + {}^{RL}_{x}I^{\alpha+n\beta}_{b} f(a)\right) \leq \frac{2(b-a)^{\alpha+n\beta}}{\alpha+n\beta} \cdot \frac{f(a)+f(b)}{2}.$$

The only possible sign problem comes from the coefficients a_n. If we assume all a_n to be real positive, then multiplying by a_n and summing over all n gives

$$f\left(\frac{a+b}{2}\right)\sum_{n=0}^{\infty} a_n \frac{2(b-a)^{\alpha+n\beta}}{\alpha+n\beta} \leq {}^{A}_{a}I^{\alpha,\beta}_{x} f(b) + {}^{A}_{x}I^{\alpha,\beta}_{b} f(a) \leq \frac{f(a)+f(b)}{2}\sum_{n=0}^{\infty} a_n \frac{2(b-a)^{\alpha+n\beta}}{\alpha+n\beta}.$$

The series on the left and right hand sides of this inequality are clearly positive, so we can divide by them to get the desired result (19).

In general, if some a_n are positive and others are negative (any that are zero can be ignored in the sum), then we get instead

$$f\left(\frac{a+b}{2}\right)\sum_{n\,:\,a_n>0} a_n \frac{2(b-a)^{\alpha+n\beta}}{\alpha+n\beta} + \frac{f(a)+f(b)}{2}\sum_{n\,:\,a_n<0} a_n \frac{2(b-a)^{\alpha+n\beta}}{\alpha+n\beta} \\ \leq {}^{A}_{a}I^{\alpha,\beta}_{x} f(b) + {}^{A}_{x}I^{\alpha,\beta}_{b} f(a) \leq \frac{f(a)+f(b)}{2}\sum_{n\,:\,a_n>0} a_n \frac{2(b-a)^{\alpha+n\beta}}{\alpha+n\beta} + f\left(\frac{a+b}{2}\right)\sum_{n\,:\,a_n<0} a_n \frac{2(b-a)^{\alpha+n\beta}}{\alpha+n\beta},$$

or more simply

$$f\left(\frac{a+b}{2}\right)B_+\left((b-a)^\beta\right) - \frac{f(a)+f(b)}{2}B_-\left((b-a)^\beta\right) \\ \leq \frac{1}{2(b-a)^\alpha}\left[{}^{A}_{a}I^{\alpha,\beta}_{x} f(b) + {}^{A}_{x}I^{\alpha,\beta}_{b} f(a)\right] \leq \frac{f(a)+f(b)}{2}B_+\left((b-a)^\beta\right) - f\left(\frac{a+b}{2}\right)B_-\left((b-a)^\beta\right),$$

and again the desired result follows. □

Corollary 3.2. *Using Theorem 3.1 with $\beta = 0$ and $A(x) = \frac{1}{\Gamma(\alpha)}$, we can deduce the Hermite–Hadamard inequality* (17) *for Riemann–Liouville fractional integrals. In this case $B(x) = \frac{1}{\Gamma(\alpha+1)}$ and we may use the version* (19) *with all a_n positive.*

Corollary 3.3. *Using Theorem 3.1 with $A(x) = E^{\gamma}_{\beta,\alpha}(\omega x)$, we can deduce the following Hermite–Hadamard inequality for Prabhakar fractional integrals:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{{}^{P}_{a}I^{\beta,\alpha,\gamma,\omega}_{x} f(b) + {}^{P}_{x}I^{\beta,\alpha,\gamma,\omega}_{b} f(a)}{2(b-a)^{\alpha} E^{\gamma}_{\beta,\alpha+1}\left(\omega(b-a)^{\beta}\right)} \leq \frac{f(a)+f(b)}{2}. \tag{23}$$

This was already established, and illustrated graphically, in [15].

Remark 3.4. *Fractional* derivative *operators with general analytic kernels were also defined in [14]. The definition, like that of the classical Riemann–Liouville derivative seen in Definition 2.1, was given by combining the general fractional integral operator* (5) *with a standard repeated differentiation operator.*

Because of this use of standard differentiation operators, we cannot extend the integral inequalities of Theorem 3.1 to results for fractional derivatives with general analytic kernels. This makes sense in terms of what we know about classical calculus: it is usually much easier to bound a function's integral than its derivative.

Why, then, was it possible in [15, Theorem 2.1] to find Hermite–Hadamard type inequalities for Atangana–Baleanu fractional derivatives? The answer lies in the series formula for the AB fractional derivative [7], expressing it as an infinite series of Riemann–Liouville integrals. *The AB derivative can be defined purely in terms of fractional integration, without using any differentiation at all, but this is not true in general for the derivative operators with analytic kernels.*

Note that the result of [15, Theorem 2.1] is essentially a special case of our Theorem 3.1, even though it is about AB derivatives rather than fractional integrals. If an operator can be written as an infinite series of Riemann–Liouville fractional integrals, then our argument above can be applied. The functions $\mathcal{K}_1$ and $\mathcal{K}_2$ appearing in [15, Theorem 2.1] correspond to the functions B_+ and B_- which we have used above.

Theorem 3.5. *Let $f : [a,b] \to \mathbb{R}$ be a convex L^1 function, and α, β, A be as in Definition 2.3 for fractional integrals with general analytic kernels.*

If all coefficients a_n of the analytic function A are real positive, then the Hermite–Hadamard inequality is as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}}{(b-a)^{\alpha} B\left(\left(\frac{b-a}{2}\right)^{\beta}\right)} \left[{}^{A}_{\frac{a+b}{2}}I^{\alpha,\beta}_{x} f(b) + {}^{A}_{x}I^{\alpha,\beta}_{\frac{a+b}{2}} f(a) \right] \leq \frac{f(a)+f(b)}{2}, \tag{24}$$

where the function B is defined by (20) *as before.*

In general, if the coefficients a_n are real but may be either positive or negative, then the Hermite–Hadamard inequality is as follows:

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) B_+\left(\left(\frac{b-a}{2}\right)^{\beta}\right) - \frac{f(a)+f(b)}{2} B_-\left(\left(\frac{b-a}{2}\right)^{\beta}\right) \\ &\quad \leq \frac{2^{\alpha-1}}{(b-a)^{\alpha}} \left[{}^{A}_{\frac{a+b}{2}}I^{\alpha,\beta}_{x} f(b) + {}^{A}_{x}I^{\alpha,\beta}_{\frac{a+b}{2}} f(a) \right] \leq \frac{f(a)+f(b)}{2} B_+\left(\left(\frac{b-a}{2}\right)^{\beta}\right) - f\left(\frac{a+b}{2}\right) B_-\left(\left(\frac{b-a}{2}\right)^{\beta}\right), \end{aligned} \tag{25}$$

where the functions B_+ and B_- are defined by (22) *as before.*

Proof. From the series definition (7), we have:

$${}^{A}_{\frac{a+b}{2}}I^{\alpha,\beta}_{x} f(b) + {}^{A}_{x}I^{\alpha,\beta}_{\frac{a+b}{2}} f(a) = \sum_{n=0}^{\infty} a_n \Gamma(\alpha + n\beta) \left({}^{RL}_{\frac{a+b}{2}}I^{\alpha+n\beta}_{x} f(b) + {}^{RL}_{x}I^{\alpha+n\beta}_{\frac{a+b}{2}} f(a) \right).$$

Now we wish to use the inequality (18), but again we must think carefully about sign. Since α, β, and $b-a$ are positive, we can deduce:

$$\frac{(b-a)^{\alpha+n\beta}}{(\alpha+n\beta)2^{\alpha+n\beta-1}}f\left(\frac{a+b}{2}\right)\leq\Gamma(\alpha+n\beta)\left[{}^{RL}_{\frac{a+b}{2}}I^{\alpha+n\beta}_{x}f(b)+{}^{RL}_{x}I^{\alpha+n\beta}_{\frac{a+b}{2}}f(a)\right]$$
$$\leq\frac{(b-a)^{\alpha+n\beta}}{(\alpha+n\beta)2^{\alpha+n\beta-1}}\cdot\frac{f(a)+f(b)}{2}. \tag{26}$$

Just like in the proof of Theorem 3.1, since $\Gamma(\alpha+n\beta)$ and $(b-a)^{\alpha+n\beta}$ are positive, the only possible sign problem comes from the coefficients a_n. If we assume all a_n to be real positive, then multiplying the above inequality by a_n and summing over all n gives

$$f\left(\frac{a+b}{2}\right)\sum_{n=0}^{\infty}a_n\frac{(b-a)^{\alpha+n\beta}}{(\alpha+n\beta)2^{\alpha+n\beta-1}}\leq{}^{A}_{\frac{a+b}{2}}I^{\alpha,\beta}_{x}f(b)+{}^{A}_{x}I^{\alpha,\beta}_{\frac{a+b}{2}}f(a)\leq\frac{f(a)+f(b)}{2}\sum_{n=0}^{\infty}a_n\frac{2(b-a)^{\alpha+n\beta}}{(\alpha+n\beta)2^{\alpha+n\beta-1}}.$$

Using the definition (20) of the function B, this can be rewritten as

$$\frac{(b-a)^{\alpha}}{2^{\alpha-1}}B\left(\left(\frac{b-a}{2}\right)^{\beta}\right)f\left(\frac{a+b}{2}\right)\leq{}^{A}_{\frac{a+b}{2}}I^{\alpha,\beta}_{x}f(b)+{}^{A}_{x}I^{\alpha,\beta}_{\frac{a+b}{2}}f(a)\leq\frac{(b-a)^{\alpha}}{2^{\alpha-1}}B\left(\left(\frac{b-a}{2}\right)^{\beta}\right)\frac{f(a)+f(b)}{2}.$$

Then, since $\alpha,\beta,b-a$ are positive and so are the coefficients in the series for B, we can deduce:

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}}{(b-a)^{\alpha}B\left(\left(\frac{b-a}{2}\right)^{\beta}\right)}\left[{}^{A}_{\frac{a+b}{2}}I^{\alpha,\beta}_{x}f(b)+{}^{A}_{x}I^{\alpha,\beta}_{\frac{a+b}{2}}f(a)\right]\leq\frac{f(a)+f(b)}{2},$$

which is the desired result (24).

In general, if some a_n are positive and others are negative (again we can ignore any zero terms in the sum), then from (26) we get instead

$$f\left(\frac{a+b}{2}\right)\sum_{n\,:\,a_n>0}a_n\frac{(b-a)^{\alpha+n\beta}}{(\alpha+n\beta)2^{\alpha+n\beta-1}}+\frac{f(a)+f(b)}{2}\sum_{n\,:\,a_n<0}a_n\frac{(b-a)^{\alpha+n\beta}}{(\alpha+n\beta)2^{\alpha+n\beta-1}}$$
$$\leq{}^{A}_{\frac{a+b}{2}}I^{\alpha,\beta}_{x}f(b)+{}^{A}_{x}I^{\alpha,\beta}_{\frac{a+b}{2}}f(a)$$
$$\leq\frac{f(a)+f(b)}{2}\sum_{n\,:\,a_n>0}a_n\frac{(b-a)^{\alpha+n\beta}}{(\alpha+n\beta)2^{\alpha+n\beta-1}}+f\left(\frac{a+b}{2}\right)\sum_{n\,:\,a_n<0}a_n\frac{(b-a)^{\alpha+n\beta}}{(\alpha+n\beta)2^{\alpha+n\beta-1}}.$$

Using the definitions (22) of the functions B_+ and B_-, this can be rewritten as

$$f\left(\frac{a+b}{2}\right)\frac{(b-a)^{\alpha}}{2^{\alpha-1}}B_+\left(\left(\frac{b-a}{2}\right)^{\beta}\right)-\frac{f(a)+f(b)}{2}\cdot\frac{(b-a)^{\alpha}}{2^{\alpha-1}}B_-\left(\left(\frac{b-a}{2}\right)^{\beta}\right)$$
$$\leq{}^{A}_{\frac{a+b}{2}}I^{\alpha,\beta}_{x}f(b)+{}^{A}_{x}I^{\alpha,\beta}_{\frac{a+b}{2}}f(a)\leq\frac{f(a)+f(b)}{2}\cdot\frac{(b-a)^{\alpha}}{2^{\alpha-1}}B_+\left(\left(\frac{b-a}{2}\right)^{\beta}\right)-f\left(\frac{a+b}{2}\right)\frac{(b-a)^{\alpha}}{2^{\alpha-1}}B_-\left(\left(\frac{b-a}{2}\right)^{\beta}\right),$$

and this gives the desired result (25) since α and $b-a$ are positive. □

Corollary 3.6. *Using Theorem 3.5 with* $\beta=0$ *and* $A(x)=\frac{1}{\Gamma(\alpha)}$, *we can deduce the Hermite–Hadamard inequality* (18) *for Riemann–Liouville fractional integrals. In this case* $B(x)=\frac{1}{\Gamma(\alpha+1)}$ *and we may use the version* (24) *with all* a_n *positive.*

Corollary 3.7. *Using Theorem 3.5 with $A(x) = E^{\gamma}_{\beta,\alpha}(\omega x)$, we can deduce the following Hermite–Hadamard inequality for Prabhakar fractional integrals:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{{}^{P}_{\frac{a+b}{2}}I^{\beta,\alpha,\gamma,\omega}_{x} f(b) + {}^{P}_{x}I^{\beta,\alpha,\gamma,\omega}_{\frac{a+b}{2}} f(a)}{2\left(\frac{b-a}{2}\right)^{\alpha} E^{\gamma}_{\beta,\alpha+1}\left(\omega\left(\frac{b-a}{2}\right)^{\beta}\right)} \leq \frac{f(a)+f(b)}{2}. \tag{27}$$

This was already established, and illustrated graphically, in [15].

Remark 3.8. *Once again, the result of Theorem 3.5 cannot be extended easily to any result for fractional* derivatives *with general analytic kernels. The existence of a standard differentiation operator in the definition of these generalised fractional derivatives means that integral inequalities can no longer be used.*

The result of [15, Theorem 2.3] gives a Hermite–Hadamard type inequality for Atangana–Baleanu fractional derivatives. Again, this is possible because AB fractional derivatives can be written according to [7] as infinite series of Riemann–Liouville integrals. The result of [15, Theorem 2.3] is essentially a special case of our Theorem 3.5, with again the functions $\mathcal{K}_1$ and $\mathcal{K}_2$ from [15, Theorem 2.3] corresponding to the functions B_+ and B_- which we have used above.

4. Further inequalities for synchronous functions

The above inequalities of Hermite–Hadamard type concern convex functions. Of course the theory of integral inequalities is much broader than just these, and in the current section we shall study some inequalities involving *synchronous* functions, which are defined as follows.

Definition 4.1. *Let $f, g : [a,b] \to \mathbb{R}$. We say these two functions are synchronous if*

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0 \tag{28}$$

for all $x, y \in [a,b]$. In other words, if the increasing and decreasing intervals are synchronised: both functions increase or decrease together.

In some previous work [10, 16], inequalities concerning fractional integrals of synchronous functions were proved in some specific types of fractional calculus. Using similar methods, it is possible to prove analogous results for general classes of fractional operators.

Theorem 4.2. *Let $f, g \in L^1[0,\infty)$ be synchronous, and let $\alpha, \beta \in \mathbb{R}^+$ be positive real parameters. If the analytic function A is such that $A(x) > 0$ for all $x > 0$, then we have the following integral inequality:*

$$x^{\alpha}B(x^{\beta})\ {}^{A}_{a}I^{\alpha,\beta}_{x}[f(x)g(x)] \geq {}^{A}_{a}I^{\alpha,\beta}_{x}[f(x)]\,{}^{A}_{a}I^{\alpha,\beta}_{x}[g(x)], \qquad x \in [0,\infty), \tag{29}$$

where the function B is as defined in (20)*.*

Proof. The definition of synchronous functions implies immediately that

$$f(\xi)g(\xi) + f(\eta)g(\eta) \geq f(\xi)g(\eta) + g(\xi)f(\eta), \qquad \xi, \eta \in [0,\infty).$$

Assuming $x \geq \xi$, this inequality can be multiplied on both sides by the positive factor $(x-\xi)^{\alpha-1}A\left((x-\xi)^{\beta}\right)$, and then integrated over the interval $\xi \in (0,x)$ to obtain, using the definition (5),

$${}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f(x)g(x)\Big] + f(\eta)g(\eta)\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[1\Big] \geq g(\eta)\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f(x)\Big] + f(\eta)\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[g(x)\Big], \tag{30}$$

for $x, \eta \in [0,\infty)$. Now the same process can be repeated: multiplying the new inequality (30) on both sides by the positive factor $(x-\eta)^{\alpha-1}A\left((x-\eta)^{\beta}\right)$ and then integrating over the interval $\eta \in (0,x)$ to obtain:

$${}^{A}_{a}I^{\alpha,\beta}_{x}[f(x)g(x)]\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[1\Big] + {}^{A}_{a}I^{\alpha,\beta}_{x}[1]\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f(x)g(x)\Big] \geq {}^{A}_{a}I^{\alpha,\beta}_{x}[f(x)]\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[g(x)\Big] + {}^{T}_{0}I^{(\alpha,\beta)}_{t}[g(x)]\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f(x)\Big],$$

or after dividing by 2,

$$ {}_{a}^{A}I_{x}^{\alpha,\beta}[f(x)g(x)]\, {}_{a}^{A}I_{x}^{\alpha,\beta}[1] \geq {}_{a}^{A}I_{x}^{\alpha,\beta}[f(x)]\, {}_{a}^{A}I_{x}^{\alpha,\beta}[g(x)]. $$

It remains only to show that

$$ {}_{a}^{A}I_{x}^{\alpha,\beta}[1] = x^{\alpha}B(x^{\beta}), \tag{31} $$

and this can be shown using the series formula (7):

$$ \begin{aligned} {}_{a}^{A}I_{x}^{\alpha,\beta}[1] &= \sum_{n=0}^{\infty} a_n \Gamma(\alpha+n\beta)\, {}_{a}^{RL}I_{x}^{\alpha+n\beta}[1] \\ &= \sum_{n=0}^{\infty} \frac{a_n x^{\alpha+n\beta}}{\alpha+n\beta} = x^{\alpha}B(x^{\beta}), \end{aligned} $$

using the definition (20) of the function B. □

Theorem 4.3. *Let $f, g \in L^1[0,\infty)$ be synchronous, and let $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}^+$ be positive real parameters. If the analytic function A is such that $A(x) > 0$ for all $x > 0$, then we have the following integral inequality:*

$$ \begin{aligned} & x^{\alpha_2}B(x^{\beta_2})\, {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}[f(x)g(x)] + x^{\alpha_1}B(x^{\beta_1})\, {}_{a}^{A}I_{x}^{\alpha_2,\beta_2}[f(x)g(x)] \\ & \qquad \geq {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}[f(x)]\, {}_{a}^{A}I_{x}^{\alpha_2,\beta_2}[g(x)] + {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}[g(x)]\, {}_{a}^{A}I_{x}^{\alpha_2,\beta_2}[f(x)], \end{aligned} \tag{32} $$

where the function B is as defined in (20)*.*

Proof. During the proof of Theorem 4.2 we found the following inequality (30), valid for all $t, v \in [0,\infty)$:

$$ {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}\Big[f(x)g(x)\Big] + f(\eta)g(\eta)\, {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}\Big[1\Big] \geq g(\eta)\, {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}\Big[f(x)\Big] + f(\eta)\, {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}\Big[g(x)\Big], $$

for $x, \eta \in [0,\infty)$. Now, instead of using the same multiplier $(x-\eta)^{\alpha_1-1}A\left((x-\eta)^{\beta_1}\right)$ as before, we instead multiply this inequality on both sides by $(x-\eta)^{\alpha_2-1}A\left((x-\eta)^{\beta_2}\right)$ and then integrate over the interval $\eta \in (0,x)$ to obtain:

$$ \begin{aligned} & {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}[f(x)g(x)]\, {}_{a}^{A}I_{x}^{\alpha_2,\beta_2}\Big[1\Big] + {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}[1]\, {}_{a}^{A}I_{x}^{\alpha_2,\beta_2}\Big[f(x)g(x)\Big] \\ & \qquad \geq {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}[f(x)]\, {}_{a}^{A}I_{x}^{\alpha_2,\beta_2}\Big[g(x)\Big] + {}_{a}^{A}I_{x}^{\alpha_1,\beta_1}[g(x)]\, {}_{a}^{A}I_{x}^{\alpha_2,\beta_2}\Big[f(t)\Big]. \end{aligned} $$

Then, making use of the formula (31) for the fractional integral of the unit function 1, this gives the desired result. □

Remark 4.4. *Putting $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ in Theorem 4.3 gives precisely the result of Theorem 4.2. However, the special case Theorem 4.2 is still useful in its own right, and we shall now use it to prove the following Theorem 4.5.*

Theorem 4.5. *Let $f_1, f_2, \ldots, f_n \in L^1[0,\infty)$ be positive increasing functions ($n \in \mathbb{N}$) and $\alpha, \beta \in \mathbb{R}^+$ be positive real parameters. If the analytic function A is such that $A(x) > 0$ for all $x > 0$, then we have the following integral inequality:*

$$ \Big[x^{\alpha}B(x^{\beta})\Big]^{n-1}\, {}_{a}^{A}I_{x}^{\alpha,\beta}\Big(f_1(x)f_2(x)\ldots f_n(x)\Big) \geq \prod_{i=1}^{n}\left({}_{a}^{A}I_{x}^{\alpha,\beta} f_i(x)\right). \tag{33} $$

Proof. We proceed by induction on n. In the case $n = 1$, the inequality is trivially true and indeed it is an equality. The case $n = 2$ was already done in Theorem 4.2.

Let us start from the case for a product of n functions as an assumption, and try to deduce from there the case for a product of $n + 1$ functions. Therefore, we have

$$\left[x^{\alpha}B(x^{\beta})\right]^{n-1}\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big(f_1(x)f_2(x)\dots f_n(x)\Big) \geq \prod_{i=1}^{n}\left({}^{A}_{a}I^{\alpha,\beta}_{x}f_i(x)\right), \tag{34}$$

and we can also apply the result of Theorem 4.2 with $f(x) = f_{n+1}(x)$ and $g(x) = f_1(t)f_2(t)\dots f_n(t)$, namely

$$x^{\alpha}B(x^{\beta})\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f_1(x)f_2(x)\dots f_n(x)f_{n+1}(x)\Big] \geq {}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f_{n+1}(t)\Big]\,{}^{A}_{a}I^{\alpha,\beta}_{x}\left[f_1(x)f_2(x)\dots f_n(x)\right].$$

Combining this inequality with the induction hypothesis (34), we get:

$$\begin{aligned}
&\left[x^{\alpha}B(x^{\beta})\right]^{n}\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f_1(x)f_2(x)\dots f_n(x)\Big]\\
&\quad\geq {}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f_{n+1}(t)\Big]\left(\left[x^{\alpha}B(x^{\beta})\right]^{n-1}\,{}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f_1(x)f_2(x)\dots f_n(x)\Big]\right)\\
&\quad\geq {}^{A}_{a}I^{\alpha,\beta}_{x}\Big[f_{n+1}(t)\Big]\prod_{i=1}^{n}\left({}^{A}_{a}I^{\alpha,\beta}_{x}f_i(x)\right) = \prod_{i=1}^{n+1}\left({}^{A}_{a}I^{\alpha,\beta}_{x}f_i(x)\right),
\end{aligned}$$

which is precisely (33) with n replaced by $n + 1$. □

Remark 4.6. *In the special case where A is an exponential function and $\beta = 1$, we recover the results obtained previously in [16] for tempered fractional calculus (also known as generalised proportional fractional calculus [20] or substantial fractional calculus [11]). In the special case where A is constant and $\beta = 0$, we recover the results obtained previously [10] for Riemann–Liouville fractional calculus.*

As well as these, of course, many other types of fractional calculus are covered by the general results proved here: Atangana–Baleanu, Prabhakar, etc. The point of considering general classes of fractional operators is so that we can prove things in a general setting instead of repeating the same proofs many times in different settings.

5. Conclusions

In this paper, we have examined some fractional integral inequalities in as broad and general a context as possible. These include both inequalities of Hermite–Hadamard type for convex functions, and inequalities of products of integrals for synchronous functions. The setting is that of fractional integrals with general analytic kernels.

The work of Section 3 may be seen as an extension of our previous work in [15], using infinite series of Riemann–Liouville integrals to prove Hermite–Hadamard type inequalities for fractional integrals with more general kernels. But in reality it is more than that: by investigating general classes of operators, rather than writing papers on each operator one by one, we are not only proving more mathematically valuable results, but also encouraging this type of research for the future. For applications, specific models may be considered according to their usefulness; but in mathematics, theorems should be proved in the most general setting possible.

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