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Chen inequalities for slant Riemannian submersions from cosymplectic space forms

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Abstract. We introduce Chen inequalities for slant Riemannian submersions from cosymplectic manifolds onto Riemannian manifolds. We provide trivial and non-trivial examples for slant Riemannian submersions, investigate some curvature relations between the total space, the base space and fibres. Moreover, we obtain Chen-Ricci inequalities on the vertical and the horizontal distributions for slant Riemannian submersions from cosymplectic space forms.

1. Introduction

In 1993, B.-Y. Chen found some relations between the extrinsic (mainly, the squared mean curvature) and intrinsic invariants (mainly, the scalar curvature and the Ricci curvature) of a submanifold in a real space form [12]. In 1999, B.-Y. Chen obtained a sharp relation between the Ricci curvature and the squared mean curvature for a submanifold in [13]. Afterwards, many geometers inspired by that fact and have obtained many results on the notion in the different ambient spaces. In 2011, B.-Y. Chen [14] published a book which consisted of a collection of the results in this direction. Recenlty, Chen-like inequalities have been studied in [5, 6, 13, 28–31, 34, 39, 41, 42].

As point out in [19, 38], a considerable interest in Riemannian geometry is to compare some geometric properties of suitable types of maps between Riemannian manifolds. In this guidance, B. O'Neill [33] and A. Gray [20] defined the concept of Riemannian submersions between Riemannian manifolds as follows: A differentiable map $F : (M, g_M) \longrightarrow (N, g_N)$ between Riemannian manifolds (M, g_M) and (N, g_N) is called a Riemannian submersion (submanifold) if F_* is onto and

$$g_N(F_*X, F_*Y) = g_M(X, Y)$$

for vector fields X, Y tangent to the horizontal space $(\ker F_*)^\perp$. The notion has some aplications in physics and in mathematics. More precisely, Riemannian submersions have applications in supergravity and superstring theories [25, 27], Kaluza-Klein theory [9, 24] and the Yang-Mills theory [8, 44]. In 1976, B. Watson [43] considered submersions between almost Hermitian manifolds by taking account of almost complex structure of total manifold and showed that, in most of the cases, the base space and each fiber have the same kind of structure as the total space.

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Inspried by Watson's article, B. Şahin [36] introduced anti-invariant Riemannian submersions and studied deeply the geomerty of the submersions. In that paper, the author showed that the vertical distribution is anti-invariant under the action of almost complex structure of the total manifold. As a generalization of anti-invariant submersions and almost Hermitian submersions, the author [37] introduced the notion of slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. For slant submersions see: [3, 18, 22, 23, 35, 40].

A simple optimal relationship between Riemannian submersions and minimal immersions were proved by B.-Y. Chen in [15] and [16]. Afterwards, a sharp relationship between the $\delta-$invariants and Riemannian submersions with totally geodesic fibers was established by P. Alegre, B.-Y. Chen and M. I. Munteanu in [4]. Recently, M. Gülbahar, Ş. E. Meriç and E. Kılıç [21] obtained sharp inequalities involving the Ricci curvature for invariant Riemannian submersions. H. Aytimur and C. Özgür obtained sharp inequalities for anti-invariant Riemannian submersions from Sasakian space forms onto Riemannian manifolds [6]. In the present paper, we are motivated to study Chen-Ricci inequalities on the vertical and the horizontal distributions for slant Riemannian submersions from cosymplectic space forms.

The present paper is organized as follows. In Section 2, we mention some basic geometric properties of Riemannian submersions and cosymplectic manifolds. In Section 3, we obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for slant Riemannian submersions from cosymplectic space forms. The equality cases are also discussed.

2. Geometrical Notations and Preliminaries

In this section, we give necessary background for Riemannian submersions and cosymplectic manifolds. A surjective C^∞-map $F:(M_1,g_1)\to(M_2,g_2)$ where (M_1,g_1) and (M_2,g_2) be C^∞-Riemannian manifolds of dimension m_1 and m_2, is a C^∞-Riemannian submersion if it has maximal rank at any point of M_1 and preserve the lenght of horizontal vector fields. We also know that the implicit function theorem states that the fibre over any $p\in M_2, F^{-1}(p)$, is a closed r-dimensional submanifold of M_1, $r=m_1-m_2$. A vector field on M_1 is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M_1 is called *basic* if X is horizontal and F-related to a vector field X_* on M_2, i.e., $F_*X_p = X_{*F(p)}$ for all $p\in M$. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $\ker F_*$, and the horizontal distribution $(\ker F_*)^\perp$, respectively. Putting $\mathcal{V}_p=\ker\pi_{*p}$, for any $p\in M_1$, we obtain an integrable distribution $\mathcal{V}$ which corresponds to the foliation of M_1 determined by the fibres of F, since each $\mathcal{V}_p$ coincides with the tangent space of $\pi^{-1}(x)$ at $p, \pi(p)=x$.

As usual, the manifold (M_1,g_1) is called *total manifold* and the manifold (M_2,g_2) is called *base manifold* of the submersion $F:(M_1,g_1)\to(M_2,g_2)$. The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_U V = \mathcal{V}\nabla_{\mathcal{V}U}\mathcal{H}V + \mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}V, \tag{1}$$

$$\mathcal{A}_U V = \mathcal{V}\nabla_{\mathcal{H}U}\mathcal{H}V + \mathcal{H}\nabla_{\mathcal{H}U}\mathcal{V}V \tag{2}$$

for any vector fields U and V on M, where ∇ is the Levi-Civita connection of g_1. It is easy to see that $\mathcal{T}_U$ and $\mathcal{A}_U$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We now summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let V,W be vertical and X,Y be horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y]. \tag{4}$$

On the other hand, from (1) and (2), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{5}$$

$$\nabla_V X = \mathcal{T}_V X + \mathcal{H}\nabla_V X, \tag{6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{8}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic $\mathcal{H}\nabla_V X = \mathcal{A}_X V$.

Remark 2.1. *In this paper, we will assume all horizontal vector fields as basic vector fields.*

Denote by R, R', $\hat{R}$ and R^* the Riemannian curvature tensor of Riemannian manifolds M, N, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, respectively. Then the Gauss-Codazzi type equations are given by

$$R(U,V,F,W) = \hat{R}(U,V,F,W) + g(\mathcal{T}_U W, \mathcal{T}_V F) - g(\mathcal{T}_V W, \mathcal{T}_U F), \tag{9}$$

$$R(X,Y,Z,H) = R^*(X,Y,Z,H) - 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) + g(\mathcal{A}_Y Z, \mathcal{A}_X H) - g(\mathcal{A}_X Z, \mathcal{A}_Y H), \tag{10}$$

$$R(X,V,Y,W) = g((\nabla_X \mathcal{T})(V,W),Y) + g((\nabla_V \mathcal{A})(X,Y),W) - g(\mathcal{T}_V X, \mathcal{T}_W Y) + g(\mathcal{A}_Y W, \mathcal{A}_X V), \tag{11}$$

where

$$\pi_*(R^*(X,Y)Z) = R'(\pi_* X, \pi_* Y)\pi_* Z \tag{12}$$

for all $U,V,F,W \in \mathcal{V}(M)$ and $X,Y,Z,H \in \mathcal{H}(M)$.

Moreover, the mean curvature vector field H of any fibre of Riemannian submersion π is given by

$$N = rH, N = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j \tag{13}$$

where $\{U_1, ..., U_r\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$. Furthermore, π has totally geodesic fibers if $\mathcal{T}$ vanishes on $\mathcal{H}(M)$ and $\mathcal{V}(M)$.

Now we recall the following lemmas:

Lemma 2.2. *Let (M, g_M) and (N, g_N) be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to N$. For $E, F, G \in \chi(M)$, we have*

$$g(\mathcal{T}_E F, G) = -g(F, \mathcal{T}_E G), \tag{14}$$

$$g(\mathcal{A}_E F, G) = -g(F, \mathcal{A}_E G). \tag{15}$$

That is, $\mathcal{A}_E$ and $\mathcal{T}_E$ are anti-symmetric with respect to g.

Lemma 2.3. *Let (M, g_M) and (N, g_N) be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$.*

(i) For $U, V \in \chi^v(M)$, $\mathcal{T}_U V = \mathcal{T}_V U$,
(ii) For $X, Y \in \chi^h(M)$, $\mathcal{A}_X Y = -\mathcal{A}_Y X$.

Let M be $(2m+1)-$dimensional smooth manifold with an endomorphism ϕ, a vector field ξ and a 1-form η which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \phi\xi = 0, \eta \circ \phi = 0, \eta(\xi) = 1. \tag{16}$$

Then M is said to be an almost contact manifold. There always exist a compatible metric g_M such that

$$g_M(\phi X, \phi Y) = g_M(X,Y) - \eta(X)\eta(Y), \eta(X) = g_M(X, \xi) \tag{17}$$

for any $X, Y \in \Gamma(TM)$. The condition for normality in terms of ϕ, ξ and η on M is $[\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. The fundamental 2-form Φ of M is defined as $\Phi(X, Y) = g_M(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g_M) is said to be cosymplectic if it is normal and both $d\Phi = 0$ and $d\eta = 0$. Then considering the covariant derivative of ϕ, the structure equation of a cosymplectic manifold is characterized by the relation

$$(\nabla_X \phi)Y = 0 \text{ and } \nabla_X \xi = 0 \tag{18}$$

for any $X, Y \in \Gamma(TM)$ [7, 26]. Recently, for some studies on cosymplectic manifolds, see: [2, 26, 32].

A plane section π in TpM of an almost contact metric manifold M is called a $\phi-$section if $\pi \perp \xi$ and $\phi(\pi) = \pi$. M is of constant $\phi-$sectional curvature if sectional curvature $K(\pi)$ does not depend on the choice of the $\phi-$section π of TpM and the choice of a point $p \in M$. A cosymplectic manifold M is said to be a cosymplectic space form if the $\phi-$sectional curvature is constant c along M. A cosymplectic space form will be denoted by $M(c)$. Then the Riemannian curvature tensor R on $M(c)$ is given by [26]

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{ & g_M(Y, Z)X - g_M(X, Z)Y + \eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + \eta(Y)g_M(X, Z)\xi - \eta(X)g_M(Y, Z)\xi \\ & + g_M(\phi Y, Z)\phi X - g_M(\phi X, Z)\phi Y - 2g_M(\phi X, Y)\phi Z\}. \end{aligned} \tag{19}$$

Definition 2.4. *Let $M(\phi, \xi, \eta, g_M)$ be a cosymplectic manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\pi : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ is said to be slant if for any nonzero vector $X \in \Gamma(\ker \pi_*) - \{\xi\}$, the angle $\theta(X)$ between ϕX and the space $\ker \pi_*$ is a constant (which is independent of the choice of $p \in M$ and of $X \in \Gamma(\ker \pi_*) - \{\xi\}$). The angle θ is called the slant angle of the slant submersion. Invariant and anti-invariant submersions are slant submersions with $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submersion which is not invariant nor anti-invariant is called proper submersion.*

Remark 2.5. *In the present paper, we suppose that the Reeb vector field ξ is vertical.*

Now, we are going to mention some examples for slant Riemannian submersions in the following.

Example 2.6. *Every anti-invariant Riemannian submersion from cosymplectic manifold onto Riemannian manifold is a slant Riemannian submersion with $\theta = \{\frac{\pi}{2}\}$.*

Example 2.7. *([1], [11], [32]) We consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates (u_i, v_i, z) $(i = 1, \ldots, n)$ and its usual contact form*

$$\eta = dz.$$

The Reeb vector field ξ is given by $\frac{\partial}{\partial z}$ and its Riemannian metric g and tensor field ϕ are given by

$$g = \sum_{i=1}^{n} \left((du_i)^2 + (dv_i)^2\right) + (dz)^2, \quad \phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, i = 1, \ldots, n.$$

This gives a cosymplectic manifold on $\mathbb{R}^{2n+1}$. For simplicity, we assume that $\frac{\partial}{\partial u_i} = \partial u_i$. In this case, the vector fields $e_i = \partial v_i, e_{n+i} = \partial u_i$, ξ form a ϕ-basis for the cosymplectic structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\phi, \xi, \eta, g)$ is a cosymplectic manifold.

Example 2.8. *Let $\left(\mathbb{R}^5, \phi, \xi, \eta, g_{\mathbb{R}^5}\right)$ be a cosymplectic manifold mentioned by Example 2.7. Let $\left(\mathbb{R}^2, g_{\mathbb{R}^2}\right)$ be a Riemannian manifold endowed with metric $g_{\mathbb{R}^2} = \sum_{i=1}^{2} dy_i^2$. Consider a map $\psi : \left(\mathbb{R}^5, \varphi, \xi, \eta, g_{\mathbb{R}^5}\right) \to \left(\mathbb{R}^3, g_{\mathbb{R}^2}\right)$ defined by*

$$\psi(u_1, u_2, v_1, v_2, t) = (-u_1 \cos\gamma + v_2 \sin\gamma, u_2).$$

Then by direct calculations, we have

$$\begin{aligned} ker\psi_* &= \operatorname{span}\{X_1 = (\sin\gamma \partial u_1 + \cos\gamma \partial v_2), X_2 = \partial v_1, X_3 = \xi = \partial z\} \\ (\ker\psi_*)^\perp &= \operatorname{span}\{H_1 = (-\cos\gamma \partial u_1 + \sin\gamma \partial v_2), H_2 = \partial u_2\}. \end{aligned}$$

Furthermore, $\psi(X_1) = (-\sin\gamma\partial u_2 + \cos\gamma\partial v_1)$ *and* $\psi(X_2) = \partial v_1$ *imply that* $\left|g_{\mathbb{R}^5}(\psi(X_1), X_2)\right| = \gamma$*. Hence, the map* ψ *is a slant Riemannian submersion with the slant angle* $\theta = \gamma$*, which means that* $0 < \gamma < \frac{\pi}{2}$*. Now, we are going to show that the fibres of the submersion are totall geodesic. Consider the Koszul formula for Levi-Civita connection* ∇ *for* $\mathbb{R}^5$

$$\begin{aligned}2g(\nabla_{X_1}X_2, X_3) &= X_1 g(X_2, X_3) + Y g(X_3, X_1) - X_3 g(X_1, X_2)\\ &\quad - g([X_2, X_3], X_1) - g([X_1, X_3], X_2) + g([X_1, X_2], X_3)\end{aligned}$$

for all $X_1, X_2, X_3 \in \Gamma(T\mathbb{R}^5)$*. By simple calculations, we obtain*

$$\nabla_{e_i}e_j = 0 \quad \text{for all} \quad i, j = 1, \ldots, 3.$$

Thus $\mathcal{T}_{X_i}X_j = \mathcal{T}_{X_j}X_i = \mathcal{T}_{X_i}X_i = 0$ *for all* X_i*,* $(i = 1, 2, 3) \in \Gamma(\ker\psi_*)$*. Therefore fibres of* ψ *are totally geodesic.*

Example 2.9. $\mathbb{R}^7$ *has a cosymplectic structure as in Example 2.7. Let* $\left(\mathbb{R}^4, g_{\mathbb{R}^4}\right)$ *be a Riemannian manifold endowed with metric* $g_{\mathbb{R}^4} = \sum_{i=1}^4 dy_i^2$*. Let* $F : \mathbb{R}^7 \to \mathbb{R}^4$ *be a map defined by* $F(u_1, u_2, u_3, v_1, v_2, v_3, t) = \left(u_1, u_2, v_3, \frac{v_1 - v_2}{\sqrt{2}}\right)$*. Then, by direct calculations, we obtain the Jacobian matrix of F as:*

$$F_* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}.$$

The rank of F_* *is equal to 3. Thus, the map F is a submersion. After some computations, we obtain*

$$\ker F_* = \operatorname{span}\left\{\bar{X}_1 = \partial u_3, \quad \bar{X}_2 = \xi = \partial z, \quad \bar{X}_3 = -\frac{1}{\sqrt{2}}(\partial v_1 + \partial v_2)\right\}$$

and

$$(\ker F_*)^\perp = \operatorname{span}\left\{\bar{Z}_1 = \partial u_1, \quad \bar{Z}_2 = \partial u_2, \quad \bar{Z}_3 = \partial v_3, \quad \bar{Z}_4 = \frac{1}{\sqrt{2}}(\partial v_1 - \partial v_2)\right\}.$$

A straight computation we obtain $\theta = \frac{\pi}{2}$*. Hence the map F is a slant Riemannian submersions from* $\mathbb{R}^7$ *to* $\mathbb{R}^4$*.*

Let $\pi : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ be a slant Riemannian submersion from a cosymplectic manifold $(M, g_M, \phi, \xi, \eta)$ to a Riemannian manifold (N, g_N). Then for any $U, V \in \Gamma(\ker\pi_*)$, we put

$$\phi U = \psi U + \omega U, \tag{20}$$

where ψU and ωU are vertical and horizontal components of ϕU, respectively. Similarly, for any $X \in \Gamma(\ker\pi_*)^\perp$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X, \tag{21}$$

where $\mathcal{B}X$ (resp. $\mathcal{C}X$) is vertical part (resp. horizontal part) of ϕX.

The following theorem is a characterization for slant submersions of a cosymplectic manifold. The proof of it exactly same with slant immersions see: [10, 17]. Therefore we omit its proof.

Theorem 2.10. *Let* π *be a Riemannian submersion from a cosymplectic manifold* $M\left(\phi, \xi, \eta, g_M\right)$ *onto a Riemannian manifold* (N, g_N)*. Then,* π *is a slant Riemannian submersion if and only if there exists a constant* $\lambda \in [0, 1]$ *such that*

$$\psi^2 = -\lambda(I - \eta \otimes \xi). \tag{22}$$

Furthermore, in such a case, if θ is the slant angle of π, it satisfies that $\lambda = \cos^2\theta$. By using (17), (20), (21) and (22), we have the following lemma.

Lemma 2.11. *Let π be a slant Riemannian submersion from a cosymplectic manifold $M\left(\phi, \xi, \eta, g_M\right)$ onto a Riemannian manifold (N, g_N) with slant angle θ. Then the following relations are valid*

$$g_M(\psi U, \psi V) = \cos^2\theta\left(g_M(U,V) - \eta(U)\eta(V)\right), \tag{23}$$

$$g_M(\omega U, \omega V) = \sin^2\theta\left(g_M(U,V) - \eta(U)\eta(V)\right), \tag{24}$$

for any $U, V \in \Gamma(\ker\pi_*)$.

3. Chen-Ricci inequality and Chen inequalities

Let $(M(c), g_M)$, (N, g_N) be a cosymplectic space form and a Riemannian manifold, respectively and $\pi : M(c) \to N$ a slant Riemannian submersion. Furthermore, let $\{U_1, ..., U_r, X_1, ..., X_n\}$ be an orthonormal basis of $T_pM(c)$ such that $\mathcal{V} = span\{U_1, ..., U_r = \xi\}$, $\mathcal{H} = span\{X_1, ..., X_n\}$. Then using (9), (10) and (19), we have

$$\begin{aligned}\hat{R}(U,V,F,W) = \frac{c}{4}\{&g_M(V,F)g_M(U,W) - g_M(U,F)g_M(V,W)\\ &+ \eta(U)\eta(F)g_M(V,W) - \eta(V)\eta(F)g_M(U,W)\\ &+ \eta(V)\eta(W)g_M(U,F) - \eta(U)\eta(W)g_M(V,F)\\ &+ g_M(\phi V,F)g_M(\phi U,W) - g_M(\phi U,F)g_M(\phi V,W)\\ &- 2g_M(\phi U,V)g_M(\phi F,W)\} - g_M(\mathcal{T}_U W, \mathcal{T}_V F)\\ &+ g_M(\mathcal{T}_V W, \mathcal{T}_U F),\end{aligned} \tag{25}$$

$$\begin{aligned}R^*(X,Y,Z,H) = \frac{c}{4}\{&g_M(Y,Z)g_M(X,H) - g_M(X,Z)g_M(Y,H)\\ &+ \eta(X)\eta(Z)g_M(Y,H) - \eta(Y)\eta(Z)g_M(X,H)\\ &+ \eta(Y)\eta(H)g_M(X,Z) - \eta(X)\eta(H)g_M(Y,Z)\\ &+ g_M(\phi Y,Z)g_M(\phi X,H) - g_M(\phi X,Z)g_M(\phi Y,H)\\ &- 2g_M(\phi X,Y)g_M(\phi Z,H)\} + 2g_M(\mathcal{A}_X Y, \mathcal{A}_Z H)\\ &- g_M(\mathcal{A}_Y Z, \mathcal{A}_X H) + g_M(\mathcal{A}_X Z, \mathcal{A}_Y H).\end{aligned} \tag{26}$$

Theorem 3.1. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then*

$$\widehat{Ric}(U) \geq \frac{c}{4}(r - 2 + 3\cos^2\theta)(1 - \eta(U)^2) - rg_M(\mathcal{T}_U U, H). \tag{27}$$

The equality case of (27) holds for a unit vertical vector $U \in \mathcal{V}_p(M(c))$ if and only if each fiber is totally geodesic.

Proof. Using (25) we derive

$$\widehat{Ric}(U) = \frac{c}{4}((r-2)(1-\eta(U)^2) + 3\sum_{i=1}^{r} g_M^2(\phi U, U_i)) - rg_M(\mathcal{T}_U U, H) + \|\mathcal{T}_U U_i\|^2, \tag{28}$$

where

$$\widehat{Ric}(U) = \sum_{i=1}^{r} \hat{R}(U, U_i, U_i, U).$$

Since

$$\sum_{i=1}^{r} g_M^2(\phi U, U_i) = \cos^2\theta(1-\eta(U)^2), \tag{29}$$

using last equation in (28), we get (27). □

Theorem 3.2. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then*

$$2\hat{\tau} \geq \frac{c}{4}(r-1)(r-2+3\cos^2\theta) - r^2\|H\|^2. \tag{30}$$

The equality case of (30) holds if and only if each fiber is totally geodesic.

Proof. Using the symmetry of $\mathcal{T}$ in (25), we have

$$2\hat{\tau} = \frac{c}{4}(r-1)(r-2+3\cos^2\theta) - r^2\|H\|^2 + \sum_{i,j=1}^{r} g_M(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_i}U_j), \tag{31}$$

which implies (30), where $\hat{\tau} = \sum_{1\leq i<j\leq r} \hat{R}(U_i, U_j, U_j, U_i)$. □

Since π is slant Riemannian submersion and ξ is vertical, using the anti-symmetry of $\mathcal{A}$ and (26), we find

$$2\tau^* = \frac{c}{4}(n(n-1) + 3\sum_{i,j=1}^{n} g_M(CX_i, X_j)g_M(CX_i, X_j)) - 3\sum_{i,j=1}^{n} g_M(\mathcal{A}_{X_i}X_j, \mathcal{A}_{X_i}X_j), \tag{32}$$

where

$$\tau^* = \sum_{1\leq i<j\leq r} \hat{R}(X_i, X_j, X_j, X_i). \tag{33}$$

Now we define

$$\|C\|^2 = \sum_{i=1}^{n} g^2(CX_i, X_j), \tag{34}$$

then from (32) and (34) we obtain

$$2\tau^* = \frac{c}{4}(n(n-1) + 3\|C\|^2) - 3\sum_{i,j=1}^{n} g_M(\mathcal{A}_{X_i}X_j, \mathcal{A}_{X_i}X_j). \tag{35}$$

From (35) we derive the following theorem :

Theorem 3.3. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then*

$$2\tau^* \leq \frac{c}{4}(n(n-1) + 3\|C\|^2). \tag{36}$$

The equality case of (36) holds if and only if $\mathcal{H}(M)$ is integrable.

Let $(M(c), g_M)$ be a cosymplectic space form and (N, g_N) a Riemannian manifold. Assume that $\pi : M(c) \to N$ is a slant Riemannian submersion and $\{U_1, ..., U_r, X_1, ..., X_n\}$ is an orthonormal basis of $TpM(c)$ such that $\mathcal{V}_p(M) = span\{U_1, ..., U_r\}$, $\mathcal{H}_p(M) = span\{X_1, ..., X_n\}$. Now we denote $\mathcal{T}^s_{ij}$ by

$$\mathcal{T}^s_{ij} = g_M(\mathcal{T}_{U_i} U_j, X_s), \tag{37}$$

where $1 \leq i, j \leq r$ and $1 \leq s \leq n$ (see [21]).

Similarly, we denote $\mathcal{A}^\alpha_{ij}$ by

$$\mathcal{A}^\alpha_{ij} = g_M(\mathcal{A}_{X_i} X_j, U_\alpha), \tag{38}$$

where $1 \leq i, j \leq n$ and $1 \leq \alpha \leq r$. From [21], we use

$$\delta(N) = \sum_{i=1}^{n} \sum_{k=1}^{r} g_M((\nabla_{X_i} \mathcal{T})_{U_k} U_k, X_i). \tag{39}$$

From the Binomial theorem there is such as the following equation between the tensor fields $\mathcal{T}$:

$$\sum_{s=1}^{n} \sum_{i,j=1}^{r} (\mathcal{T}^s_{ij})^2 = \frac{1}{2} r^2 \|H\|^2 + \frac{1}{2}(\mathcal{T}^s_{11} - \mathcal{T}^s_{22} - ... - \mathcal{T}^s_{rr})^2 + 2\sum_{s=1}^{n} \sum_{j=2}^{r} (\mathcal{T}^s_{1j})^2 - 2\sum_{s=1}^{n} \sum_{2 \leq i < j \leq r} (\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{40}$$

Theorem 3.4. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\widehat{Ric}(U_1) \geq \frac{c}{4}(r - 2 + 3\cos^2\theta)(1 - \eta(U_1)^2) - \frac{1}{4} r^2 \|H\|^2. \tag{41}$$

The equality case of (41) holds if and only if

$$\mathcal{T}^s_{11} = \mathcal{T}^s_{22} + ... + \mathcal{T}^s_{rr},$$
$$\mathcal{T}^s_{1j} = 0, \quad j = 2, ..., r.$$

Proof. Using (37) in (31), we can write

$$2\hat{\tau} = \frac{c}{4}(r-1)(r - 2 + 3\cos^2\theta) - r^2\|H\|^2 + \sum_{s=1}^{n} \sum_{i,j=1}^{r} (\mathcal{T}^s_{ij})^2. \tag{42}$$

Thus using (40) in (42) can be written as

$$2\hat{\tau} = \frac{c}{4}(r-1)(r - 2 + 3\cos^2\theta) - \frac{1}{2} r^2\|H\|^2 + \frac{1}{2}(\mathcal{T}^s_{11} - \mathcal{T}^s_{22} - ... - \mathcal{T}^s_{rr})^2 + 2\sum_{s=1}^{n} \sum_{j=2}^{r} (\mathcal{T}^s_{1j})^2$$
$$- 2\sum_{s=1}^{n} \sum_{2 \leq i < j \leq r} (\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{43}$$

Then from (43) we have

$$2\hat{\tau} \geq \frac{c}{4}(r-1)(r - 2 + 3\cos^2\theta) - \frac{1}{2} r^2\|H\|^2 - 2\sum_{s=1}^{n} \sum_{2 \leq i < j \leq r} (\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{44}$$

Moreover, taking $U = W = U_i$, $V = F = U_j$ in (9) and using (37), we obtain

$$2\sum_{2 \leq i < j \leq r} R(U_i, U_j, U_j, U_i) = 2\sum_{2 \leq i < j \leq r} \hat{R}(U_i, U_j, U_j, U_i) + 2\sum_{s=1}^{n} \sum_{2 \leq i < j \leq r} (\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{45}$$

Using (45) in (44), we get

$$2\hat{\tau} \geq \frac{c}{4}(r-1)(r-2+3\cos^2\theta) - \frac{1}{2}r^2\|H\|^2 + 2\sum_{2\leq i<j\leq r}\hat{R}(U_i,U_j,U_j,U_i) - 2\sum_{2\leq i<j\leq r}R(U_i,U_j,U_j,U_i). \tag{46}$$

Furthermore, we have

$$2\hat{\tau} = 2\sum_{2\leq i<j\leq r}\hat{R}(U_i,U_j,U_j,U_i) + 2\sum_{j=1}^{r}\hat{R}(U_1,U_j,U_j,U_1). \tag{47}$$

Considering (47) in (46), we get

$$2\widehat{Ric}(U_1) \geq \frac{c}{4}(r-1)(r-2+3\cos^2\theta) - \frac{1}{2}r^2\|H\|^2 - 2\sum_{2\leq i<j\leq r}R(U_i,U_j,U_j,U_i). \tag{48}$$

Since $M(c)$ is a cosymplectic space form, its curvature tensor R satisfies the equality (19), we have

$$\sum_{2\leq i<j\leq r}R(U_i,U_j,U_j,U_i) = \frac{c}{4}(\frac{r-3}{2} + \eta(U)^2)(r-2+3\cos^2\theta). \tag{49}$$

From (48) and (49), we obtain (41). □

Theorem 3.5. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$Ric^*(X_1) \leq \frac{c}{4}(n-1+3\|CX_1\|^2). \tag{50}$$

The equality case of (50) holds if and only if

$$\mathcal{A}^{\alpha}_{1j} = 0, \quad j = 2, ..., n.$$

Proof. Using (38) in (35), we have

$$2\tau^* = \frac{c}{4}(n(n-1) + 3\|C\|^2) - 3\sum_{\alpha=1}^{r}\sum_{i,j=1}^{n}(\mathcal{A}^{\alpha}_{ij})^2. \tag{51}$$

Using that $\mathcal{A}$ is anti-symmetric on $\mathcal{H}(M(c))$, (51) can be written as

$$2\tau^* = \frac{c}{4}(n(n-1) + 3\|C\|^2) - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}^{\alpha}_{1j})^2 - 6\sum_{\alpha=1}^{r}\sum_{2\leq i<j\leq n}(\mathcal{A}^{\alpha}_{ij})^2. \tag{52}$$

Moreover, taking $X = H = X_i$, $Y = Z = X_j$ in (10) and using (38), we derive

$$2\sum_{2\leq i<j\leq n}R(X_i,X_j,X_j,X_i) = 2\sum_{2\leq i<j\leq n}R^*(X_i,X_j,X_j,X_i) + 6\sum_{\alpha=1}^{r}\sum_{2\leq i<j\leq n}(\mathcal{A}^{\alpha}_{ij})^2. \tag{53}$$

Using (53) in (52), we get

$$2\tau^* = \frac{c}{4}(n(n-1) + 3\|C\|^2) - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}^{\alpha}_{1j})^2 + 2\sum_{2\leq i<j\leq n}R^*(X_i,X_j,X_j,X_i) - 2\sum_{2\leq i<j\leq n}R(X_i,X_j,X_j,X_i). \tag{54}$$

Besides, from (19) we obtain

$$\sum_{2\leq i<j\leq n} R(X_i, X_j, X_j, X_i) = \frac{c}{4}(\frac{(n-2(n-1)}{4} + 3\sum_{2\leq i<j\leq n} g^2(CX_i, X_j)). \tag{55}$$

Then from (54) and (55)

$$2Ric^*(X_1) = \frac{c}{2}(n-1+3\,\|CX_1\|^2) - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}^{\alpha}_{1j})^2, \tag{56}$$

which gives (50). □

Now, we compute the Chen-Ricci inequality between the vertical and horizontal distributions for the case of ξ is vertical. For the scalar curvature τ of $M(c)$, we derive

$$2\tau = \sum_{s=1}^{n} Ric(X_s, X_s) + \sum_{k=1}^{r} Ric(U_k, U_k), \tag{57}$$

$$\begin{aligned} 2\tau = &\sum_{j,k=1}^{r} R(U_j, U_k, U_k, U_j) + \sum_{i=1}^{n}\sum_{k=1}^{r} R(X_i, U_k, U_k, X_i) \\ &+ \sum_{i,s=1}^{n} R(X_i, X_s, X_s, X_i) + \sum_{s=1}^{n}\sum_{j=1}^{r} R(U_j, X_s, X_s, U_j). \end{aligned} \tag{58}$$

Let denote

$$\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g_M(\mathcal{T}_{U_k}X_i, \mathcal{T}_{U_k}X_i), \tag{59}$$

$$\left\|\mathcal{T}^{\mathcal{H}}\right\|^2 = \sum_{j,k=1}^{r} g_M(\mathcal{T}_{U_j}U_k, \mathcal{T}_{U_j}U_k), \tag{60}$$

$$\left\|\mathcal{A}^{\mathcal{V}}\right\|^2 = \sum_{i,j=1}^{n} g_M(\mathcal{A}_{X_i}X_j, \mathcal{A}_{X_i}X_j), \tag{61}$$

$$\left\|\mathcal{A}^{\mathcal{H}}\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g_M(\mathcal{A}_{X_i}U_k, \mathcal{A}_{X_i}U_k). \tag{62}$$

Theorem 3.6. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\begin{aligned} &\frac{c}{4}(nr + r + 3(\|\mathcal{B}\|^2 + \|CX_1\|^2 - \sin^2\theta) - (r - 2 + 3\cos^2\theta)\eta(U_1)^2) \\ &\leq \widehat{Ric}(U_1) + Ric^*(X_1) + \frac{1}{4}r^2\,\|H\|^2 + 3\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}^{\alpha}_{1s})^2 - \delta(N) + \left\|\mathcal{T}^{\mathcal{V}}\right\|^2 - \left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \end{aligned} \tag{63}$$

The equality case of (63) holds if and only if

$$\mathcal{T}^s_{11} = \mathcal{T}^s_{22} + ... + \mathcal{T}^s_{rr},$$
$$\mathcal{T}^s_{1j} = 0, \quad j = 2, ..., r.$$

Proof. Since $M(c)$ is a cosymplectic space form, using (58) we derive

$$2\tau = \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\sum_{i=1}^{n}\sum_{k=1}^{r} g^2(\mathcal{B}X_i, U_k) + \|C\|^2)). \tag{64}$$

Now, we define

$$\|\mathcal{B}\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g^2(\mathcal{B}X_i, U_k). \tag{65}$$

On the other hand, using the Gauss-Codazzi type equations (9), (10) and (11), we obtain

$$\begin{aligned}
2\tau = {} & 2\hat{\tau} + 2\tau^* + r^2\|H\|^2 - \sum_{k,j=1}^{r} g_M(\mathcal{T}_{U_k}U_j, \mathcal{T}_{U_k}U_j) + 3\sum_{i,s=1}^{n} g_M(\mathcal{A}_{X_i}X_s, \mathcal{A}_{X_i}X_s) \\
& - \sum_{i=1}^{n}\sum_{k=1}^{r} g_M((\nabla_{X_i}\mathcal{T})_{U_k}U_k, X_i) + \sum_{i=1}^{n}\sum_{k=1}^{r}(g_M(\mathcal{T}_{U_k}X_i, \mathcal{T}_{U_k}X_i) \\
& - g_M(\mathcal{A}_{X_i}U_k, \mathcal{A}_{X_i}U_k)) - \sum_{s=1}^{n}\sum_{j=1}^{r} g_M((\nabla_{X_s}\mathcal{T})_{U_j}U_j, X_s) \\
& + \sum_{s=1}^{n}\sum_{j=1}^{r}(g_M(\mathcal{T}_{U_j}Xs, \mathcal{T}_{U_j}Xs) - g_M(\mathcal{A}_{X_s}U_j, \mathcal{A}_{X_s}U_j)).
\end{aligned} \tag{66}$$

Then using (40) and (66), we obtain

$$\begin{aligned}
2\tau = {} & 2\hat{\tau} + 2\tau^* + \frac{1}{2}r^2\|H\|^2 - \frac{1}{2}(\mathcal{T}_{11}^s - \mathcal{T}_{22}^s - \dots - \mathcal{T}_{rr}^s)^2 - 2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^s)^2 \\
& + 2\sum_{s=1}^{n}\sum_{2\le i<j\le r}(\mathcal{T}_{ii}^s\mathcal{T}_{jj}^s - (\mathcal{T}_{ij}^s)^2) + 6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^\alpha)^2 + 6\sum_{\alpha=1}^{r}\sum_{2\le i<s\le n}(\mathcal{A}_{is}^\alpha)^2 \\
& + \sum_{i=1}^{n}\sum_{k=1}^{r}(g_M(\mathcal{T}_{U_k}X_i, \mathcal{T}_{U_k}X_i) - g_M(\mathcal{A}_{X_i}U_k, \mathcal{A}_{X_i}U_k)) - 2\delta(N) \\
& + \sum_{s=1}^{n}\sum_{j=1}^{r}(g_M(\mathcal{T}_{U_j}Xs, \mathcal{T}_{U_j}Xs) - g_M(\mathcal{A}_{X_s}U_j, \mathcal{A}_{X_s}U_j)).
\end{aligned} \tag{67}$$

Using (45), (53), (64) and (65) in (67), we get

$$\begin{aligned}
& \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) \\
& = 2\widehat{Ric}(U_1) + 2Ric^*(X_1) + \frac{1}{2}r^2\|H\|^2 - \frac{1}{2}(\mathcal{T}_{11}^s - \mathcal{T}_{22}^s - \dots - \mathcal{T}_{rr}^s)^2 \\
& - 2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^s)^2 + 6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^\alpha)^2 + \sum_{i=1}^{n}\sum_{k=1}^{r}(g_M(\mathcal{T}_{U_k}X_i, \mathcal{T}_{U_k}X_i) \\
& - g_M(\mathcal{A}_{X_i}U_k, \mathcal{A}_{X_i}U_k)) - 2\delta(N) + \sum_{s=1}^{n}\sum_{j=1}^{r}(g_M(\mathcal{T}_{U_j}Xs, \mathcal{T}_{U_j}Xs) \\
& - g_M(\mathcal{A}_{X_s}U_j, \mathcal{A}_{X_s}U_j)) + 2\sum_{2\le i<j\le r} R(U_i, U_j, U_j, U_i) + 2\sum_{2\le i<s\le n} R(X_i, X_s, X_s, X_i).
\end{aligned} \tag{68}$$

Using (49), (55), (59) and (62) in (68) we obtain

$$\frac{c}{4}(nr + r - 3 + 3(\|\mathcal{B}\|^2 + \|CX_1\|^2 + \cos^2\theta) - (r - 2 + 3\cos^2\theta)\eta(U_1)^2)$$
$$\leq \widehat{Ric}(U_1) + Ric^*(X_1) + \frac{1}{4}r^2\|H\|^2 + 3\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}^{\alpha}_{1s})^2 - \delta(N) + \left\|\mathcal{T}^{\mathcal{V}}\right\|^2 - \left\|\mathcal{A}^{\mathcal{H}}\right\|^2 \tag{69}$$

which gives (63). □

From (59)-(62), (64), (65) and (66) we obtain

$$\frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) = 2\hat{\tau} + 2\tau^*$$
$$+ r^2\|H\|^2 - \left\|\mathcal{T}^{\mathcal{H}}\right\|^2 + 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2 - 2\delta(N) + 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 - 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{70}$$

From (70) we obtain following theorem.

Theorem 3.7. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$2\hat{\tau} + 2\tau^* \leq \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2))$$
$$- r^2\|H\|^2 + \left\|\mathcal{T}^{\mathcal{H}}\right\|^2 + 2\delta(N) - 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 + 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2, \tag{71}$$

$$2\hat{\tau} + 2\tau^* \geq \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2))$$
$$- r^2\|H\|^2 + \left\|\mathcal{T}^{\mathcal{H}}\right\|^2 - 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2 + 2\delta(N) - 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2. \tag{72}$$

Equality cases of (71) and (72) hold for all $p \in M$ if and only if horizontal distribution $\mathcal{H}$ is integrable.

From Theorem 3.7, we have the following corollary.

Corollary 3.8. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N) such that each fibres be totally geodesic. Then we have*

$$2\hat{\tau} + 2\tau^* \leq \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) + 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2, \tag{73}$$

$$2\hat{\tau} + 2\tau^* \geq \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) - 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2. \tag{74}$$

Equality cases of (73) and (74) hold for all $p \in M$ if and only if horizontal distribution $\mathcal{H}$ is integrable.

From (70) we obtain following theorem.

Theorem 3.9. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$2\hat{\tau} + 2\tau^* \geq \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2))$$
$$- r^2\|H\|^2 + 2\delta(N) - 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 + 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2 - 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2, \tag{75}$$

$$2\hat{\tau} + 2\tau^* \leq \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2))$$
$$- r^2\|H\|^2 + \left\|\mathcal{T}^{\mathcal{H}}\right\|^2 + 2\delta(N) + 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2 - 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2. \tag{76}$$

Equality cases of (75) and (76) hold for all $p \in M$ if and only if the fibre through p of π is a totally geodesic submanifold of M.

From Theorem 3.9, we have the following corollary.

Corollary 3.10. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N) such that $\mathcal{H}$ be integrable. Then we have*

$$2\hat{\tau} + 2\tau^* \geq \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) - r^2\|H\|^2 + 2\delta(N) - 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2, \tag{77}$$

$$2\hat{\tau} + 2\tau^* \leq \frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) - r^2\|H\|^2 + 2\delta(N) + \left\|\mathcal{T}^{\mathcal{H}}\right\|^2. \tag{78}$$

Equality cases of (77) and (78) hold for all $p \in M$ if and only if the fibre through p of π is a totally geodesic submanifold of M.

Lemma 3.11. *Let a and b be non-negative real numbers, then*

$$\frac{a+b}{2} \geq \sqrt{ab}$$

with equality iff $a = b$.

Using Lemma 3.11 in (70), we obtain following theorems.

Theorem 3.12. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) \leq 2\hat{\tau} + 2\tau^* + r^2\|H\|^2 + 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 + 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2 - 2\delta(N) - 2\sqrt{2}\left\|\mathcal{A}^{\mathcal{H}}\right\|\left\|\mathcal{T}^{\mathcal{H}}\right\|. \tag{79}$$

Equality cases of (79) hold for all $p \in M$ if and only if $\left\|\mathcal{A}^{\mathcal{H}}\right\| = \left\|\mathcal{T}^{\mathcal{H}}\right\|$.

Theorem 3.13. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) \geq 2\hat{\tau} + 2\tau^* + r^2\|H\|^2 - \left\|\mathcal{T}^{\mathcal{H}}\right\|^2 - 2\delta(N) - 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2 + 2\sqrt{6}\left\|\mathcal{A}^{\mathcal{V}}\right\|\left\|\mathcal{T}^{\mathcal{V}}\right\|. \tag{80}$$

Equality cases of (80) hold for all $p \in M$ if and only if $\left\|\mathcal{A}^{\mathcal{V}}\right\| = \left\|\mathcal{T}^{\mathcal{V}}\right\|$.

Lemma 3.14. *[41] Let $a_1, a_2, \ldots, a_n$ be $n-$real numbers $(n > 1)$, then*

$$\frac{1}{n}\Big(\sum_{i=1}^{n} a_i\Big)^2 \leq \sum_{i=1}^{n} a_i^2$$

with equality iff $a_1 = a_2 = \ldots = a_n$.

Theorem 3.15. *Let $\pi : M(c) \to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c), g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\frac{c}{4}((n+r)(n+r-3) + 2 + 3((r-1)\cos^2\theta + 2\|\mathcal{B}\|^2 + \|C\|^2)) \leq 2\hat{\tau} + 2\tau^* + r(r-1)\|H\|^2 + 3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2 - 2\delta(N) + 2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 - 2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{81}$$

Equality case of (81) holds for all $p \in M$ if and only if we have the following statements:
i) π is a Riemannian submersion that has a totally umbilical fibres.
ii) $\mathcal{T}_{ij} = 0$, for $i \neq j \in \{1, 2, \ldots, r\}$.

Proof. From (70) we have

$$\frac{c}{4}((n+r)(n+r-3)+2+3((r-1)\cos^2\theta+2\|\mathcal{B}\|^2+\|C\|^2))$$
$$=2\hat{\tau}+2\tau^*+r^2\|H\|^2-\sum_{i=1}^{n}\sum_{j=1}^{r}(\mathcal{T}_{jj}^{s})^2-2\sum_{i=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^{s})^2+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(N)+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{82}$$

Using Lemma 3.14 in (82), we obtain

$$\frac{c}{4}((n+r)(n+r-3)+2+3((r-1)\cos^2\theta+2\|\mathcal{B}\|^2+\|C\|^2))$$
$$\leq 2\hat{\tau}+2\tau^*+r^2\|H\|^2-\frac{1}{r}\sum_{s=1}^{n}(\sum_{j=1}^{r}\mathcal{T}_{jj}^{s})^2-2\sum_{s=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^{s})^2+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(N)+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2, \tag{83}$$

which is equivalent to (81). Equality case of (81) holds for all $p\in M$ if and only if

$$\mathcal{T}_{11}=\mathcal{T}_{22}=...=\mathcal{T}_{rr} \text{ and } \sum_{s=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^{s})^2=0 \tag{84}$$

which completes proof of the theorem. □

Using by similar proof way of Theorem 3.15, we have the following theorem:

Theorem 3.16. *Let $\pi: M(c)\to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c),g_M)$ onto a Riemannian manifold (N,g_N). Then we have*

$$\frac{c}{4}((n+r)(n+r-3)+2+3((r-1)\cos^2\theta+2\|\mathcal{B}\|^2+\|C\|^2))\geq 2\hat{\tau}+2\tau^*$$
$$+r^2\|H\|^2-\left\|\mathcal{T}^{\mathcal{H}}\right\|^2+\frac{3}{n}tr(\mathcal{A}^{\mathcal{V}})^2-2\delta(N)+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{85}$$

Equality case of (85) holds for all $p\in M$ if and only if $\mathcal{A}_{11}=\mathcal{A}_{22}=...=\mathcal{A}_{nn}$ and $\mathcal{A}_{ij}=0$, for $i\neq j\in\{1,2,...,n\}$.

From Theorem 3.16, we have the following corollary.

Corollary 3.17. *Let $\pi: M(c)\to N$ be a slant Riemannian submersion from a cosymplectic space form $(M(c),g_M)$ onto a Riemannian manifold (N,g_N) such that each fiber is totally geodesic. Then we have*

$$\frac{c}{4}((n+r)(n+r-3)+2+3((r-1)\cos^2\theta+2\|\mathcal{B}\|^2+\|C\|^2))$$
$$\geq 2\hat{\tau}+2\tau^*+\frac{3}{n}tr(\mathcal{A}^{\mathcal{V}})^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{86}$$

Equality case of (86) holds for all $p\in M$ if and only if $\mathcal{A}_{11}=\mathcal{A}_{22}=...=\mathcal{A}_{nn}$ and $\mathcal{A}_{ij}=0$, for $i\neq j\in\{1,2,...,n\}$.

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