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Existence of solutions of some boundary value problems with impulsive conditions and ABC-fractional order

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Abstract. This article discusses on topics about the integral boundary value problems with impulsive conditions. Using a generalized contraction, the existence of solutions of an initial boundary value problem involving Atangana Baleanu Capotu-fractional order (in Mittag–Lefler kernel sense) will also be investigated and some important results will be presented. Two examples are provided to illustrate the results.

1. Introduction

Fractional differential equations are strong tools to demonstrate many natural phenomena. Refer to [5–7, 9, 10, 17, 20, 29] to see some important results. The study of implicit differential equations is one of the most important studies in the theory of differential equations. These equations have applications in many area such as managerial and economic sciences. Refer to some recent works on implicit differential equations [1–4, 8, 16, 19, 23, 26–28].

In [11] the authors have studied the existence of solutions and Hyers-Ulam stability of the following problem:

$$\begin{cases} {}_0^cD^{\alpha}_{\varsigma_n}\pi(\varsigma) = f(\varsigma, \pi(\varsigma), \pi(m\varsigma), {}_0^cD^{\alpha}_{\varsigma_n}\pi(\varsigma)),\ \varsigma \in \kappa, \\ \varsigma \neq \varsigma_n \text{ for } n = 1, 2, \dots, k,\ 0 < \alpha \leq 1,\ 0 < m < 1, \\ \pi(0) = \pi_0, \\ \Delta\pi(\varsigma_n) = I_n(\pi(\varsigma_n)),\ n = 1, 2, \dots, k, \end{cases} \tag{1}$$

where $\kappa = [0, T]$, $T > 0$ and ${}_0^cD^{\alpha}_{\varsigma_n}$ denotes the Caputo derivative at points other than ς_n in κ, while $f : \kappa \times \mathbb{R}^3 \to \mathbb{R}$ and $\pi : C(\kappa, \mathbb{R}) \to \mathbb{R}$ are given continuous functions. Further, $I_n : \mathbb{R} \to \mathbb{R}$ are the nonlinear impulsive mappings and $\Delta\pi(\varsigma_n) = \pi(\varsigma_n^+) - \pi(\varsigma_n^-)$, where ς_n^+, ς_n^- are the right and left limits of ς_n at n, respectively. In this article, we study the existence and uniqueness of solutions for such integral boundary value problems by a new fixed point theorem.

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F. Jarad et al. in [18] have studied the fractional differential equations in the sense of Atangana-Baleanu-Caputo fractional derivative (ABC-fractional derivative) and presented the existence and uniqueness of solutions of the type:

$$\begin{cases} {}^{\text{ABC}}_{a}D^{\nu}\varsigma(\iota) = \Upsilon(\iota, \varsigma(\iota)), \\ \varsigma(a) = \varsigma_a, \end{cases} \tag{2}$$

where $\nu \in (0,1)$, ${}^{\text{ABC}}_{a}D^{\nu}$ is ABC-fractional order differential operator and $\Upsilon(\iota, \varsigma(\iota)) \in C[a,b]$. The generalization of this work have been presented in some articles (see [7, 21]). Moreover, many authors have studied and generalized the fixed point results of some mappings satisfying the generalized (θ, ψ)-contraction conditions in the context of partially ordered b-metric spaces (see [12–15]).

In this research, we examine the existence of solutions of the following initial boundary value problem by utilizing the new generalized contraction.

$$\begin{cases} {}^{\text{ABC}}{}_{a}\mathcal{D}^{\vartheta}\varsigma(\iota) = \Upsilon(\iota, \varsigma(\iota)) + \int_a^\iota \mathcal{K}(\iota, s, \varsigma(s))ds, \\ \varsigma(a) = \xi, \end{cases} \tag{3}$$

where $\vartheta \in (0,1]$, $\Upsilon(\iota, \varsigma(\iota)) \in C([a,b] \times \mathbb{R}, \mathbb{R})$ and $\mathcal{K}(\iota, s, \varsigma(\iota)) \in C([a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $\Upsilon(\iota, \varsigma(\iota)) \mid_a = 0$.

2. Preliminaries

Let $\kappa = [0,T]$ and $0 = \varsigma_0 < \varsigma_1 < \varsigma_2 < \cdots < \varsigma_n = T$, $(n \in \mathbb{N})$. Denote the space of all piecewise continuous function by $PC(\kappa, \mathbb{R})$ and set $\kappa = \kappa_0 \cup \kappa_1 \cup \kappa_2 \cup \cdots \cup \kappa_n$ where $\kappa_0 = [\varsigma_0, \varsigma_1]$, $\kappa_1 = (\varsigma_1, \varsigma_2]$, $\kappa_2 = (\varsigma_2, \varsigma_3], \ldots, \kappa_n = (\varsigma_n, \varsigma_{n+1}]$ and $\kappa' = \kappa \setminus \{\varsigma_1, \varsigma_2, \ldots, \varsigma_n\}$. Define

$$E = \{\pi : \kappa \to \mathbb{R} : \pi \in C(\kappa_n, \mathbb{R})\},$$

and put $\Delta\pi(\varsigma_n) = \pi(\varsigma_n^+) - \pi(\varsigma_n^-)$ for $\pi \in E$.
Here, (E, d) is a Banach space with respect to d be defined by:

$$d(y, \pi) = \|(y - \pi)^2\|_\infty = \sup_{t \in \kappa}(y(\iota) - \pi(\iota))^2.$$

Definition 2.1. *([25]) The Reimann fractional integral of function $\pi \in L^1([0,T], \mathbb{R}_+)$ is defined as*

$${}_0I^{\alpha}_{\varsigma}\pi(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma} (\varsigma - \tau)^{\alpha - 1}\pi(\tau)d\tau.$$

Definition 2.2. *([22]) The Caputo derivative of $z : (0, \infty) \to \mathbb{R}$ is defined as*

$${}^{c}_{0}D^{\alpha}_{\varsigma}\pi(\varsigma) = \frac{1}{\Gamma(n - \alpha)} \int_0^{\varsigma} (\varsigma - \tau)^{n - \alpha - 1}\pi^{(n)}(\tau)d\tau, \;\; n = [\alpha] + 1.$$

We denote by Φ all the functions ϕ defined on $[0, \infty)$ that satisfies the conditions:

(i) ϕ is continuous,

(ii) ϕ is nondecreasing,

(iii) $\phi(\iota) = 0$ if and only if $t = 0$.

Also, let Ψ denotes the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions:

(a) ψ is lower-continuous,

(b) $\psi(\iota) = 0$ if and only if $t = 0$.

Moreover, suppose ϕ and ψ satisfies the inequality $\phi(\frac{\iota}{2}) \geq \psi(\iota)$. Then, we have the following results.

Definition 2.3. *([24]) Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with $s > 1$, $\phi \in \Phi$ and $\psi \in \Psi$. The mapping $S : P \to P$ is a generalized (ϕ, ψ)-contractive if it satisfies:*

$$\phi(sd(Sv, S\xi)) \leq \phi(d(v, \xi)) - \psi(d(v, \xi)),$$

for any $v, \xi \in P$ with $v \preceq \xi$.

Theorem 2.4. *([24]) For the mentioned $(P, d, s, \preceq)$, let $S : P \to P$ be an almost generalized (ϕ, ψ)-contractive, continuous and nondecreasing with regards to $\preceq$. If there exists $v_0 \in P$ with $v_0 \preceq Sv_0$, then S has a fixed point.*

3. Existence of solutions: Main results

Here, we drive some conditions for the existence of at least one solution for problems (1) and (3).

Lemma 3.1. *([11]) Let $\psi : \kappa \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, $\pi \in E$ is the solution of the problem:*

$$\begin{cases} {}_0^cD^{\alpha}_{\varsigma_n}\pi(\varsigma) = \psi(\varsigma), \ \ \varsigma \in \kappa, \ \varsigma \neq \varsigma_n \text{ for } n = 1, 2, \dots, k, \\ \pi(0) = \pi_0, \\ \Delta\pi(\varsigma_n) = I_n(\pi(\varsigma_n)), \ \ n = 1, 2, \dots, k, \end{cases} \tag{4}$$

if and only if π satisfies the integral equation as

$$\pi(\varsigma) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\varsigma}(\varsigma - \tau)^{\alpha-1}\psi(\tau)d\tau + \pi_0, \ \ \varsigma \in \kappa_0, \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varsigma_n}^{\varsigma}(\varsigma - \tau)^{\alpha-1}\psi(\tau)d\tau + \sum_{i=1}^{n}\Bigg[\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varsigma_{i-1}}^{\varsigma_i}(\varsigma_i - \tau)^{\alpha-1}\psi(\tau)d\tau \\ \qquad + I_i(\pi(\varsigma_i))\Bigg] + \pi_0, \ \ \varsigma \in \kappa_n, \ n = 1, 2, \dots, k. \end{cases} \tag{5}$$

Corollary 3.2. *([11]) From Lemma 3.1, the solution of* (1) *is given as follows:*

$$\pi(\varsigma) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\varsigma}(\varsigma - \tau)^{\alpha-1}f(\varsigma, \pi(\varsigma), \pi(m\varsigma), {}_0^cD^{\alpha}_{\varsigma_n}\pi(\varsigma))d\tau + \pi_0, \ \ \varsigma \in \kappa_0, \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varsigma_n}^{\varsigma}(\varsigma - \tau)^{\alpha-1}f(\varsigma, \pi(\varsigma), \pi(m\varsigma), {}_0^cD^{\alpha}_{\varsigma_n}\pi(\varsigma))d\tau \\ \qquad + \displaystyle\sum_{i=1}^{n}\Bigg[\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varsigma_{i-1}}^{\varsigma_i}(\varsigma_i - \tau)^{\alpha-1}f(\varsigma, \pi(\varsigma), \pi(m\varsigma), {}_0^cD^{\alpha}_{\varsigma_n}\pi(\varsigma))d\tau \\ \qquad + I_i(\pi(\varsigma_i))\Bigg] + \pi_0, \ \ \varsigma \in \kappa_n, \ n = 1, 2, \dots, k. \end{cases} \tag{6}$$

We use the following notations

$$u_{\pi}(\varsigma) = f(\varsigma, \pi(\varsigma), \pi(m\varsigma), \quad {}_0^cD^{\alpha}_{\varsigma_n}\pi(\varsigma)) = f(\varsigma, \pi(\varsigma), \ \pi(m\varsigma), u_{\pi}(\varsigma)).$$

The following assumptions are necessary to obtain the main results.

(A_1) $f : \kappa \times \mathbb{R}^3 \to \mathbb{R}$ is continuous;

(A_2) There exist $M_f > 0$ and $0 < N_f < 1$ such that for $\varsigma \in \kappa$ and for $\pi, \bar{\pi} \in \mathbb{R}$ the following relation holds.

$$|f(\varsigma, \pi(\varsigma), \pi(m\varsigma), u_\pi(\varsigma)) - f(\varsigma, \bar{\pi}(\varsigma), \bar{\pi}(m\varsigma), u_{\bar{\pi}}(\varsigma))| \leq M_f(|\pi(\varsigma) - \bar{\pi}(\varsigma)| + |\pi(m\varsigma) - \bar{\pi}(m\varsigma)|) + N_f|u_\pi(\varsigma) - u_{\bar{\pi}}(\varsigma)|;$$

(A_3) For any $\pi, \bar{\pi} \in E$, there exists $A_I^* > 0$ with

$$|I_i(\pi(\varsigma_i)) - I_i(\bar{\pi}(\varsigma_i))| \leq A_I^*|\pi(\varsigma_i) - \bar{\pi}(\varsigma_i)|;$$

(A_4) For $\Psi(\iota, \varsigma(\iota)) \in L[a, b]$ and continuous functions ς and ς_1 there exist constants $\lambda_i > 0, (i = 1, 2)$ such that

$$\|\Psi(s, \varsigma(s)) - \Upsilon(s, \varsigma_1(s))\| \leq \lambda_1\|\varsigma - \varsigma_1\|, \quad \|\mathcal{K}(\iota, s, \varsigma(\iota)) - \mathcal{K}(\iota, s, \varsigma_1(\iota))\| \leq \lambda_2\|\varsigma - \varsigma_1\|.$$

Define $W : C(\kappa, \mathbb{R}) \to C(\kappa, \mathbb{R})$ as follows:

$$\begin{cases} (W\pi)(\varsigma) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\varsigma (\varsigma - \tau)^{\alpha-1} u_\pi(\tau) d\tau + \pi_0, \ \ \varsigma \in \kappa_0, \\ (W\pi)(\varsigma) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\varsigma_n}^\varsigma (\varsigma - \tau)^{\alpha-1} u_\pi(\tau) d\tau + \sum_{i=1}^n \left[\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\varsigma_{i-1}}^{\varsigma_i} (\varsigma_i - \tau)^{\alpha-1} u_\pi(\tau) d\tau + I_i(\pi(\varsigma_i))\right] + \pi_0, n = 1, 2, \ldots, k. \end{cases}$$

Theorem 3.3. *If the mentioned conditions $(A_1) - (A_3)$ together with the inequality*

$$\left(\frac{2M_f T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} + \frac{2M_f n T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} + A_I^* n\right) < \frac{1}{2} \tag{7}$$

hold and there exists $\pi \in C(\kappa, \mathbb{R})$ with $\pi \leq W\pi$, then (1) *has a solution.*

Proof. We first show that W is continuous. Let $\{\pi_n\} \in C(\kappa, \mathbb{R})$ with $\pi_n \to \pi \in C(\kappa, \mathbb{R})$. For each $\varsigma \in \kappa_n$, we have

$$\begin{aligned} |(W\pi_n)(\varsigma) - (W\pi)(\varsigma)| \leq & \frac{1}{\Gamma(\alpha)} \int_{\varsigma_n}^\varsigma (\varsigma - \tau)^{\alpha-1} |u_{\pi_n}(\tau) - u_\pi(\tau)| d\tau \\ & + \sum_{i=1}^n \frac{1}{\Gamma(\alpha)} \int_{\varsigma_{i-1}}^{\varsigma_i} (\varsigma_i - \tau)^{\alpha-1} |u_{\pi_n}(\tau) - u_\pi(\tau)| d\tau + \sum_{i=1}^n |I_i(\pi_n(\varsigma_i)) - I_i(\pi(\varsigma_i))|, \end{aligned} \tag{8}$$

where $u_{\pi_n}, u_\pi \in C(\kappa, \mathbb{R})$ satisfy

$$u_{\pi_n}(\varsigma) = f(\varsigma, \pi_n(\varsigma), \pi_n(m\varsigma), u_{\pi_n}(\varsigma)) \ \text{ and } \ u_\pi(\varsigma) = f(\varsigma, \pi(\varsigma), \pi(m\varsigma), u_\pi(\varsigma)).$$

By (A_2), we get

$$|u_{\pi_n}(\varsigma) - u_\pi(\varsigma)| \leq \frac{2M_f}{1 - N_f} |\pi_n(\varsigma) - \pi(\varsigma)|.$$

Now, $\pi_n \to \pi$ as $n \to \infty$ implies $u_{\pi_n}(\varsigma) \to u_\pi(\varsigma)$, $\varsigma \in \kappa_n$. Let $\aleph > 0$ such that for $\varsigma \in \kappa_n$ we have $|u_{\pi_n}(\varsigma)| \leq \aleph$ and $|u_\pi(\varsigma)| \leq \aleph$. Thus,

$$(\varsigma - \tau)^{\alpha-1} |u_{\pi_n}(\tau) - u_\pi(\tau)| \leq (\varsigma - \tau)^{\alpha-1} \Big(|u_{\pi_n}(\tau)| + |u_\pi(\tau)|\Big) \leq 2\aleph(\varsigma - \tau)^{\alpha-1},$$

$$(\varsigma_i - \tau)^{\alpha-1} |u_{\pi_n}(\tau) - u_\pi(\tau)| \leq (\varsigma_i - \tau)^{\alpha-1} \Big(|u_{\pi_n}(\tau)| + |u_\pi(\tau)|\Big) \leq 2\aleph(\varsigma_i - \tau)^{\alpha-1}.$$

For each $\varsigma \in \kappa_n$, $\tau \to 2\aleph(\varsigma - \tau)^{\alpha-1}$ and $\tau \to 2\aleph(\varsigma_i - \tau)^{\alpha-1}$ are integrable. Also, f and I are continuous. Therefore, applying Lebesgue dominated convergence theorem, we get $|(W\pi_n)(\varsigma) - (W\pi)(\varsigma)| \to 0$ as $n \to \infty$. Hence, particularly $\max_{\varsigma \in \kappa} |W\pi_n(\varsigma) - W\pi(\varsigma)| \to 0$ which implies that $\|W\pi_n - W\pi\|_E \to 0$. Similarly, for $\varsigma \in \kappa_0$, we can show that $\|W\pi_n - W\pi\|_E \to 0$. So, W is continuous.

For $\pi, \bar{\pi} \in E$ and $\varsigma \in \kappa_n$, we have

$$\begin{aligned}|(W\pi)(\varsigma) - (W\bar{\pi})(\varsigma)| &\leq \frac{1}{\Gamma(\alpha)} \int_{\varsigma_n}^{\varsigma} (\varsigma - \tau)^{\alpha-1} |u_\pi(\tau) - w_{\bar{\pi}}(\tau)| d\tau \\ &+ \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \int_{\varsigma_i - 1}^{\varsigma_i} (\varsigma_i - \tau)^{\alpha-1} |u_\pi(\tau) - w_{\bar{\pi}}(\tau)| d\tau \\ &+ \sum_{i=1}^{n} |I(\pi(\varsigma_i)) - I(\bar{\pi}(\varsigma_i))|, \end{aligned} \tag{9}$$

where $u_\pi, w_\pi \in C(\kappa, \mathbb{R})$ are given by

$$\begin{aligned} u_\pi(\varsigma) &= f(\varsigma, \pi(\varsigma), \pi(m\varsigma), u_\pi(\varsigma)), \\ w_{\bar{\pi}}(\varsigma) &= f(\varsigma, \bar{\pi}(\varsigma), \bar{\pi}(m\varsigma), w_{\bar{\pi}}(\varsigma)). \end{aligned}$$

By (A_2), we have

$$\begin{aligned} |u_\pi(\varsigma) - w_{\bar{\pi}}(\varsigma)| &= |f(\varsigma, \pi(\varsigma), \pi(m\varsigma), u_\pi(\varsigma)) - f(\varsigma, \bar{\pi}(\varsigma), \bar{\pi}(m\varsigma), w_{\bar{\pi}}(\varsigma))| \\ &\leq M_f(|\pi(\varsigma) - \bar{\pi}(\varsigma)| + |\pi(m\varsigma) - m\bar{z}(\varsigma)|) + N_f |u_\pi(\varsigma) - w_{\bar{\pi}}(\varsigma)| \\ &\leq 2M_f(|\pi(\varsigma) - \bar{\pi}(\varsigma)| + N_f |u_\pi(\varsigma) - w_{\bar{\pi}}(\varsigma)|. \end{aligned}$$

Then,

$$|u_\pi(\varsigma) - w_{\bar{\pi}}(\varsigma)| \leq \frac{2M_f}{1 - N_f} |\pi(\varsigma) - \bar{\pi}(\varsigma)|.$$

Thus, using assumptions $(A_2) - (A_3)$, inequality (9) implies

$$\begin{aligned} |(W\pi)(\varsigma) - (W\bar{\pi})(\varsigma)| &\leq \frac{2M_f}{(1 - N_f)\Gamma(\alpha)} \int_{\varsigma_n}^{\varsigma} (\varsigma - \tau)^{\alpha-1} |\pi(\tau) - \bar{\pi}(\tau)| d\tau \\ &+ \sum_{i=1}^{n} \frac{2M_f}{(1 - N_f)\Gamma(\alpha)} \int_{\varsigma_i - 1}^{\varsigma_i} (\varsigma_i - \tau)^{\alpha-1} |\pi(\tau) - \bar{\pi}(\tau)| d\tau + \sum_{i=1}^{n} A_I^* |\pi(\varsigma) - \bar{\pi}(\varsigma)| \\ &\leq \left(\frac{2M_f T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} + \frac{2M_f n T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} + A_I^* n \right) |\pi(\varsigma) - \bar{\pi}(\varsigma)|. \end{aligned}$$

So, we have

$$\|W\pi - W\bar{\pi}\|_E \leq \left(\frac{2M_f T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} + \frac{2M_f n T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} + A_I^* n \right) \|\pi - \bar{\pi}\|_E. \tag{10}$$

Similarly, for $\pi, \bar{\pi} \in E$ and $\varsigma \in \kappa_0$, we get

$$\|W\pi - W\bar{\pi}\|_E \leq \left(\frac{2M_f T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} \right) \|\pi - \bar{\pi}\|_E. \tag{11}$$

Therefore,

$$\|W\pi - W\bar{\pi}\|_E{}^2 \leq \left(\frac{2M_f T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} \right)^2 \|\pi - \bar{\pi}\|_E{}^2.$$

Since

$$\left(\frac{2M_f T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} \right) \leq \left(\frac{2M_f T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} + \frac{2M_f n T^\alpha}{(1 - N_f)\Gamma(\alpha + 1)} + A_I^* n \right) < \frac{1}{2},$$

then

$$\|W\pi - W\bar{\pi}\|_E^2 \leq \frac{1}{4}\|\pi - \bar{\pi}\|_E^2.$$

So, we have

$$2\|W\pi - W\bar{\pi}\|_E^2 \leq \frac{\|\pi - \bar{\pi}\|_E^2}{2} = \|\pi - \bar{\pi}\|_E^2 - \frac{\|\pi - \bar{\pi}\|_E^2}{2}.$$

Then,

$$\varphi(2\|W\pi - W\bar{\pi}\|_E^2) \leq \varphi(\|\pi - \bar{\pi}\|_E^2) - \varphi(\frac{\|\pi - \bar{\pi}\|_E^2}{2}).$$

Since $\varphi(\frac{\iota}{2}) \geq \psi(\iota)$, we obtain

$$\varphi(2\|W\pi - W\bar{\pi}\|_E^2) \leq \varphi(\|\pi - \bar{\pi}\|_E^2) - \psi(\frac{\|\pi - \bar{\pi}\|_E^2}{2}).$$

Hence, W is a generalized (φ, ψ)−contractive operator and therefore, by using Theorem 2.4, the problem (1) has a solution. □

Now we study problem (3) and derive some results.

Theorem 3.4. *([21]) Assume that* $\vartheta \in (0, 1], \Upsilon(\iota, \varsigma(\iota)) \in C([a, b] \times \mathbb{R}, \mathbb{R})$ *and* $\mathcal{K}(\iota, s, \varsigma(\iota)) \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *with* $\Upsilon(\iota, \varsigma(\iota)) \mid_a = 0$. *Then, the solution of*

$$\begin{cases} {}^{ABC}{}_a\mathcal{D}^{\vartheta}\varsigma(\iota) = \Upsilon(\iota, \varsigma(\iota)) + \int_a^{\iota} \mathcal{K}(\iota, s, \varsigma(s))ds, \\ \varsigma(a) = \xi. \end{cases}$$

is given by

$$\varsigma(\iota) = \xi + \frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\Upsilon(\iota, \varsigma(\iota)) + \int_a^{\iota} \mathcal{K}(\iota, s, \varsigma(s))ds\right] + \frac{\vartheta}{\mathbb{B}(\vartheta)\Gamma(\vartheta)}\int_a^{\iota} (\iota - s)^{\vartheta-1}\left[\Upsilon(s, \varsigma(s)) + \int_a^{s} \mathcal{K}(s, r, \varsigma(r))dr\right]ds,$$

where $\mathbb{B}$ *is the normalized function which is defined as*

$$\mathbb{B}(\vartheta) = 1 - \vartheta + \frac{\vartheta}{\Gamma(\vartheta)}.$$

Define $\mathcal{F} : C(\kappa, \mathcal{R}) \to C(\kappa, \mathcal{R})$ as

$$\begin{aligned} \mathcal{F}(\iota, \varsigma(\iota)) =& \xi + \frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\Upsilon(\iota, \varsigma(\iota)) + \int_a^{\iota} \mathcal{K}(\iota, s, \varsigma(s))ds\right] \\ &+ \frac{\vartheta}{\mathbb{B}(\vartheta)\Gamma(\vartheta)}\int_a^{\iota} (\iota - s)^{\vartheta-1}\left[\Upsilon(s, \varsigma(s)) + \int_a^{s} \mathcal{K}(s, r, \varsigma(r))dr\right]ds. \end{aligned}$$

Theorem 3.5. *If the assumed condition* (A_4) *together with the inequality*

$$\left(\frac{1-\vartheta}{\mathbb{B}(\vartheta)} + \frac{\vartheta(b^{\vartheta} - a^{\vartheta})}{\mathbb{B}(\vartheta)\Gamma(\vartheta+1)}\right)[\lambda_1 + \lambda_2(b-a)] < \frac{1}{2} \tag{12}$$

hold and there exists $\varsigma \in C(\kappa, \mathbb{R})$ *such that* $\varsigma \leq \mathcal{F}\varsigma$, *then* (3) *has a solution.*

Proof. First, we show that $\mathcal{F}$ is continuous.

$$\begin{aligned}|\varsigma_n(\iota)-\varsigma(\iota)| &= |\xi+\frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\Upsilon(\iota,\varsigma_n(\iota))+\int_a^{\iota}\mathcal{K}(\iota,s,\varsigma_n(s))ds\right]\\
&+\frac{\vartheta}{\mathbb{B}(\vartheta)\Gamma(\vartheta)}\int_a^{\iota}(\iota-s)^{\vartheta-1}\left[\Upsilon(s,\varsigma_n(s))+\int_a^{s}\mathcal{K}(s,r,\varsigma_n(r))dr\right]ds\\
&-\xi-\frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\Upsilon(\iota,\varsigma(\iota))+\int_a^{\iota}\mathcal{K}(\iota,s,\varsigma(s))ds\right]\\
&-\frac{\vartheta}{\mathbb{B}(\vartheta)}\Gamma(\vartheta)\int_a^{\iota}(\iota-s)^{\vartheta-1}\left[\Upsilon(s,\varsigma(s))+\int_a^{s}\mathcal{K}(s,r,\varsigma(r))dr\right]ds|\\
&\leq\frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[|\Upsilon(\iota,\varsigma_n(\iota))-\Upsilon(\iota,\varsigma(\iota))|+\int_a^{\iota}|\mathcal{K}(\iota,s,\varsigma_n(s))-|\mathcal{K}(\iota,s,\varsigma(s))|ds\right]\\
&+\frac{\vartheta}{\mathbb{B}(\vartheta)\Gamma(\vartheta)}\int_a^{\iota}(\iota-s)^{\vartheta-1}\,[|\Upsilon(s,\varsigma_n(s))-\Upsilon(s,\varsigma(s))|\\
&+\int_a^{s}|\mathcal{K}(s,r,\varsigma_n(r))-\mathcal{K}(s,r,\varsigma(r))|dr\Bigg]ds\\
&\leq\frac{1-\vartheta}{\mathbb{B}(\vartheta)}\,[\lambda_1+(b-a)\lambda_2]\,|\varsigma_n-\varsigma|+\frac{\vartheta(b^{\vartheta}-a^{\vartheta})}{\mathbb{B}(\vartheta)\Gamma(\vartheta+1)}\,[\lambda_1+\lambda_2(b-a)]\,|\varsigma_n-\varsigma|\\
&=\left(\frac{1-\vartheta}{\mathbb{B}(\vartheta)}+\frac{\vartheta(b^{\vartheta}-a^{\vartheta})}{\mathbb{B}(\vartheta)\Gamma(\vartheta+1)}\right)[\lambda_1+\lambda_2(b-a)]\,|\varsigma_n-\varsigma|\leq\triangle^n|\varsigma_1-\varsigma|,\end{aligned}$$

where $\triangle=\left(\frac{1-\vartheta}{\mathbb{B}(\vartheta)}+\frac{\vartheta(b^{\vartheta}-a^{\vartheta})}{\mathbb{B}(\vartheta)\Gamma(\vartheta+1)}\right)[\lambda_1+\lambda_2(b-a)]<\frac{1}{2}$. Therefore, we can obtain $\varsigma_n\to\varsigma$. Applying Lebesgue dominated convergence theorem, we have $\|\mathcal{F}(\iota,\varsigma_n(\iota))-\mathcal{F}(\iota,\varsigma(\iota))\|\to 0$, as $n\to\infty$. So, $\mathcal{F}$ is continuous. Now, we show that $\mathcal{F}(\iota,\varsigma)$ satisfies in the conditions of the generalized (φ,ψ)−contractive operator. For this, we have

$$\begin{aligned}\|\mathcal{F}(\iota,\varsigma(\iota))-\mathcal{F}(\iota,\varsigma_1(\iota))\| &=\|\,\xi+\frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\Upsilon(\iota,\varsigma(\iota))+\int_a^{\iota}\mathcal{K}(\iota,s,\varsigma(s))ds\right]\\
&+\frac{\vartheta}{\mathbb{B}(\vartheta)\Gamma(\vartheta)}\int_a^{\iota}(\iota-s)^{\vartheta-1}\left[\Upsilon(s,\varsigma(s))+\int_a^{s}\mathcal{K}(s,r,\varsigma(r))dr\right]ds\\
&-\xi-\frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\Upsilon(\iota,\varsigma_1(\iota))+\int_a^{\iota}\mathcal{K}(\iota,s,\varsigma_1(s))ds\right]\\
&-\frac{\vartheta}{\mathbb{B}(\vartheta)}\Gamma(\vartheta)\int_a^{\iota}(\iota-s)^{\vartheta-1}\left[\Upsilon(s,\varsigma_1(s))+\int_a^{s}\mathcal{K}(s,r,\varsigma_1(r))dr\right]ds\,\|\\
&\leq\frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\|\,\Upsilon(\iota,\varsigma(\iota))-\Upsilon(\iota,\varsigma_1(\iota))\,\|+\int_a^{\iota}\|\,\mathcal{K}(\iota,s,\varsigma(s))-\mathcal{K}(\iota,s,\varsigma_1(s))\,\|\,ds\right]\\
&+\frac{\vartheta}{\mathbb{B}(\vartheta)\Gamma(\vartheta)}\int_a^{\iota}(\iota-s)^{\vartheta-1}\,[\|\Upsilon(s,\varsigma(s))-\Upsilon(s,\varsigma_1(s))\|\\
&+\int_a^{s}\|\mathcal{K}(s,r,\varsigma(r))-\mathcal{K}(s,r,\varsigma_1(r))\|dr\Bigg]ds\\
&\leq\frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\lambda_1\|\varsigma-\varsigma_1\|+\int_a^{\iota}\lambda_2\|\varsigma-\varsigma_1\|ds\right]\\
&+\frac{\vartheta}{\mathbb{B}(\vartheta)\Gamma(\vartheta)}\int_a^{\iota}(\iota-s)^{\vartheta-1}\left[\lambda_1\|\varsigma-\varsigma_1\|+\int_a^{s}\lambda_2\|\varsigma-\varsigma_1\|dr\right]ds\end{aligned}$$

$$\leq \frac{1-\vartheta}{\mathbb{B}(\vartheta)}\left[\lambda_1+(b-a)\lambda_2\right]\|\varsigma-\varsigma_1\|+\frac{\vartheta(b^{\vartheta}-a^{\vartheta})}{\mathbb{B}(\vartheta)\Gamma(\vartheta+1)}\left[\lambda_1+\lambda_2(b-a)\right]\|\varsigma-\varsigma_1\|$$
$$=\left(\frac{1-\vartheta}{\mathbb{B}(\vartheta)}+\frac{\vartheta(b^{\vartheta}-a^{\vartheta})}{\mathbb{B}(\vartheta)\Gamma(\vartheta+1)}\right)\left[\lambda_1+\lambda_2(b-a)\right]\|\varsigma-\varsigma_1\|.$$

Therefore,

$$\| \mathcal{F}(\iota,\varsigma(\iota))-\mathcal{F}(\iota,\varsigma_1(\iota))\| \leq \frac{1}{2}\|\varsigma_1-\varsigma\|.$$

Hence,

$$\| \mathcal{F}(\iota,\varsigma(\iota))-\mathcal{F}(\iota,\varsigma_1(\iota))\|^2 \leq \frac{1}{4}\|\varsigma_1-\varsigma\|^2.$$

So,

$$2\| \mathcal{F}(\iota,\varsigma(\iota))-\mathcal{F}(\iota,\varsigma_1(\iota))\|^2 \leq \frac{1}{2}\|\varsigma_1-\varsigma\|^2=\|\varsigma_1-\varsigma\|^2-\frac{\|\varsigma_1-\varsigma\|^2}{2}.$$

Thus,

$$\varphi(2\| \mathcal{F}(\iota,\varsigma(\iota))-\mathcal{F}(\iota,\varsigma_1(\iota))\|^2)\leq \varphi(\|\varsigma_1-\varsigma\|^2)-\varphi(\frac{\|\varsigma_1-\varsigma\|^2}{2}).$$

Since $\varphi(\frac{\iota}{2})\geq\psi(\iota)$, we have

$$\varphi(2\| \mathcal{F}(\iota,\varsigma(\iota))-\mathcal{F}(\iota,\varsigma_1(\iota))\|^2)\leq \varphi(\|\varsigma_1-\varsigma\|^2)-\Upsilon(\|\varsigma_1-\varsigma\|^2).$$

So, $\mathcal{F}$ is a generalized $(\varphi,\psi)-$contractive operator. All the condition of Theorem 2.4 hold. Therefore, the problem (3) has a solution. □

Example 3.6. *Consider the following problem.*

$$\begin{cases} {}^c_0D^{\frac{1}{2}}_{\varsigma}\pi(\varsigma)=\frac{e^{-\pi x}}{15}+\frac{e^{-x}}{38+\varsigma^2}\Big(\cos(|\pi(\varsigma)|)+z\Big(\frac{1}{4}\varsigma\Big)+\cos(|{}^c_0D_{\varsigma}{}^{\frac{1}{2}}\pi(\varsigma)|)\Big),\\ \varsigma\in[0,1],\ \varsigma\neq\frac{1}{3},\ k=1,\\ \pi(0)=0,\\ \Delta z\Big(\frac{1}{3}\Big)=\frac{1}{10}\pi(\frac{1}{3}). \end{cases} \tag{13}$$

Here, $\alpha=\frac{1}{2}$, $m=\frac{1}{4}$, $\kappa_0=[0,\frac{1}{3}]$ and $\kappa_1=(\frac{1}{3},1]$. Set

$$f(\varsigma,\pi(\varsigma),\pi(m\varsigma),u_{\pi}(\varsigma))=\frac{e^{-\pi x}}{15}+\frac{e^{-x}}{38+\varsigma^2}\Big(\cos(|\pi(\varsigma)|)+z\Big(\frac{1}{4}\varsigma\Big)+\cos(|{}^c_0D_{\varsigma}{}^{\frac{1}{2}}\pi(\varsigma)|)\Big).$$

Function f is continuous. Using (A_2) for any $\pi,\bar{\pi}\in\mathbb{R}$, we have

$$|f(\varsigma,\pi(\varsigma),\pi(m\varsigma),{}^c_0D^{\alpha}_{\varsigma}\pi(\varsigma))-f(\varsigma,\bar{\pi}(\varsigma),\bar{\pi}(m\varsigma),{}^c_0D^{\alpha}_{\varsigma}\bar{\pi}(\varsigma))|\leq\frac{1}{19}|\pi(\varsigma)-\bar{\pi}(\varsigma)|+\frac{1}{38}|{}^c_0D_{\varsigma}{}^{\frac{3}{2}}\pi(\varsigma)-{}^c_0D_{\varsigma}{}^{\frac{3}{2}}\bar{\pi}(\varsigma)|.$$

Hence, (A_2) holds with $M_f=\frac{1}{19}$ and $N_f=\frac{1}{38}$. Set

$$I_k(v)=\frac{1}{10}v,$$

where $v \in \mathbb{R}$. Then, for $v, \bar{v} \in \mathbb{R}$ and $k = 1$, we have

$$|I_1(v) - I_1(\bar{v})| \leq |\frac{1}{10}v - \frac{1}{10}\bar{v}| = \frac{1}{10}|v - \bar{v}|.$$

Hence, (A_3) holds with $A_I^* = \frac{1}{10}$. Also, the condition

$$\Big(\frac{M_f T^\alpha}{(1-N_f)\Gamma(\alpha+1)} + \frac{M_f n T^\alpha}{(1-N_f)\Gamma(\alpha+1)} + A_I^* n\Big) = 0.169 < 1,$$

satisfies with $T = 1$ and $n = 1$. Thus, from Theorem 3.3, problem (13) has a solution.

Example 3.7. *Let $\Upsilon(\iota, \varsigma(\iota)) = \iota \sin \varsigma(\iota) \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\mathcal{K}(\iota, s, \varsigma(\iota)) = \iota\varsigma(\iota) \sin s \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\mathbb{B}(\iota) = \frac{1}{(2\iota-1)^2}$. Consider*

$$\begin{cases} {}^{ABC}{}_0\mathcal{D}^{0.9}\varsigma(\iota) = \iota \sin \varsigma(\iota) + \int_0^\iota \iota\varsigma(\iota) \sin s ds, \\ \varsigma(0) = 0 \end{cases} \tag{14}$$

where $\vartheta = 0.9, a = 0, b = 1$ and $\lambda_1 = \lambda_2 = 1$. We have

$$\Big(\frac{1-\vartheta}{\mathbb{B}(\vartheta)} + \frac{\vartheta(b^\vartheta - a^\vartheta)}{\mathbb{B}(\vartheta)\Gamma(\vartheta+1)}\Big)[\lambda_1 + \lambda_2(b-a)] < \frac{1}{2}.$$

Hence, by Theorem 3.5, Problem (14) has at least one solution.

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