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A finite graph is the Reeb graph of a Morse circle-valued function

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Abstract. We show that any non-trivial finite connected graph (allowing loop edges and multiple edges) is isomorphic to the Reeb graph of a Morse circle-valued function on a closed n-manifold of a given dimension $n \geq 2$; this manifold roughly resembles a thick version of the graph, we present its construction and study its properties. In the case of surfaces ($n = 2$), we prove a criterion for when a finite graph can be realized as the Reeb graph of such a function on a given surface.

1. Introduction

The *Reeb graph* of a smooth circle-valued function $f : M \to S^1$ is defined similarly to the Reeb graph of a real-valued function: It is a topological space obtained by contracting the connected components of the level sets of f (called *contours*) to points, endowed with the quotient topology. This quotient space has marked points called *vertices*: they are images of the *critical contours* (containing critical points of f).

The Reeb graph of a function shows the evolution of its level sets, providing information on the function behavior. Therefore, Reeb graphs of circle-valued functions are used for the topological classification of these functions and in the study of their topological properties [1, 5, 16, 17, 19, 20]. In the theory of dynamical systems, Lyapunov functions are especially important, and their Reeb graphs are known under the name of *Lyapunov graphs*; the case of circle-valued Lyapunov functions is considered in [4, 18].

By definition, the Reeb graph is a quotient space, but in some cases this space has the structure of a finite graph: for example, this is true for simple Morse circle-valued functions, see [2, Proposition 2.1] and [4, Proposition 3.1]. Moreover, any smooth circle-valued function with a finite non-zero number of critical values on a closed manifold defines the Reeb graph that is a finite graph [7, Theorem 5.7]. An example of a smooth circle-valued function whose Reeb graph is not a finite graph, and is not even a Hausdorff space can be seen in [7, Example 5.1]. In this paper, we study the inverse problem:

Realization problem: When a given finite graph is the Reeb graph of some circle-valued function?

This problem can be considered in a specific class of functions, or up to isomorphism or homeomorphism of graphs viewed as 1-dimensional CW-complexes; it also can be considered on a given manifold.

In the class of real-valued functions, the realization problem has been extensively studied by many authors, e.g., [11–13, 21–24, 26–28]. Masumoto and Saeki [22, Theorem 2.1] showed that any finite graph without loop edges can be realized as the Reeb graph of a smooth function with a finite number of critical values. If the realization problem is considered in the class of Morse functions, this imposes additional

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conditions on the graph structure: not every graph without loop edges can be the Reeb graph of a Morse function; see [12, 28]. If the realization problem is considered up to homeomorphism, then there are no any restrictions on the graph structure: Any finite graph (even with loop edges) is homeomorphic to the Reeb graph of a Morse–Bott function [11, Theorem 3.4].

For circle-valued functions, there are few results on this topic; they concern the realization problem only in the class of simple Morse and simple Morse–Bott functions; see [1, 4].

In this paper, we show that in contrast to the case of real-valued functions, any non-trivial finite connected graph (allowing loop edges and multiple edges) is isomorphic to the Reeb graph of a Morse circle-valued function on a closed n-manifold, for any given $n \geq 2$ (Theorem 5.5). This manifold roughly resembles a thick version of the graph, we present its construction (Lemmas 4.1 and 5.1) and study its properties (Propositions 4.2 and 5.2).

We also briefly consider the realization problem in the classes of Morse–Bott and round functions (Theorem 6.1, Proposition 6.2), as well as on a given manifold (Theorem 7.1, Conjecture 7.2).

In 2-dimensional case, we prove a criterion for a finite graph to be realizable as the Reeb graph of a Morse circle-valued function on a given surface (Theorem 8.3).

The paper is organized as follows. In Section 2, we give necessary definitions and known facts. In Section 3, we study properties of atoms, which are the building blocks for constructing manifolds. In Section 4, we construct a manifold M_G resembling a given graph G and study its properties. In Section 5, we show that any non-trivial finite graph is the Reeb graph of a Morse circle-valued function. In Section 6, we consider the realization problem in the classes of Morse–Bott and round circle-valued functions. In Section 7, some facts on the realization problem on a given manifold are presented. Finally, in Section 8, we prove a criterion on realization of a finite graph as the Reeb graph of a Morse circle-valued function on a given surface.

2. Definitions and useful facts

In this section, we introduce, for future reference, some necessary notions and facts.

2.1. *Graphs*

We consider finite connected graphs that allow multiple edges (also called *parallel edges* or a *multi-edge*) and loop edges (also called a *self-loops* or *buckles*); a *loop edge* is an edge incident to only one vertex. Any graph has a geometric realization as a 1-dimensional CW complex, in which 0-cells correspond to vertices, and 1-cells correspond to edges. A graph is called *trivial* if it has only one vertex and no edges; note that a graph must have at least one vertex.

A complete graph on 2 vertices is denoted by K_2; it has one edge and geometrically represents a 1-simplex or a closed interval.

The *cycle rank* $b_1(G)$ of a graph G is the first Betti number of the graph considered as a one-dimensional CW complex; in computational geometry this value is called the *number of loops*.

A vertex v is called a *cut vertex* if the number of connected components of $G \setminus v$ is greater than that of G. A graph is *biconnected* if it is connected and has no cut vertices. A *block* of a graph is its maximal biconnected subgraph. A *leaf block* is a block containing exactly one cut vertex.

An *orientation* of a graph G is an assignment of a direction to each of its edges; a graph endowed with an orientation is called a *directed graph* or *digraph*. The *indegree* $\deg_{in} v$ of a vertex v in a digraph is the number of edges incoming to v and the *outdegree* $\deg_{out} v$ of a vertex v is the number of edges outgoing from v; obviously, $\deg v = \deg_{in} v + \deg_{out} v$. A vertex v is a *source* if $\deg_{in} v = 0$, and a *sink* if $\deg_{out} v = 0$; so a vertex with $\deg v = 1$, is either a source or a sink.

Lemma 2.1. *([11, Proposition 3.1]) Any finite non-trivial graph G admits an orientation with all its sources and sinks having degree 1.*

2.2. Reeb graph

The *Reeb graph* R_f of a smooth circle-valued function $f : M \to S^1$ on a manifold M is the quotient space $M/\sim$ endowed with the quotient topology, where the equivalence relation $x \sim y$ holds whenever x and y belong to the same *contour* (connected component of a level set) of f; the image of a *critical contour* (containing a critical point) is called a *vertex*.

Thus the Reeb graph of a smooth circle-valued function is the quotient space with marked points.

Definition 2.2. *A Reeb graph R_f* has the structure of a finite graph G, *or R_f is isomorphic to G, or R_f is G, if there exists a homeomorphism $R_f \to G$ mapping one-to-one the vertices of R_f to the vertices of G. The notation is $R_f = G$.*

Generally this quotient space is not a graph; see [7, Example 5.1]. But for a smooth circle-valued function on a closed manifold with a finite non-zero number of critical values, the Reeb graph has the structure of a finite graph [7, Theorem 5.7]. In particular, Morse circle-valued functions on a closed manifold define Reeb graphs that are finite graphs.

2.3. Morse form foliations

A *Morse form* ω is a closed one-form that is locally the differential of a Morse function. On a closed manifold M, the set of its singularities $\operatorname{Sing}\omega$ is finite. On $M \setminus \operatorname{Sing}\omega$, the form ω defines a foliation that can be extended to a singular foliation $\mathcal{F}_\omega$ on the whole M; singular leaves of $\mathcal{F}_\omega$ contain singularities of ω, while non-singular leaves do not.

The union of all compact non-singular leaves of $\mathcal{F}_\omega$ is open, and has a finite number of connected components C_i; each C_i is a cylinder over a compact leaf $L_i \subset C_i$, i.e., $C_i \cong L_i \times (0, 1)$, where the diffeomorphism maps $L_i \times t$ to leaves of $\mathcal{F}_\omega$. Denote $\Delta = M \setminus \bigcup_i C_i$, then $\partial C_i \subset \Delta$, and each C_i adjoins one or two connected components Δ_j of the set Δ. We obtain a decomposition of the manifold:

$$M = \bigcup_i C_i \cup \bigcup_j \Delta_j.$$

This decomposition allows to represent the manifold as a finite graph (admitting multiple edges and loop edges), called the *foliation graph*; for details of this construction, see [8]. We denote the foliation graph by $\Gamma_{\mathcal{F}_\omega}$; its edges C_i consist of compact non-singular leaves, its vertices are sets Δ_j.

A foliation is *compact*, if all its leaves, both singular and non-singular, are compact. For a compact foliation, the sets Δ_j are exactly its singular leaves, i.e., the vertices of its foliation graph are compact singular leaves.

Recall that the *form's rank*, $\operatorname{rk}\omega$, is the rank of its group of periods over $\mathbb{Q}$:

$$\operatorname{rk}\omega = \operatorname{rk}_{\mathbb{Q}}\{\int_{z_1}\omega, \dots, \int_{z_k}\omega\},$$

where $z_1, \dots, z_k$ is a basis of $H_1(M)$. Note that $\operatorname{rk}\omega = 0$ only for exact forms, $\omega = df$. If $\operatorname{rk}\omega = 1$, the form ω is called *rational*.

Proposition 2.3. *([9, Corollary 5.1]) Let ω be a Morse form on a closed orientable manifold. The singular foliation $\mathcal{F}_\omega$ is compact if and only if there exists a Morse form ω' with $\operatorname{rk}\omega' \le 1$ defining the same foliation, $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$.*

3. Building blocks: atoms

In this section, we generalize and adapt for the realization problem the concept of an *atom* introduced for Morse functions on surfaces in [3, Definition 2.4].

3.1. The concept of an atom

Definition 3.1. *An* atom *is a pair* (A, f), *where*

(i) *A is a compact connected manifold with boundary;*
(ii) *$f : A \to \mathbb{R}$ is a smooth function that is constant and regular on each boundary component, and has exactly one critical contour $C \subset \operatorname{Int} A$.*

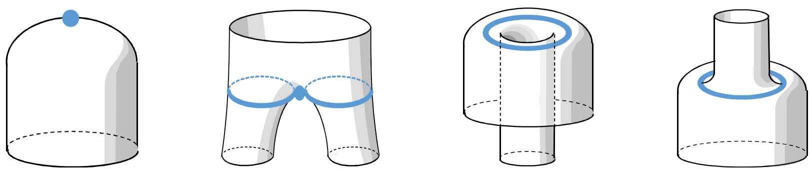
Remark 3.2. *Note that for an atom (A, f) with $\partial A = \emptyset$, it holds $f|_A \equiv const$.*

When this does not cause ambiguity, we will refer to the manifold A as an *atom*. Note that A is an f-saturated neighborhood of the critical contour C. There are simple properties:

Lemma 3.3. *Let (A, f) be an atom. Then*

(i) *If $\partial A \neq \emptyset$, then f is a height function.*
(ii) *A critical level of f is connected.*

Depending on the class of the function f, an atom (A, f) can be Morse, Morse–Bott, round, etc; see Figure 1. Recall that a *round function* is a smooth function whose critical set consists of circles.



(*a*) (*b*) (*c*) (*d*)

Figure 1: Different types of atoms: (*a*), (*b*) atoms corresponding to a Morse function; (*c*) an atom corresponding to a Morse–Bott or a round function; (*d*) an atom corresponding to a round function. Critical points and critical levels are shown in blue.

The Reeb graph R_f of the function $f : A \to \mathbb{R}$ has exactly one vertex v corresponding to the critical contour $C \subset \operatorname{Int} A$; this quotient space has the structure of a star with v being its center.

3.2. Atoms corresponding to vertices of a digraph

Consider an atom (A, f) with $C \subset A$ being a critical contour of $f : A \to \mathbb{R}$. Denote by $\partial_+ A$ the set of boundary components $\partial_i \subset \partial A$ such that $f|_{\partial_i} > f|_C$, and by $\partial_- A$ the set of boundary components ∂_i such that $f|_{\partial_i} < f|_C$; we have $\partial A = \partial_- A \cup \partial_+ A$. If the function f has a maximum (minimum) at C, then $\partial_+ A = \emptyset$ $(\partial_- A = \emptyset)$.

Definition 3.4. *An atom (A, f)* corresponds to a vertex *v in a digraph $\vec{G}$ if*

(i) $\deg_{in} v = |\partial_- A|$
(ii) $\deg_{out} v = |\partial_+ A|$

We denote such an atom by (A_v, f_v).

Remark 3.5. *If v is an isolated vertex, then the atom (A_v, f_v) consists of a closed manifold A_v, $\partial A_v = \emptyset$, and a constant function $f_v|_{A_v} \equiv const$; see Remark 3.2.*

The vertex $v \in G$ can correspond to different manifolds A_v; in particular, the type of the manifold A_v depends both on the class of the function f_v and on the orientation of the edges incident to the vertex v; see Figure 2.

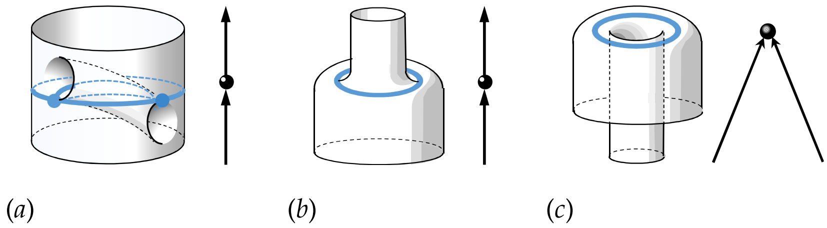


Figure 2: 2-vertices and different types of corresponding atoms: (*a*), (*b*) while the vertex v has $\deg_{in} v = \deg_{out} v = 1$, the atom shown in (*a*) corresponds to a Morse function, it is a torus with boundary; the atom shown in (*b*) corresponds to a round function, it is a sphere with boundary. (*c*): here $\deg_{in} v = 2$ and $\deg_{out} v = 0$, the vertex v is a sink; the atom corresponds to a Morse–Bott or a round function, it is a sphere with boundary. Critical points and critical levels are shown in blue.

3.3. *Morse atoms*

If $\dim A_v = 2$, then the connected components of ∂A_v are circles S^1; if $\dim A_v \geq 3$, then the boundary components can be different submanifolds. Since we use atoms to construct a closed manifold by gluing them together, we need all the boundary components of the atoms to be diffeomorphic to each other.

We are especially interested in the case when all the boundary components are spheres S^{n-1}. Reformulating [23, Lemma 4.1] in terms of digraphs and atoms, we obtain the following result:

Lemma 3.6. *Let $\vec{G}$ be a finite non-trivial digraph, and v be its vertex that is not a source or a sink. Then for any $n \geq 2$, there exists an n-dimensional Morse atom (A_v, f_v) corresponding to the vertex v such that all the boundary components of A_v are spheres S^{n-1}.*

The following proposition shows how to construct such an atom: we can take almost any closed manifold and cut out some balls from it to obtain a manifold with boundary. Indeed, any compact manifold with boundary other than a cylinder can be considered as a Morse atom:

Proposition 3.7. *Let $\vec{G}$ be a finite non-trivial digraph, v be its vertex that is not a source or a sink, and A be a compact connected manifold with boundary such that A is not a cylinder and the number of its boundary components $|\partial A| = \deg v$. Then there exists a Morse function $f : A \to \mathbb{R}$ such that the pair (A, f) is an atom corresponding to the vertex v.*

If $\deg v = 1$, then $A \cong D^n$, and f is a Morse function chosen as the height function of an embedding $D^n \hookrightarrow \mathbb{R}^{n+1}$.

Proof. Let $\partial A = \partial_- A \cup \partial_+ A$, where $|\partial_- A| = \deg_{in} v$ and $|\partial_+ A| = \deg_{out} v$. Since v is not a source or a sink, then $\partial_- A \neq \emptyset$ and $\partial_+ A \neq \emptyset$. Consider some $a, b \in \mathbb{R}$ such that $a < b$. By [25, Theorem 2.5], there exists a Morse function $g : A \to \mathbb{R}$ such that $g^{-1}(a) = \partial_- A$, and $g^{-1}(b) = \partial_+ A$, and $\mathrm{Crit}(g) \subset \mathrm{Int}\, A$. Since A is not a cylinder, by [14, 2.2. Theorem, p.153], we have $\mathrm{Crit}(g) \neq \emptyset$. Then we move all critical points of g to one critical level using the rearrangement method from [25, Theorem 4.1], and obtain a Morse function $f : A \to \mathbb{R}$ with one critical level. By construction $f^{-1}(a) = \partial_- A$, and $f^{-1}(b) = \partial_+ A$, so the pair (A, f) is an atom corresponding to v.

Suppose $\deg v = 1$, and (A, f) is the corresponding Morse atom. Then the critical contour $C \subset A$ is a maximum or a minimum of f and consists of one point with A being its f-saturated neighborhood. Thus $A \cong D^n$ and f can be chosen as the height function of an embedding $D^n \hookrightarrow \mathbb{R}^{n+1}$. □

In particular, Proposition 3.7 allows one to choose as atoms A_v the manifolds with a very simple homological structure:

Corollary 3.8. *For a vertex v, n-dimensional Morse atoms are:*

$$\textit{orientable } A_v = \begin{cases} D^n, & \text{if } \deg v = 1, \\ S^{n-1} \times S^1 \setminus D_1^n \cup D_2^n, & \text{if } \deg v = 2, \\ S^n \setminus \cup_{i=1}^{\deg v} D_i^n, & \text{if } \deg v \geq 3. \end{cases}$$

$$\textit{non-orientable } A_v = \begin{cases} \mathbb{R}P^n \setminus \cup_{i=1}^{\deg v} D_i^n, & \textit{if } n = 2k, \quad k \geq 1 \\ \mathbb{R}P^{n-1} \times S^1 \setminus \cup_{i=1}^{\deg v} D_i^n, & \textit{if } n = 2k+1, k \geq 1. \end{cases}$$

For $n = 2$, the orientable surfaces from Corollary 3.8 are shown in Figure 5, the non-orientable surface is shown in Figure 3. The corresponding functions f_v are also described there.

3.4. Morse atoms, $\dim A_v = 2$

In this section, we consider in detail the properties of 2-dimensional Morse atoms, i.e., we assume that A_v is a surface, and f_v is a Morse function. If $\deg v = 1$, then the surface is orientable, namely, it is a disk, see Figure 1 (a).

Proposition 3.9. *Consider an atom* (A_v, f_v)*, where* A_v *is a surface and* f_v *is a Morse function. If* $\deg v = 1$*, then the surface is orientable and its genus* $g(A_v) = 0$*; otherwise for an orientable surface*

$$2g(A_v) = 2 - \deg v + |\operatorname{Crit}(f_v)|,$$

for a non-orientable surface

$$g(A_v) = 2 - \deg v + |\operatorname{Crit}(f_v)|.$$

Proof. If $\deg v = 1$, a Morse function defines an atom shown in Figure 5(a); it is a disk, so $g(A_v) = 0$.

If $\deg v > 1$, let us glue up each boundary component of A_v with a disk, denote this closed surface by $\hat{A}_v$; obviously, $g(\hat{A}_v) = g(A_v)$. Then extend f_v to these disks to have one center singularity in each of them, denote this extension by $\hat{f}_v$, this function has $\deg v$ additional centers. We have

$$\chi(\hat{A}_v) = \mu_0(\hat{f}_v) - \mu_1(\hat{f}_v) + \mu_2(\hat{f}_v) = b_0(\hat{A}_v) - b_1(\hat{A}_v) + b_2(\hat{A}_v),$$

where $b_i(\hat{A}_v)$ are the Betti numbers, $\mu_i(\hat{f}_v)$ is the number of critical points of index i. By construction, the number of centers $\mu_0(\hat{f}_v) + \mu_2(\hat{f}_v) = \deg v$, and $\mu_1(\hat{f}_v) = |\operatorname{Crit}(f_v)|$. Thus, in the orientable case,

$$\deg v - |\operatorname{Crit}(f_v)| = 2 - 2g(A_v),$$

in the non-orientable case,

$$\deg v - |\operatorname{Crit}(f_v)| = 2 - g(A_v).$$

□

While $g(A_v) \geq 1$ always holds for a non-orientable surface, an orientable atom can be a sphere with boundary, and in this case $g(A_v) = 0$. However, orientable Morse atoms corresponding to vertices of degree 2 have non-zero genus:

Corollary 3.10. *Consider an atom* (A_v, f_v)*, where* A_v *is an orientable surface and* f_v *is a Morse function. If* $\deg v = 2$*, then the surface genus* $g(A_v) \geq 1$*, otherwise* $g(A_v) \geq 0$*.*

Indeed, for $\deg v = 2$, Proposition 3.9 implies $2g(A_v) = |\operatorname{Crit}(f_v)| \geq 1$. Thus $g(A_v) \geq 1$.

Proposition 3.9 admits to bound from below the number of critical points on a critical contour of an atom:

Corollary 3.11. *Consider an atom* (A_v, f_v)*, where* A_v *is a surface and* f_v *is a Morse function. If* A_v *is orientable, then the number of critical points on the critical contour* $C_v \subset \operatorname{Int} A_v$*:*

$$|C_v \cap \operatorname{Crit}(f_v)| \textit{ is } \begin{cases} 1, & \textit{if } \deg v = 1, \\ \geq 2, & \textit{if } \deg v = 2, \\ \geq \deg v - 2, & \textit{otherwise}. \end{cases}$$

If A_v *is non-orientable, then* $|C_v \cap \operatorname{Crit}(f_v)| \geq \deg v - 1$*.*

Below we give examples of non-orientable atoms (A_v, f_v). By Corollary 3.8, the simplest non-orientable atoms are the following

$$A_v = \mathbb{R}P^2 \setminus \cup_{i=1}^{\deg v} D_i^2.$$

Example 3.12. *Let* $\deg v = 2$*, and* $A_v = \mathbb{R}P^2 \setminus (D_1^2 \cup D_2^2)$*. We consider* $\mathbb{R}P^2$ *as Boy's surface—an immersion of* $\mathbb{R}P^2$ *in* $\mathbb{R}^3$*; the corresponding height function is Morse and has exactly one minimum, one maximum and one saddle-type critical point, see Figure 3(a). To obtain* A_v*, we remove small spherical neighborhoods of the extrema, see Figure 3(b); then* f_v *is the restriction of the height function to* A_v*, in particular,* $|\mathrm{Crit}(f_v) = 1|$*, and the surface genus* $g(A_v) = 1$*, cf. Proposition 3.9.*

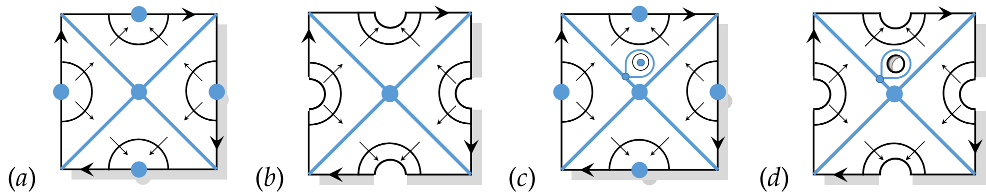


Figure 3: The surfaces—the projective plane $\mathbb{R}P^2$ and the projective plane with boundary—are shown as a square with the sides identified according to the arrows; thin arrows indicate the gradient direction; critical points and a critical contour are shown in blue. (*a*) The projective plane $\mathbb{R}P^2$ with a Morse function having two extrema and one critical point of index 1. (*b*) The atom $A_v = \mathbb{R}P^2 \setminus (D_1^2 \cup D_2^2)$, so $\deg v = 2$. (*c*) The projective plane $\mathbb{R}P^2$ with a Morse function having three extrema and two critical points of index 1; in the same way, we can construct a function with $\deg v \geq 3$ extrema and $(\deg v - 1)$ critical points of index 1 lying on the same critical contour. (*d*) The atom $A_v = \mathbb{R}P^2 \setminus (D_1^2 \cup D_2^2 \cup D_3^2)$, so $\deg v = 3$.

Example 3.13. $\deg v \geq 3$ *and* $A_v = \mathbb{R}P^2 \setminus \cup_{i=1}^{\deg v} D_i^2$*. As above, we consider* $\mathbb{R}P^2$ *as Boy's surface and slightly transform the height function to obtain exactly* $\deg v$ *extrema, see Figure 3(c). Then similarly to Example 3.12, we construct* A_v *removing small spherical neighborhoods of the extrema, see Figure 3(d), and denote by* f_v *the restriction to* A_v *of the transformed height function.*

4. Construction of a manifold resembling a thick graph

In this section we construct a closed manifold M_G resembling a given graph G. For this construction, we use atoms (A_v, f_v) corresponding to vertices of the graph and tubes τ_e corresponding to its edges. We suppose that all atoms and tubes have the same dimension, and all the boundary components of A_v and τ_e are diffeomorphic to a closed manifold L of codimension one; in particular, $\tau_e \cong e \times L$. If atoms and tubes are two-dimensional, then $L = S^1$.

The following simple lemma clarifies the construction of the manifold M_G corresponding to a finite graph G.

Lemma 4.1. *Let G be a finite connected graph, $G = (V, E)$, and $n \geq 2$, $n \in \mathbb{Z}$. Then there exists a closed n-dimensional manifold*

$$M_G = \Big(\bigcup_{v\in V} A_v\Big) \cup \Big(\bigcup_{e\in E} \tau_e\Big), \tag{1}$$

where A_v are n-dimensional atoms corresponding to vertices $v \in V$, and $\tau_e \cong e \times L$ are tubes corresponding to edges $e \in E$ with L being a closed codimension one manifold. The atoms A_v are interconnected by the tubes τ_e according to the structure of the graph G, namely,

$$A_v \cap \tau_e = \begin{cases} \text{one or two } L, & \text{if } e \text{ is incident to } v, \\ \emptyset, & \text{otherwise} \end{cases} \tag{2}$$

(two L above correspond to the case of a loop edge).

In the case $n = 2$, we have $L = S^1$.

Proof. We represent each vertex v of the graph by an atom A_v; recall that the number of boundary components $|\partial A_v| = \deg v$. Then we connect the atoms A_v by tubes $\tau_e \cong e \times L$ along the edges e of the graph; it is possible, since all boundary components of the atoms A_v are diffeomorphic to L. So we obtain a closed manifold M_G. □

The manifold M_G resembles a thick version of the graph G, so we call it a *thick graph*; see Figure 4. Note that G can have different manifolds as a thick graph; it depends on the choice of the atoms (A_v, f_v), e.g., see Figure 2.

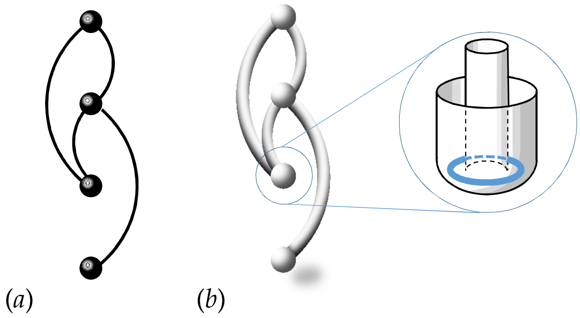


Figure 4: Construction of a manifold M_G roughly resembling a thick version of a graph G: (*a*) a finite graph G; (*b*) a surface M_G constructed from atoms corresponding to vertices of G and tubes corresponding to edges of G. Atoms are shown stylized as small spheres; one of them is shown enlarged. The figure is adapted from [13].

For each $A_v \subset M_G$, there is a local function $f_v : A_v \to \mathbb{R}$. By definition, a tube $\tau_e \cong e \times L$, so for each $\tau_e \subset M_G$, we define a function $f_e : \tau_e \to \mathbb{R}$ such that $f_e(x, y) = x$; it has no critical points. If the graph is directed, the direction of the function growth coincides with the direction of the edge e.

The local functions $f_v : A_v \to \mathbb{R}$ and $f_e : \tau_e \to \mathbb{R}$ define a singular foliation on the thick graph M_G; this foliation has important properties:

Proposition 4.2. *Let G be a finite graph, $G = (V, E)$, and M_G be a closed manifold that is a thick graph of G. Then the local functions $\{f_v | v \in V\}$ and $\{f_e | e \in E\}$ define a compact singular foliation $\mathcal{F}$ on M_G such that*

(i) *this foliation can be defined by a closed 1-form ω, i.e., $\mathcal{F} = \mathcal{F}_\omega$, and* $\operatorname{Sing}\omega = \cup_{v\in V} \operatorname{Crit}(f_v)$;
(ii) *the foliation graph $\Gamma_{\mathcal{F}}$ is isomorphic to G, i.e., $\Gamma_{\mathcal{F}} = G$.*

Proof. Suppose that M_G has the form of (1), and all boundary components of A_v and τ_e are diffeomorphic to a closed manifold L of codimension one. Then all atoms A_v are foliated by contours of the functions f_v, these foliations are singular; all tubes τ_e are foliated by contours of the functions f_e, these foliations are regular. If $A_v \cap \tau_e \neq \emptyset$, the boundary components are leaves of both foliations. We obtain a singular foliation $\mathcal{F}$ of the manifold M_G; all leaves of this foliation are compact.

Now consider open sets $U_v \supset A_v$ and $U_e \supset \tau_e$ such that $U_v \cap U_{v'} = \emptyset$ for $v \neq v'$, and $U_e \cap U_{e'} = \emptyset$ for $e \neq e'$. In addition, $U_v \cap U_e \neq \emptyset$ if and only if v in incident to e. We will extend the functions f_v and f_e to open sets U_v and U_e so that their contours coincide at the intersection, i.e., $df_v = df_e$ on $U_v \cap U_e \neq \emptyset$. Thus we obtain an open cover $\{ U_v, U_e \}$ of M_G with local exact 1-forms df_v and df_e. On the whole M_G, these 1-forms define a closed 1-form ω whose singularities are critical points of the functions f_v. This form ω defines a singular foliation $\mathcal{F}_\omega$; by construction, $\mathcal{F}_\omega$ coincides with $\mathcal{F}$.

By definition, vertices of the foliation graph $\Gamma_{\mathcal{F}}$ are singular leaves of $\mathcal{F}$, which are critical contours $C_v \subset A_v$ of the functions f_v. Edges of $\Gamma_{\mathcal{F}}$ are cylinders filled by non-singular leaves of $\mathcal{F}$, which are regular contours of the functions f_v and f_e; in particular, each edge $\mathcal{E}$ contains a unique tube τ_e, so we denote it by $\mathcal{E}_e$. Thus $\Gamma_{\mathcal{F}} = (\{C_v\}, \{\mathcal{E}_e\})$.

By construction of M_G, there is a one-to-one correspondence between vertices $v \in G$ and critical contours $C_v \subset A_v$, which are vertices $C_v \in \Gamma_{\mathcal{F}}$. Similarly, there is one-to-one correspondence between edges $e \in G$ and edges $\mathcal{E}_e \in \Gamma_{\mathcal{F}}$. Obviously, $\partial\mathcal{E}_e \cap C_v \neq \emptyset$ if and only if $\tau_e \cap A_v \neq \emptyset$, and by (2), this holds only if e is incident to v. So graphs $\Gamma_{\mathcal{F}}$ and G have the same structure, $\Gamma_{\mathcal{F}} = G$. □

5. Realization problem in the class of Morse functions

In this section, we show that for any non-trivial graph G, the manifold M_G can be constructed only on the basis of Morse atoms (whose local functions f_v are Morse).

Lemma 5.1. *Given a finite non-trivial graph G (allowing multiple edges and loop edges) and an integer $n \geq 2$, there exists a closed n-manifold M_G constructed as a thick graph based on Morse atoms. If $G = K_2$, then $M_G \cong S^n$, otherwise M_G can be chosen orientable or non-orientable; moreover, the construction guarantees that M_G is orientable if and only if all its atoms are orientable.*

Proof. If $G = K_2$, then by Proposition 3.7, we obtain $M_G \cong S^n$.

Now let $G \neq K_2$. Since G is non-trivial, it admits an orientation from Lemma 2.1; namely, all sinks and sources of the digraph $\vec{G}$ have degree 1. By Proposition 3.7, this orientation of $\vec{G}$ allows us to represent its vertices v by Morse atoms (A_v, f_v); we choose orientable Morse atoms from Corollary 3.8.

Consider an embedding of the digraph $\vec{G}$ into $\mathbb{R}^{n+1}$. For example, for $n = 2$, we have an embedding $\vec{G} \hookrightarrow \mathbb{R}^3$; atoms (A_v, f_v) corresponding to the vertices of any degree are shown in Figure 5. These atoms are embedded into $\mathbb{R}^3$ with $\deg_{in} v$ boundary components at the bottom and $\deg_{out} v$ boundary components at the top, so that the height function $f_v : A_v \to \mathbb{R}$ increases along the direction of the edges incident to v. Note that both surfaces A_v and tubes $\tau_e \cong S^1 \times [0, 1]$ have an orientation induced by the orientation of $\mathbb{R}^3$; we use orientation-reversing homeomorphisms to glue the boundary components of A_v and τ_e, this keeps the orientation of these surfaces. For $n \geq 3$, the construction is similar.

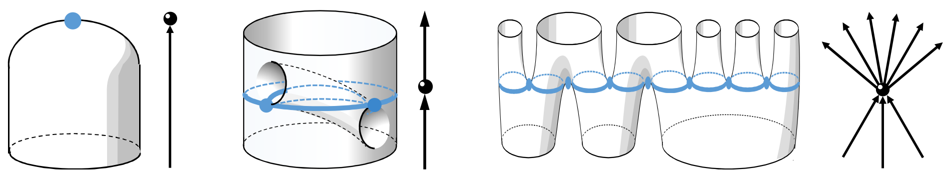
Then we interconnect the orientable manifolds A_v by the tubes τ_e according to the structure of the graph G, and obtain the manifold M_G as shown in Lemma 4.1; see the construction in Figure 4. Since $\vec{G}$ is embedded into $\mathbb{R}^{n+1}$, the manifold M_G is also embedded into $\mathbb{R}^{n+1}$, and thus it is orientable.

To obtain a non-orientable thick graph, it suffices to replace one of the atoms in M_G with a non-orientable one and adjust the local function of this new atom. □

Proposition 5.2. *Given a finite non-trivial graph G (allowing multiple edges and loop edges), any manifold M_G constructed as a thick graph based on Morse atoms, admits a Morse form foliation $\mathcal{F}_\omega$ whose foliation graph is isomorphic to G, i.e., $\Gamma_{\mathcal{F}_\omega} = G$.*

Proof. By Proposition 4.2, on M_G there exists a closed 1-form ω defining a singular foliation whose foliation graph is isomorphic to G, i.e., $\Gamma_{\mathcal{F}_\omega} = G$. Since the construction of M_G is based on Morse atoms, the form ω has only Morse singularities, i.e., it is a Morse form, and $\mathcal{F}_\omega$ is a Morse form foliation. □

Corollary 5.3. *A finite non-trivial graph G (allowing multiple edges and loop edges) is the foliation graph of a Morse form foliation on a closed n-manifold, for any $n \geq 2$.*



(*a*) (*b*) (*c*)

Figure 5: Examples of embedded atoms A_v, surfaces with different number of holes at the top and bottom, used to construct a surface M_G: (*a*) corresponds to a vertex of degree 1 that is a sink of $\vec{G}$, the height function has a maximum, A_v is a sphere with boundary; (*b*) corresponds to a vertex of degree 2, A_v is a torus with boundary; (*c*) corresponds to a vertex with $\deg v \geq 3$, A_v is a sphere with boundary. On all atoms, the associated height function is of Morse type. Singularities and singular levels are shown in blue.

Proposition 5.4. *Given a finite non-trivial graph G (allowing multiple edges and loop edges), on any orientable manifold M_G constructed as a thick graph based on Morse atoms, there exists a Morse function $f : M_G \to S^1$ such that its Reeb graph R_f is isomorphic to G, i.e., $R_f = G$.*

Proof. By Proposition 5.2, any orientable manifold M_G constructed as a thick graph based on Morse atoms admits a Morse form foliation $\mathcal{F}_\omega$ whose foliation graph is isomorphic to G, i.e., $\Gamma_{\mathcal{F}_\omega} = G$. All leaves of this foliation are compact, so by Proposition 2.3, there exists a Morse form ω' with $\operatorname{rk}\omega' = 1$, such that $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$, i.e., the foliation $\mathcal{F}_\omega$ can be defined by the rational Morse form ω'.

Consider $f(x) = e^{2\pi i \int_a^x \omega'}$, it is a circle-valued function, $f : M_G \to S^1$. Contours of this function coincide with leaves of $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$, so its Reeb graph is $\Gamma_{\mathcal{F}_\omega}$, and by Proposition 5.2, we obtain $R_f = G$. □

This proposition implies the main result of this paper:

Theorem 5.5. *A graph G (allowing multiple edges and loop edges) is the Reeb graph R_f of a Morse circle-valued function $f : M \to S^1$ on a closed n-manifold, for any $n \geq 2$, if and only if G is finite and non-trivial.*
If $G = K_2$, then $M \cong S^n$, otherwise M can be chosen orientable or non-orientable.

Remark 5.6. *The function in Theorem 5.5 can be chosen Morse real-valued if and only if the graph G has no loop edges and all its leaf blocks are K_2 [12, Theorem 12].*

Proof. Let G be finite and non-trivial. By Lemma 5.1, if $G = K_2$, then $M \cong S^n$, otherwise there exists an orientable manifold $M = M_G$ constructed as a thick graph based on Morse atoms. By Proposition 5.4, there is a Morse function $f : M \to S^1$ such that $R_f = G$.

To obtain a non-orientable $\widetilde{M}$, we replace an atom A_v, $\deg v \neq 1$, in M_G with a non-orientable atom B_v, and adjust the local function f_v of the new atom to obtain a Morse function $\widetilde{f} : \widetilde{M} \to S^1$, namely, $\widetilde{M} = (M_G \setminus A_v) \cup B_v$, and

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in M_G \setminus A_v, \\ f_v(x), & x \in B_v. \end{cases}$$

By construction, the Reeb graph $R_{\widetilde{f}} = G$.

In the opposite direction: since f is Morse, it is not constant, so R_f is non-trivial. Since M is compact, f has a finite number of critical points, so R_f has a finite number of vertices. Their degrees are finite, thus $R_f = G$ is a finite graph. □

Corollary 5.7. *A finite non-trivial graph G (allowing multiple edges and loop edges) is the Reeb graph of a Morse circle-valued function on a closed n-manifold, for any $n \geq 2$.*

6. On realization by Morse–Bott or round circle-valued functions

A few words should be said on the graph realization in other classes of functions, namely, Morse–Bott and round circle-valued functions on a closed manifold. The realization problem in these classes is well-posed because, by [7, Theorem 5.7], the Reeb graphs of these functions are finite graphs.

Since the class of Morse–Bott functions includes the class of Morse functions, and the trivial function is Morse–Bott, then the following result is a simple generalization of Theorem 5.5:

Theorem 6.1. *A graph G (allowing multiple edges and loop edges) is the Reeb graph R_f of a Morse–Bott circle-valued function $f : M \to S^1$ on a closed n-manifold, for any $n \geq 2$, if and only if G is finite.*

So any finite graph—even trivial—can be realized as the Reeb graph of a Morse–Bott circle-valued function.

Now consider round functions; recall that a *round function* is a smooth function whose critical set consists of a finite number of circles. This is an interesting class of functions: unlike Morse or Morse–Bott functions, round functions are not dense in the space of smooth functions, moreover, they do not exist on all manifolds, see Proposition 6.2 below.

We are not aware of any results on the realization problem in the class of round circle-valued functions, but the following simple properties of these functions and their Reeb graphs give necessary conditions for the realization of a graph:

Proposition 6.2. *Let $f : M^n \to S^1$ be a round circle-valued function on a closed n-manifold, $n \geq 2$. Then*

(i) *R_f is finite and non-trivial,*
(ii) *R_f does not have vertices of degree 1, if M^2 is an orientable surface,*
(iii) *$M^2 \neq S^2$, $\mathbb{R}P^2$,*

Proof. (i) Since $n \geq 2$, R_f is non-trivial, by [7, Theorem 5.7], it is finite.

(ii) Let M^2 be an orientable surface. Since local extrema of f are circles, then a small f-saturated neighborhood of such an extremum has two boundary components; see the atom in Figure 1 (c). Thus all sources and sinks of the Reeb digraph have degree 2, i.e., R_f has no vertices of degree 1.

(iii) Since both S^2 and $\mathbb{R}P^2$ have finite fundamental groups, then, by [7, Lemma 3.3], all circle-valued functions f have lifts $\hat{f} : S^2 \to \mathbb{R}$ and $\hat{f} : \mathbb{R}P^2 \to \mathbb{R}$ such that $\hat{f} = exp \circ f$. By construction, $\text{Crit}(\hat{f}) = \text{Crit}(f)$, so the lift $\hat{f}$ of a round circle-valued function f is also a round function. On the other hand, by [15, Proposition 1], round real-valued functions do not exist on S^2 or $\mathbb{R}P^2$; thus round circle-valued functions also do not exist on these surfaces. □

For $n \geq 3$, the Reeb graphs of round real-valued functions $f : M^n \to \mathbb{R}$ have the same structure as the Reeb graphs of Morse real-valued functions [13, Theorem 6.2]; the case $n = 2$ is special. Taking into account this fact, as well as Theorem 5.5 and Proposition 6.2, we can make the conjecture that Proposition 6.2 is a criterion:

Conjecture 6.3. *A graph G (allowing multiple edges and loop edges) is the Reeb graph R_f of a round circle-valued function $f : M^n \to S^1$ on a closed manifold, for any $n \geq 2$, if and only if*

(i) *G is finite and non-trivial,*
(ii) *G does not have vertices of degree 1, if M^2 is an orientable surface,*
(iii) *$M^2 \neq S^2$, $\mathbb{R}P^2$.*

7. Realization problem for a given manifold

On a given manifold M, the realization of a graph as the Reeb graph of a function $f : M \to S^1$ is much more complicated. We are not aware of any criterion on the realization in the class of circle-valued functions on a given manifold (except for surfaces, see Section 8), although there exists a necessary condition:

Theorem 7.1. *([7, Theorem 4.7]) Let M be a closed manifold, $f : M \to S^1$ a smooth function with finite number of critical values. Then*

$$b_1(R_f) \leq \text{corank}(\pi_1(M)). \tag{3}$$

In addition, if f is not null-homotopic, then $b_1(R_f) \geq 1$.

Recall that the *co-rank* of a finitely generated group G, corank(G), is the maximum rank of a free quotient group of G; some methods of calculating of the co-rank of the fundamental group of a manifold can be found in [10].

In the spirit of the criterion proved by Saeki [27, Corollary 6.4] for real-valued functions, and taking into account Theorem 7.1, we can suppose that Theorem 7.1 is a part of a criterion:

Conjecture 7.2. *Given a closed manifold M, a finite graph G can be realized as the Reeb graph of a smooth function $f : M \to S^1$ with finite number of critical values if and only if*

$$b_1(G) \leq \operatorname{corank}(\pi_1(M)).$$

If G has no loop edges, the function f can be chosen real-valued.

For a specific class of smooth circle-valued functions, e.g., for Morse functions, inequality (3) can be strengthened. Namely, Theorem 8.3 below shows that for a Morse function f on a closed surface M^2 of genus g, it holds

$$b_1(R_f) \leq \operatorname{corank}(\pi_1(M^2)) - N_2,$$

where N_2 is the number of vertices of degree 2 of the Reeb graph R_f. Note that by [10, (4.1)], for an orientable surface M^2 of genus g, $\operatorname{corank}(\pi_1(M^2)) = g$, for a non-orientable one, $\operatorname{corank}(\pi_1(M^2)) = \left[\frac{g}{2}\right]$.

8. Realization problem for a given surface by Morse functions

In this section, we will prove a criterion on realizability of a finite graph as the Reeb graph of a Morse circle-valued function on a given surface.

8.1. Properties of the thick graphs that are surfaces

If the thick graph M_G is a surface, then its genus is defined by characteristics of the graph G and the atoms of M_G. From here we will denote by A_v orientable atoms, and by B_v non-orientable ones:

Proposition 8.1. *([11, Proposition 5.3]) Let $G = (V, E)$ be a finite connected graph, where $V = V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$, and*

$$M_G = \Big(\bigcup_{v\in V_1} A_v\Big) \cup \Big(\bigcup_{v\in V_2} B_v\Big) \cup \Big(\bigcup_{e\in E} \tau_e\Big),$$

be a closed surface, where A_v are orientable and B_v are non-orientable atoms; $\tau_e \cong e \times S^1$ are tubes; and the "vertices" A_v, B_v are interconnected by the "edges" τ_e according to the structure of the graph G, namely,

$$\left.\begin{matrix} A_v \\ B_v \end{matrix}\right\} \cap \tau_e = \begin{cases} \text{one or two } S^1, & \text{if } e \text{ is incident to } v, \\ \emptyset, & \text{otherwise} \end{cases}$$

(two S^1 above correspond to the case of a loop edge). Then the surface genus

$$g(M_G) = \begin{cases} b_1(G) + \sum_{v\in V} g(A_v), & \text{if } M_G \text{ is orientable}, \\ 2(b_1(G) + \sum_{v\in V_1} g(A_v)) + \sum_{v\in V_2} g(B_v), & \text{otherwise}. \end{cases}$$

Here $b_1(G)$ is the cycle rank of the graph.

In the orientable case, obviously $V = V_1$.

Proposition 8.2. *For a surface M_G constructed as a thick graph based on Morse atoms, there are exact lower bounds on the surface genus:*

$$g(M_G) \geq \begin{cases} b_1(G) + N_2, & \text{if } M_G \text{ is orientable}, \\ \max\{2b_1(G) + N_2, 1\}, & \text{otherwise}. \end{cases}$$

Here $b_1(G)$ is the cycle rank of the graph, and N_2 is the number of vertices of degree 2.

Proof. Let M_G be orientable. By Proposition 8.1,

$$g(M_G) = b_1(G) + \sum_{\deg v \neq 2} g(A_v) + \sum_{\deg v = 2} g(A_v). \tag{4}$$

By Corollary 3.10, for vertices with $\deg v = 2$, it holds $g(A_v) \geq 1$, and for all other vertices, $g(A_v) \geq 0$. The atoms with the minimum genus are shown in Figure 5. Thus (4) implies the exact bound:

$$g(M_G) \geq b_1(G) + N_2.$$

Now let M_G be non-orientable; obviously, $g(M_G) \geq 1$. By Proposition 8.1,

$$g(M_G) = 2b_1(G) + 2\sum_{\deg v \neq 2} g(A_v) + 2\sum_{\deg v = 2} g(A_v) + \sum_{v} g(B_v). \tag{5}$$

If $2b_1(G) + N_2 < 1$, i.e., G is a tree without vertices of degree 2, then we choose all atoms except one to be orientable, $g(A_v) \geq 0$, and one non-orientable atom has $g(B_v) \geq 1$. In this case, M_G has the minimum genus, $g(M_G) = 1$, if all orientable atoms are spheres with boundary, and the non-orientable atom is $\mathbb{R}P^2$ with boundary, see Example 3.13.

If $2b_1(G) + N_2 \geq 1$, then as above, for orientable atoms, $g(A_v) \geq 0$, if $\deg v \neq 2$, and $g(A_v) \geq 1$, if $\deg v = 2$; for non-orientable atoms, $g(B_v) \geq 1$. Thus M_G has the minimum genus, if all vertices of degree 2 correspond to non-orientable atoms with $g(B_v) = 1$ (see Example 3.12 for details), and all other vertices correspond to spheres with boundary, $g(A_v) = 0$. Then (5) implies the boundary:

$$g(M_G) \geq 2b_1(G) + N_2.$$

If $N_2 = 0$, then all vertices correspond to spheres with boundary, $g(A_v) = 0$, one of the tubes τ_e is glued to an atom invertedly to provide non-orientability of M_G; in this case, $g(M_G) \geq 2b_1(G)$. □

8.2. Criterion on realization for a given surface

For closed surfaces, the theorem below generalizes the result of de Rezende *et al.* [4, Theorem 5.5] on realization of a finite digraph in the class of simple Morse circle-valued functions, and the criterion of Michalak on realization in the class of Morse real-valued functions [23, Theorem 5.6].

Theorem 8.3. *Given a finite connected graph G and an integer $g \geq 0$, the graph G can be realized as the Reeb graph of a Morse circle-valued function $f : M^2 \to S^1$ on a closed surface of genus g if and only if*

(i) *G is non-trivial,*

(ii) *for orientable M^2:* $\quad g$ *is* $\begin{cases} 0, & \text{if } G = K_2, \\ \geq b_1(G) + N_2, & \text{otherwise,} \end{cases}$

(iii) *for non-orientable M^2:* $\quad g \geq \max\{2b_1(G) + N_2, 1\}$.

where K_2 is the complete graph on 2 vertices (closed interval), and N_2 is the number of vertices of degree 2.

Remark 8.4. *The function f in Theorem 8.3 can be chosen Morse real-valued if and only if the graph G has no loop edges and all its leaf blocks are K_2 [12, Theorem 12]. In this case, for orientable surface, f can be constructed as a height function of an embedding $M^2 \hookrightarrow \mathbb{R}^3$ [6, Proposition 6.1].*

Proof. For a given graph G and an integer g satisfying (i) and (ii) (or (i) and (iii)), we will construct an orientable (or non-orientable) surface of genus g and a Morse circle-valued function f on it such that $R_f = G$.

First consider the orientable case, with G and g satisfying (i) and (ii). By Proposition 8.2, there exists an orientable surface M_G with

$$g(M_G) = b_1(G) + N_2, \tag{6}$$

and, in terms of atoms, Proposition 8.1 states

$$g(M_G) = b_1(G) + \sum_v g(A_v),$$

the types of atoms are specified in Proposition 8.2.

If $g = b_1(G) + N_2$, then the desired surface $M^2 = M_G$. In particular, if $G = K_2$, then $g(M_G) = 0 = g$, i.e., $M^2 \cong S^2$. By Proposition 5.4, there exists a Morse function $f : M^2 \to S^1$ such that $R_f = G$.

If $g > b_1(G) + N_2$, then there exists a vertex v with $\deg v \neq 1$. We will construct a new thick graph $\widetilde{M}_G$ in such a way: in the surface M_G constructed above, we replace a Morse atom A_v, $\deg v \neq 1$, with a new Morse atom $\widetilde{A}_v$ having a higher genus, $g(\widetilde{A}_v) > g(A_v) \geq 0$. Namely, if we add to the critical contour $C_v \subset A_v$ a handle shown in Figure 5 (b), this will increase the atom genus by 1, without changing $\deg v$ and the graph; see Figure 6. Denote by $\widetilde{A}_v$ an atom obtained by adding to the critical contour $k = g - b_1(G) - N_2 > 0$ handles; by construction, $g(\widetilde{A}_v) = g(A_v) + k$. Proposition 8.1 and (6) imply that the surface genus:

$$g(\widetilde{M}_G) = b_1(G) + g(\widetilde{A}_v) + \sum_{v' \neq v} g(A_{v'}) = g.$$

We have constructed $M^2 = \widetilde{M}_G$ with $g(M^2) = g$. By Proposition 5.4, there exists a Morse function $f : M^2 \to S^1$ such that $R_f = G$.

Now consider the non-orientable case, with G and g satisfying (i) and (iii). If $2b_1(G) + N_2 < 1$, i.e., G is a tree without vertices of degree 2, then by Proposition 8.2, there exists a non-orientable surface M_G with the minimum genus $g(M_G) = 1$, i.e., $M_G \cong \mathbb{R}P^2$. If $g = 1$, the desired surface $M^2 = \mathbb{R}P^2$. If $2b_1(G) + N_2 \geq 1$, then by Proposition 8.2, there exists a non-orientable surface M_G with

$$g(M_G) = 2b_1(G) + N_2, \tag{7}$$

and, in terms of atoms, Proposition 8.1 states

$$g(M_G) = 2(b_1(G) + \sum_{v \in V_1} g(A_v)) + \sum_{v \in V_2} g(B_v),$$

with orientable atoms A_v and non-orientable B_v, the types of atoms are specified in Proposition 8.2.

If $g = 2b_1(G) + N_2$, then the desired surface $M^2 = M_G$. If $g > 2b_1(G) + N_2$, denote $k = g - 2b_1(G) - N_2 > 0$.

Similarly to the orientable case, we construct a new thick graph $\widetilde{M}_G$ with $g(\widetilde{M}_G) = g$, by replacing one atom (with $\deg v \neq 1$) in M_G with a new atom of higher genus. If $k = 2p$, we can take either orientable or non-orientable atom and replace it with a new one having p additional handles glued to the critical contour, see Figure 6 (c); we obtain $g(\widetilde{M}_G) = g(M_G) + 2p$. Now let $k = 2p + 1$. We can take an orientable atom and replace it with a non-orientable one having one cross-cap and p additional handles, then $g(\widetilde{M}_G) = g(M_G) + 2p + 1$. Or we can take a non-orientable atom with a cross-cap and replace it with a new one without this cross-cap and having $p + 1$ additional handles glued to the critical contour, one of these handles being inverted; the new atom has $2(p + 1) - 1$ cross-caps; then $g(\widetilde{M}_G) = g(M_G) + 2p + 1$.

In both cases, we obtain $g(\widetilde{M}_G) = g(M_G) + k$, then (7) implies $g(\widetilde{M}_G) = g$, so $M^2 = \widetilde{M}_G$.

Now we will construct a Morse function $f : M^2 \to S^1$ such that $R_f = G$.

In the thick graph M^2, we replace all non-orientable atoms B_v with the suitable orientable atoms A_v^B. By Lemma 5.1, we can choose this new thick graph to be orientable, so we denote it by M_G^{or}. By Proposition 5.4, there exists a Morse function $f_1 : M_G^{or} \to S^1$ such that $R_{f_1} = G$. To construct the function $f : M^2 \to S^1$, we

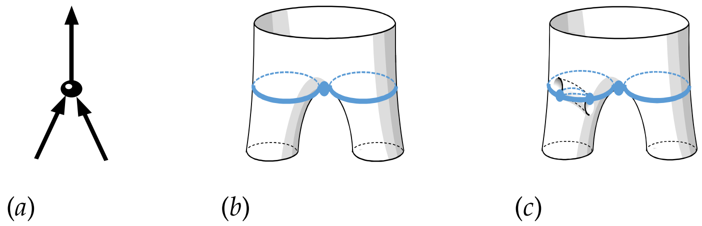


Figure 6: Morse atoms of different genus corresponding to a given vertex: (*a*) vertex v with $\deg_{in} v = 2$ and $\deg_{out} v = 1$; (*b*) Morse atom A_v with one critical point on its critical contour, it is a sphere with boundary; (*c*) atom with a handle glued to its critical contour, it has 3 critical points on it. This atom is a torus with boundary. We can glue any number of handles to the critical contour to obtain an atom $\widetilde{A}_v$ of any desired genus. Critical points and critical levels are shown in blue.

adjust the local functions $f_v : B_v \to \mathbb{R}$ so that $f_v = f_1$ on each boundary component of B_v, and the gluing of the functions would be smooth; on the orientable atoms $A_v \subset M^2$ and the tubes $\tau_e \subset M^2$, the function $f \equiv f_1$, namely,

$$f(x) = \begin{cases} f_1(x), & x \in A_v \text{ or } x \in \tau_e, \\ f_v(x), & x \in B_v. \end{cases}$$

By construction, $f : M^2 \to S^1$ is Morse, and $R_f = R_{f_1} = G$.

In the opposite direction: assume that G is realized as the Reeb graph of a Morse function $f : M^2 \to S^1$ on a surface M^2 of genus g. Since a Morse function is not constant, then $G = R_f$ is not trivial, and (i) is true. To prove inequalities (ii) and (iii), we represent M^2 as a thick graph: indeed, consider critical contours of f, their pairwise disjoint regular neighborhoods are Morse atoms. Then Proposition 8.2 completes the proof. $\square$

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