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# Characterization of the essential approximation S-spectrum and the essential defect S-spectrum in a right quaternionic Hilbert space

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**Abstract.** In this paper, we introduce and study the essential approximation S-spectrum and the essential defect S-spectrum in a right quaternionic Hilbert space. Our results are used to describe the investigation of the stability of the essential approximation S-spectrum and the essential defect S-spectrum of linear operator $A$ subjected to additive perturbation $K$ such that $(AK + KA + K^2 - 2Re(\mathbf{q})K)R_{\mathbf{q}}(A + K)^{-1}$ or $R_{\mathbf{q}}(A + K)^{-1}(AK + KA + K^2 - 2Re(\mathbf{q})K)$ is a quasi-compact operator in the right quaternionic Hilbert space.

## 1. Introduction

We start from basic issues regarding the general notion of essential S-spectrum of an operator on a quaternionic Hilbert space. For the definition of the spectrum we follow the viewpoint adopted in [6] for quaternionic Banach modules. A pivotal tool in our investigation is the notion of slice function [7]. For quaternionic Hilbert spaces, a formulation of the spectral theorem already exists [15] without any systematic investigation of the continuous functional calculus. In fact, the spectral theory is applied to quantum theories through functional calculus [1, 4, 5, 8]. However, several difficulties arise with quaternionic linear operators due to their non-commutativity with quaternions, which hinder the generalization of results valid in the complex set to the quaternionic space.

Due to the non-commutativity, in the quaternionic case, there are three types of Hilbert spaces: left, right, and two-sided, depending on how vectors are multiplied by scalars. This fact can entail several problems. For example, when a Hilbert space $\mathbb{H}$ is onesided (either left or right) the set of linear operators acting on it does not have a linear structure. Moreover, in a one-sided quaternionic Hilbert space, given a linear operator $A$ and a quaternion $\mathbf{q} \in \mathbb{H}$, in general, we have that $(\mathbf{q}A)^* \neq \overline{\mathbf{q}}A$. As a solution, a notion of multiplication can be introduced on both sides with a fixed arbitrary Hilbert basis of $\mathbb{H}$, which allows us to have a linear structure on the set of linear operators and consequently develop a full theory. Thus, the framework of this paper is, in part, a right quaternionic Hilbert space equipped with a left multiplication, introduced by fixing a Hilbert basis. We will finally bring our own interpretation of the operator theory of Fredholm and the S-essential spectrum has not yet been studied in the quaternionic framework. This interpretation is based on the numerical study of the interactions between the spatial domain and the quaternion frequency

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domain. We will see that there are notably symmetry properties both on the spectrum and on its original signal due to the use of this particular transform.

In the complex setting, in a Hilbert space $\mathfrak{h}$, a bounded linear operator, $A$, is not invertible if it is not bounded below. The set of approximate eigenvalues which are $\lambda \in \mathbb{C}$ such that $A - \lambda \mathbb{I}_{\mathfrak{h}}$, where $\mathbb{I}_{\mathfrak{h}}$ is the identity operator on $\mathfrak{h}$, is not bounded below, equivalently, the set of $\lambda \in \mathbb{C}$ for which there is a sequence of unit vectors $\phi_1, \phi_2 ..$ such that $\lim\limits_{n\to\infty} \|A\phi_n - \lambda\phi_n\| = 0$. The set of approximate eigenvalues is known as the approximate spectrum. In the quaternionic setting, let $V_{\mathbb{H}}^R$ be a separable right Hilbert space, $A$ be a bounded right linear operator, and $R_{\mathbf{q}}(A) = A^2 - 2Re(\mathbf{q})A + |\mathbf{q}|^2\mathbb{I}_{V_{\mathbb{H}^R}}$, with $\mathbf{q} \in \mathbb{H}$, the set of all quaternions, be the pseudo-resolvent operator, the set of right eigenvalues of $A$ coincide with the point S-spectrum (see proposition 4.5 in [6]). In this regard, it will be appropriate to define and study the quaternionic approximate S-point spectrum as the quaternions for which $R_{\mathbf{q}}(A)$ in not bounded below.

Quaternions: Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 i + \mathbf{q}_2 j + \mathbf{q}_3 k, \qquad \mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathbb{R}$$

where $i, j, k$ are the three quaternionic imaginary units, satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. The quaternionic conjugate of $\mathbf{q}$ is

$$\overline{\mathbf{q}} = \mathbf{q}_0 - \mathbf{q}_1 i - \mathbf{q}_2 j - \mathbf{q}_3 k,$$

while $|q| = (q\overline{q})^{1/2}$ denotes the usual norm of the quaternion $q$. If $q$ is a non-zero element, it has inverse $q^{-1} = \frac{\overline{q}}{|q|^2}$. Finally, the set

$$\mathbb{S} = \left\{ I = x_1 i + x_2 j + x_3 k : x_1, x_2, x_3 \in \mathbb{R} \text{ and } x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

contains all the elements whose square is $-1$. It is a 2-dimensional sphere in $\mathbb{H}$ identified with $\mathbb{R}^4$.

Quaternionic Hilbert spaces: In this subsection we discuss right quaternionic Hilbert spaces. For more details we refer the reader to [1, 6, 15].
Right quaternionic Hilbert Space: Let $V_{\mathbb{H}}^R$ be a vector space under right multiplication by quaternions. For $\phi, \psi, \omega \in V_{\mathbb{H}}^R$ and $\mathbf{q} \in \mathbb{H}$, the inner product

$$< \cdot \mid \cdot >_{V_{\mathbb{H}}^R} : V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$$

satisfies the following properties

1. $\overline{< \phi \mid \psi >_{V_{\mathbb{H}}^R}} = < \psi \mid \phi >_{V_{\mathbb{H}}^R}$
2. $\|\phi\|^2_{V_{\mathbb{H}}^R} = < \phi \mid \phi >_{V_{\mathbb{H}}^R} > 0$ unless $\phi = 0$, a real norm
3. $< \phi \mid \psi + \omega >_{V_{\mathbb{H}}^R} = < \phi \mid \psi >_{V_{\mathbb{H}}^R} + < \phi \mid \omega >_{V_{\mathbb{H}}^R}$
4. $< \phi \mid \psi\mathbf{q} >_{V_{\mathbb{H}}^R} = < \phi \mid \psi >_{V_{\mathbb{H}}^R} \mathbf{q}$
5. $< \phi\mathbf{q} \mid \psi >_{V_{\mathbb{H}}^R} = \overline{\mathbf{q}} < \phi \mid \psi >_{V_{\mathbb{H}}^R}$, where $\overline{\mathbf{q}}$ stands for the quaternionic conjugate.

It is always assumed that the space $V_{\mathbb{H}}^R$ is complete under the norm given above and separable. Then, together with $< \cdot \mid \cdot >_{V_{\mathbb{H}}^R}$ this defines a right quaternionic Hilbert space. Quaternionic Hilbert spaces share many of the standard properties of complex Hilbert spaces.

**Proposition 1.1.** [6, Proposition 2. 5] *Let $O = \{\varphi_k : k \in \mathbb{N}\}$ be an orthonormal subset of $V_{\mathbb{H}}^R$. Then, the following conditions are pairwise equivalent:*

1. *The closure of the linear combinations of elements in $O$ with coefficients on the right is $V_{\mathbb{H}}^R$.*

2. *For every* $\phi, \psi \in V_{\mathbb{H}}^R$, *the series* $\sum_{k\in\mathbb{N}} <\phi|\varphi_k>_{V_{\mathbb{H}}^R} <\varphi_k|\psi>_{V_{\mathbb{H}}^R}$ *converges absolutely and it holds:*

$$<\phi|\varphi_k>_{V_{\mathbb{H}}^R} = \sum_{k\in\mathbb{N}} <\phi|\varphi_k>_{V_{\mathbb{H}}^R} <\varphi_k|\psi>_{V_{\mathbb{H}}^R} .$$

3. *For every* $\phi \in V_{\mathbb{H}}^R$, *it holds:*

$$\|\phi\|^2_{V_{\mathbb{H}}^R} = \sum_{k\in\mathbb{N}} | <\varphi_k|\phi>_{V_{\mathbb{H}}^R} |^2.$$

4. $O^{\perp} = \{0\}$.

**Definition 1.2.** *The set* $O$ *is called a Hilbert basis of* $V_{\mathbb{H}}^R$, *if it satisfies the equivalent conditions stated in Proposition 1.1.*

**Remark 1.3.** *Every quaternionic Hilbert space* $V_{\mathbb{H}}^R$ *has a Hilbert basis. All the Hilbert bases of* $V_{\mathbb{H}}^R$ *have the same cardinality. Furthermore, if* $O$ *is a Hilbert basis of* $V_{\mathbb{H}}^R$, *then every* $\phi \in V_{\mathbb{H}}^R$ *can be uniquely decomposed as follows:*

$$\phi = \sum_{k\in\mathbb{N}} \varphi_k <\varphi_k|\phi>_{V_{\mathbb{H}}^R},$$

*where the series* $\sum_{k\in\mathbb{N}} \varphi_k <\varphi_k|\phi>_{V_{\mathbb{H}}^R}$ *converges absolutely in* $V_{\mathbb{H}}^R$ *(see* [6]*).*
*The field of quaternions* $\mathbb{H}$ *it self can be turned into a left quaternionic Hilbert space by defining the inner product* $<\mathbf{q}, \mathbf{q}'>= \mathbf{q}\overline{\mathbf{q}'}$ *or into a right quaternionic Hilbert space with* $<\mathbf{q}, \mathbf{q}'>= \overline{\mathbf{q}}\mathbf{q}'$.

We follow the work by Ghiloni et al. [6] to study the quaternionic version of some spectral properties of compact operators. First, we recall the next definition from [6].

We end this introduction by mentioning, in order of expected difficulty. In Section 2, some notations, basic concepts about right quaternionic linear operators, S-spectrum, and Fredholm operators are recalled. in section 3, we interested in studying the stability problem of the essential approximation S-spectrum and the essential defect S-spectrum. In Section 4, the concept of the quasi-compact in a right quaternionic Hilbert space is introduced and its properties are studied, which we will need to characterize the essential approximation S-spectrum and the essential defect S-spectrum in terms of the quasi-compact operators.

## 2. Preliminary results

### 2.1. Right quaternionic linear operators and some basic properties

In this subsection,we shall define right linear operators and recall some basic properties. Most of them are very well known.

**Definition 2.1.** *Let* $V_{\mathbb{H}}^R$ *be a right quaternionic Hilbert space. A right* $\mathbb{H}$*-linear operator, for simplicity, right linear operator, is a map* $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ *such that*

$$A(\phi\mathbf{a} + \psi\mathbf{b}) = (A\phi)\mathbf{a} + (A\psi)\mathbf{b} \ \textit{and} \ \mathbf{a}, \mathbf{b} \in \mathbb{H}$$

*where the domain* $\mathcal{D}(A)$ *of* $A$ *is a right* $\mathbb{H}$*-linear subspace of* $V_{\mathbb{H}}^R$.

The set of all right linear operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ will be denoted by $\mathcal{L}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and the identity linear operator on $V_{\mathbb{H}}^R$ will be denoted by $\mathbb{I}_{\mathbb{H}}^R$. For a given $A \in \mathcal{L}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, the range and the kernel will be

$$\begin{aligned} R(A) &= \Big\{\psi \in U_{\mathbb{H}}^R : A\phi = \psi \text{ for } \phi \in \mathcal{D}(A)\Big\} \\ N(A) &= \Big\{\phi \in \mathcal{D}(A) : A\phi = 0\Big\}. \end{aligned}$$

We call an operator $A \in \mathcal{L}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ bounded if

$$\|A\| = \sup_{\|\phi\|_{V_{\mathbb{H}}^R}=1} \|A\phi\|_{U_{\mathbb{H}}^R} < \infty,$$

or equivalently, there exist $K \geq 0$ such that $\|A\phi\|_{U_{\mathbb{H}}^R} \leq K\|\phi\|_{V_{\mathbb{H}}^R}$ for all $\phi \in \mathcal{D}(A)$. The set of all bounded right linear operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ will be denoted by $\mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$. Set of all invertible bounded right linear operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ will be denoted by $\mathcal{G}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$. Assume that $V_{\mathbb{H}}^R$ is a right quaternionic Hilbert space, $A$ be an operator with dense domain acting on it. Then, there exists a unique linear operator $A^*$ such that

$$< \psi \mid A\phi >_{U_{\mathbb{H}}^R} = < A^*\psi \mid \phi >_{V_{\mathbb{H}}^R} \quad \text{for all } \phi \in \mathcal{D}(A), \psi \in \mathcal{D}(A^*).$$

where the domain $\mathcal{D}(A^*)$ of $A^*$ is defined by

$$\mathcal{D}(A^*) = \left\{ \psi \in U_{\mathbb{H}}^R : \exists \varphi \text{ such that } < \psi \mid A\phi >_{U_{\mathbb{H}}^R} = < \varphi \mid \phi >_{V_{\mathbb{H}}^R} \right\}.$$

We define the natural domains of the sum $A+B$ and of the composition $AB$ by setting $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and $\mathcal{D}(AB) = \left\{ \phi \in \mathcal{D}(B) : B\phi \in \mathcal{D}(A) \right\}$. Note that $\mathcal{B}(V_{\mathbb{H}}^R)$ has a natural structure of real algebra, in which the sum is the usual point wise sum, the product is the composition and the real scalar multiplication

$$\begin{array}{ccc} \mathcal{B}(V_{\mathbb{H}}^R) \times \mathbb{H} & \longrightarrow & \mathcal{B}(V_{\mathbb{H}}^R) \\ (A, \lambda) & \longmapsto & A\lambda \end{array}$$

is defined by setting

$$(A\lambda)(\phi) = A(\phi)\lambda.$$

**Lemma 2.2.** [6, Proposition 2.11 (e)] *Let $A : \mathcal{D}(A) \subset V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ be a right linear operator. If $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is surjective, then $A$ is open. In particular, if $A$ is bijective, then $A^{-1} \in \mathcal{B}(V_{\mathbb{H}}^R)$.*

**Definition 2.3.** *Let $V_{\mathbb{H}}^R$ and $U_{\mathbb{H}}^R$ be right quaternionic Hilbert spaces. A bounded operator $K : V_{\mathbb{H}}^R \longrightarrow U_{\mathbb{H}}^R$ is compact if $K$ maps bounded sets into precompact sets. That is, $K(U)$ is compact in $U_{\mathbb{H}}^R$, where $U = \left\{ \phi \in V_{\mathbb{H}}^R : |\phi| < 1 \right\}$. Equivalently, for all bounded sequences $(f_n) \in V_{\mathbb{H}}^R$ the sequence $(Kf_n)$ has a convergence subsequence in $U_{\mathbb{H}}^R$.*

We denote the set of all compact operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ by $\mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and the compact operators from $V_{\mathbb{H}}^R$ from $V_{\mathbb{H}}^R$ will be denoted by $\mathcal{B}_0(V_{\mathbb{H}}^R)$.

*2.2. Left scalar multiplications*

We shall extract the definition and some properties of left scalar multiplication of vectors on $U_{\mathbb{H}}^R$ from [6] as needed for the development of the manuscript. The left scalar multiplication of vectors on a right quaternionic Hilbert space is an extremely non-canonical operation associated with a choice of preferred Hilbert basis. From the Remark 1.3, $U_{\mathbb{H}}^R$ has a Hilbert basis

$$O = \left\{ \phi_k : k \in \mathbb{N} \right\}.$$

**Definition 2.4.** *Let $V_{\mathbb{H}}^R$ be a separable right Hilbert space and let $O$ its Hilbert basis. The left scalar multiplication on $V_{\mathbb{H}}^R$ induced by $O$ is defined as the map*

$$\begin{array}{ccc} \mathbb{R} \times V_{\mathbb{H}}^R & \longrightarrow & V_{\mathbb{H}}^R \\ (\mathbf{q}, \phi) & \longmapsto & \mathbf{q}\phi = \sum_{k\in\mathbb{N}} \varphi_k \mathbf{q} < \varphi_k, \phi > \end{array}$$

Let $V_{\mathbb{H}}^R$ be a separable right Hilbert space and let $O$ its Hilbert basis. Then, the left scalar multiplication on $V_{\mathbb{H}}^R$ satisfies the following properties.

**Proposition 2.5.** [6, Proposition 3.1] *For every $\phi, \psi \in V_{\mathbb{H}}^R$ and $\mathbf{q}, \mathbf{p} \in \mathbb{H}$, we have*

1. $\mathbf{q}(\phi + \psi) = \mathbf{q}\phi + \mathbf{q}\psi$ *and* $\mathbf{q}(\phi\mathbf{p}) = (\mathbf{q}\phi)\mathbf{p}$.
2. $\|\mathbf{q}\phi\| = |\mathbf{q}|\|\phi\|$.
3. $\mathbf{q}(\mathbf{p}\phi) = (\mathbf{q}\mathbf{p})\phi$.
4. $< \overline{\mathbf{q}}\phi, \psi > = < \phi, \mathbf{q}\psi >$.
5. $\mathbf{q}\phi_k = \phi_k\mathbf{q}$ *for all* $k \in \mathbb{N}$.

The quaternionic left scalar multiplication of linear operators is also defined in [6].

*2.3. Spectral Mapping Theorem for Quaternionic Hilbert Space*

In this subsection, we study some properties of various essential S-spectra of a quaternionic Hilbert space.

**Definition 2.6.** *Let $A : \mathcal{D}(A) \subset V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ be a right linear operator. The S-resolvent set (also called spherical resolvent set) of A is the set $\sigma^S(A)(\subset \mathbb{H})$ such that the three following conditions hold true:*
*(a)* $N(R_{\mathbf{q}}(A)) = \{0\}$.
*(b)* $R(R_{\mathbf{q}}(A))$ *is dense in* $V_{\mathbb{H}}^R$.
*(c)* $R_{\mathbf{q}}(A)^{-1} : R(R_{\mathbf{q}}(A)) \longrightarrow \mathcal{D}(A^2)$ *is bounded.*
*The S-spectrum (also called spherical spectrum) $\sigma^S(A)$ of A is defined by setting $\sigma^S(A) := \mathbb{H} \backslash \rho^S(A)$. For a bounded linear operator A we can write the resolvent set as*

$$\begin{aligned}\rho^S(A) &= \left\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \in \mathcal{G}(V_{\mathbb{H}}^R)\right\}\\ &= \left\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \text{ has an inverse in } \mathcal{B}(V_{\mathbb{H}}^R)\right\}\\ &= \left\{\mathbf{q} \in \mathbb{H} : N(R_{\mathbf{q}}(A)) = \{0\} \text{ and } R(R_{\mathbf{q}}(A)) = V_{\mathbb{H}}^R\right\}\end{aligned}$$

*and the spectrum can be written as*

$$\begin{aligned}\sigma^S(A) &= \mathbb{H} \backslash \rho^S(A)\\ &= \left\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \text{ has no inverse in } \mathcal{B}(V_{\mathbb{H}}^R)\right\}\\ &= \left\{\mathbf{q} \in \mathbb{H} : N(R_{\mathbf{q}}(A)) \neq \{0\} \text{ or } R(R_{\mathbf{q}}(A)) \neq V_{\mathbb{H}}^R\right\}.\end{aligned}$$

*The right S-spectrum $\sigma_r^S$ and the left S-spectrum $\sigma_l^S(A)$ are defined respectively as*

$$\begin{aligned}\sigma_r^S &= \left\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \text{ in not right invertible in } \mathcal{B}(V_{\mathbb{H}}^R)\right\}\\ &= \left\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \text{ in not left invertible in } \mathcal{B}(V_{\mathbb{H}}^R)\right\}\end{aligned}$$

*The spectrum $\sigma^S(A)$ decomposes into three major disjoint subsets as follows:*
*(i) The spherical point spectrum of A:*

$$\sigma_p^S(A) := \left\{\mathbf{q} \in \mathbb{H} : N(R_{\mathbf{q}}(A)) \neq \{0\}\right\}.$$

*(ii) The spherical residual spectrum of A:*

$$\sigma_r^S(A) := \left\{\mathbf{q} \in \mathbb{H} : N(R_{\mathbf{q}}(A)) = \{0\}, \overline{R(R_{\mathbf{q}}(A)} \neq V_{\mathbb{H}}^R\right\}.$$

*(iii) The spherical continuous spectrum of A:*

$$\sigma_c^S(A) := \left\{\mathbf{q} \in \mathbb{H} : N(R_{\mathbf{q}}(A)) = \{0\}, \overline{R(R_{\mathbf{q}}(A))} = V_{\mathbb{H}}^R, R_{\mathbf{q}}(A)^{-1} \notin \mathcal{B}(V_{\mathbb{H}}^R)\right\}.$$

In the complex setting Fredholm operators are studied in Banach spaces and Hilbert spaces for bounded and even to unbounded linear operators. In this section we shall study the theory for quaternionic bounded linear operators on separable Hilbert spaces. In this regard let $V_{\mathbb{H}}^{R}$ and $U_{\mathbb{H}}^{R}$ be two separable right quaternionic Hilbert spaces. A Fredholm operator is an operator $A \in \mathcal{B}(V_{\mathbb{H}}^{R}, U_{\mathbb{H}}^{R})$ such that $N(A)$ and $coR(A) = U_{\mathbb{H}}^{R}/R(A)$ are finite dimensional. The dimension of the cokernel is called the codimension, and it is denoted by $codim(A)$. For $A \in V_{\mathbb{H}}^{R}$, the nullity, $\alpha(A)$, of $A$ is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of $A$ is defined as the codimension of $R(A)$ in $V_{\mathbb{H}}^{R}$.
When it is the case of $A$ be a sequence of bounded right linear operators on $V_{\mathbb{H}}^{R}$, the set of upper semi-Fredholm relation is defined by:

$$\Phi_{+}(V_{\mathbb{H}}^{R}) = \left\{\mathbf{q} \in \mathbb{H} : \alpha(R_{\mathbf{q}}(A)) < \infty \text{ and } R(R_{\mathbf{q}}(A)) \text{ is closed in } V_{\mathbb{H}}^{R}\right\},$$

and the set of lower semi-Fredholm operator is defined by:

$$\Phi_{-}(V_{\mathbb{H}}^{R}) = \left\{\mathbf{q} \in \mathbb{H} : \beta(R_{\mathbf{q}}(A)) < \infty \text{ and } R(R_{\mathbf{q}}(A)) \text{ is closed in } V_{\mathbb{H}}^{R}\right\}.$$

$\Phi(V_{\mathbb{H}}^{R}) := \Phi_{+}(V_{\mathbb{H}}^{R}) \cap \Phi_{-}(V_{\mathbb{H}}^{R})$ denotes the set of bounded right linear operators on $V_{\mathbb{H}}^{R}$ and $\Phi_{\pm}(V_{\mathbb{H}}^{R}) := \Phi_{+}(V_{\mathbb{H}}^{R}) \cup \Phi_{-}(V_{\mathbb{H}}^{R})$ denotes the set of semi-Fredholm operator from $V_{\mathbb{H}}^{R}$.

**Theorem 2.7.** [11, Theorem 6.16] *Let $A \in \mathcal{B}(V_{\mathbb{H}}^{R}, U_{\mathbb{H}}^{R})$, be a Fredholm operator, then for any compact operator $K \in \mathcal{B}(V_{\mathbb{H}}^{R}, U_{\mathbb{H}}^{R})$, $A + K$ is a Fredholm operator and $i(A + K) = i(A)$.*

**Theorem 2.8.** [11, Theorem 6.13] *Let $V_{\mathbb{H}}^{R}$, $U_{\mathbb{H}}^{R}$ and $W_{\mathbb{H}}^{R}$ be right quaternionic Hilbert spaces. If $A_1 \in \mathcal{B}(V_{\mathbb{H}}^{R}, U_{\mathbb{H}}^{R})$ and $A_2 \in \mathcal{B}(U_{\mathbb{H}}^{R}, W_{\mathbb{H}}^{R})$ are two Fredholm operators, then $A_2A_1 \in \mathcal{B}(V_{\mathbb{H}}^{R}, W_{\mathbb{H}}^{R})$ is also a Fredholm operator, and it satisfies $i(A_2A_1) = i(A_1) + i(A_2)$.*

**Proposition 2.9.** [11, Proposition 4.2] *If $A \in \mathcal{B}(V_{\mathbb{H}}^{R})$, $B \in \mathcal{B}(V_{\mathbb{H}}^{R})$ and $K \in \mathcal{B}_0(V_{\mathbb{H}}^{R})$, then $AK$ and $KB$ are compact operators.*

**Corollary 2.10.** [11, Corollary 6.14.] *Let $A \in \mathcal{B}(V_{\mathbb{H}}^{R}, U_{\mathbb{H}}^{R})$ and $B \in \mathcal{B}(U_{\mathbb{H}}^{R}, V_{\mathbb{H}}^{R})$ such that $AB = F$ and $BA = G$ are Fredholm operators. Then $A$ and $B$ are Fredholm operators and $i(AB) = i(A) + i(B)$.*

Among these essential S-spectra, the following sets are defined for a bounded linear operator $A$:

$$\begin{aligned}
\sigma_{e1}^{S}(A) &:= \left\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \notin \Phi_{+}(V_{\mathbb{H}}^{R})\right\}\\
\sigma_{e2}^{S}(A) &:= \left\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \notin \Phi_{-}(V_{\mathbb{H}}^{R})\right\}\\
\sigma_{e3}^{S}(A) &:= \sigma_{e1}^{S}(A) \cap \sigma_{e2}^{S}(A)\\
\sigma_{e4}^{S}(A) &:= \left\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \notin \Phi(V_{\mathbb{H}}^{R})\right\}\\
\sigma_{e5}^{S}(A) &:= \bigcap_{K \in \mathfrak{B}_0(V_{\mathbb{H}}^{R})} \sigma^{S}(A + K)\\
\sigma_{eap}^{S}(A) &:= \bigcap_{K \in \mathfrak{B}_0(V_{\mathbb{H}}^{R})} \sigma_{ap}^{S}(A + K)\\
\sigma_{e\delta}^{S}(A) &:= \bigcap_{K \in \mathfrak{B}_0(V_{\mathbb{H}}^{R})} \sigma_{\delta}^{S}(A + K).
\end{aligned}$$

$\sigma_{ap}^{S}(A) := \{\mathbf{q} \in \mathbb{H} :\ R_{\mathbf{q}}(A) \text{ not bounded below}\}$, where bounded below is injective and open and $\sigma_{\delta}^{S}(A) := \{\mathbf{q} \in \mathbb{H} :\ R_{\mathbf{q}}(A) \text{ is not surjective}\}$. $\sigma_{eap}(\cdot)$ was introduced by V. Rakocevic in [12] and denotes the essential approximate point spectrum and $\sigma_{e\delta}(\cdot)$ is the essential defect spectrum and was introduced by C. Schmoeger [13]. Note that all these sets are closed and in general satisfy the following inclusions

$$\sigma_{e1}^{S}(A) \cap \sigma_{e2}^{S}(A) = \sigma_{e3}^{S}(A) \subseteq \sigma_{e4}^{S}(A) \subseteq \sigma_{e5}^{S}(A).$$

$$\sigma^S_{e1}(A) \subseteq \sigma^S_{eap}(A),\ \sigma^S_{e2}(A) \subseteq \sigma^S_{e\delta}(A) \text{ and } \sigma^S_{e5}(A) \subseteq \sigma^S_{eb}(A).$$

The purpose of this section is to discuss the essential approximate point S-spectrum and the essential defect S-spectrum of a bounded right quaternionic linear operator. We begin with the following useful result.

**Lemma 2.11.** *Let $A \in \Phi_+(V^R_{\mathbb{H}})$. Then, the following statements are equivalent:*
(*i*) $i(A) \leq 0$.
(*ii*) *A can be expressed in the form $A = U + K$ where $K \in \mathcal{B}_0(V^R_{\mathbb{H}})$ and $U \in \mathcal{B}(V^R_{\mathbb{H}})$ an operator with closed range and $\alpha(U) = 0$.*

This lemma is well known for bounded upper semi-Fredholm operators. The proof is a straightforward adaption of the proof of Theorem 3.9 in [17]. The results of the next proposition were established in [12] and [13] for bounded linear operators. We will improve it for a bounded right quaternionic linear operator, in a right quaternionic Hilbert space with a left multiplication s. This result is a characterization of the essential approximate point S-spectrum (resp. the essential defect S-spectrum) by means of upper semi-Fredholm (resp. lower semi-Fredholm).

**Theorem 2.12.** *Let $A \in \mathcal{B}(V^R_{\mathbb{H}})$, then*

(*i*) $\mathbf{q} \notin \sigma^S_{eap}(A)$ *if, and only if,* $R_{\mathbf{q}}(A) \in \Phi_+(V^R_{\mathbb{H}})$ *and* $i(R_{\mathbf{q}}(A)) \leq 0$.

(*ii*) $\mathbf{q} \notin \sigma^S_{e\delta}(A)$ *if, and only if,* $R_{\mathbf{q}}(A) \in \Phi_-(V^R_{\mathbb{H}})$ *and* $i(R_{\mathbf{q}}(A)) \geq 0$.

*Proof.* (*i*) Let $R_{\mathbf{q}}(A) \in \Phi_+(V^R_{\mathbb{H}})$ such that $i(R_{\mathbf{q}}(A)) \leq 0$. Then by Lemma 2.11, $R_{\mathbf{q}}(A)$ can be expressed in the form $R_{\mathbf{q}}(A) = U + K$ where $K \in \mathcal{B}_0(V^R_{\mathbb{H}})$ and $U \in \mathcal{B}(V^R_{\mathbb{H}})$ an operator with closed range and $\alpha(U) = 0$. Hence by Theorem 5.1 p. 70 in [14], there exists a constant $c > 0$ such that $\|Ux\| \geq c\|x\|$, for all $x \in \mathcal{D}(A)$. Thus $\mathbf{q} \notin \sigma^S_{ap}(A + K)$ and therefore $\mathbf{q} \notin \sigma^S_{eap}(A)$. Conversely, if $\mathbf{q} \notin \sigma^S_{eap}(A)$, then there exists $K \in \mathcal{B}_0(V^R_{\mathbb{H}})$ such that

$$\inf_{\|x\|=1,\ x\in\mathcal{D}(A)} \left\|(R_{\mathbf{q}}(A) - K)x\right\| > 0.$$

The use of Theorem 5.1 p. 70 in [14] leads to $R_{\mathbf{q}}(A) - K \in \Phi_+(V^R_{\mathbb{H}})$ and $\alpha(R_{\mathbf{q}}(A) - K) = 0$, hence it follows from Theorem 2.7 that $R_{\mathbf{q}}(A) \in \Phi_+(V^R_{\mathbb{H}})$ and $\alpha(R_{\mathbf{q}}(A)) = 0$. This completes the proof of (*i*).

(*ii*) This assertion follows, immediately, from (*i*). □

**Corollary 2.13.** *Let $A \in \mathcal{B}(V^R_{\mathbb{H}})$, then*

(*i*) $\sigma^S_{eap}(A) = \sigma^S_{e1}(A) \bigcup \left\{\mathbf{q} \in \mathbb{H} : i\left(R_{\mathbf{q}}(A)\right) > 0\right\}$.

(*ii*) $\sigma^S_{e\delta}(A) = \sigma^S_{e2}(A) \bigcup \left\{\mathbf{q} \in \mathbb{H} : i\left(R_{\mathbf{q}}(A)\right) < 0\right\}$.

**Proposition 2.14.** (*i*) *Let $A \in \mathcal{B}(V^R_{\mathbb{H}})$ and assume that $\sigma^S_{e1}(A)$ is connected and $\rho^S(A) \neq \emptyset$. Then, $\sigma^S_{e1}(A) = \sigma^S_{eap}(A)$.*

(*ii*) *Let $A \in \mathcal{B}(V^R_{\mathbb{H}})$ and assume that $\sigma^S_{e2}(A)$ is connected and $\rho^S(A) \neq \emptyset$. Then, $\sigma^S_{e2}(A) = \sigma^S_{e\delta}(A)$.*

*Proof.* (*i*) The first inclusion is a consequence of Corollary 2.13, then it easy to check that $\sigma^S_{e1}(A) \subset \sigma^S_{eap}(A)$. We prove that $\sigma^S_{eap}(A) \subset \sigma^S_{e1}(A)$. Consequently, to establish the result, it suffices to show that:

$$\sigma^S_{eap}(A) \cap \left(\mathbb{H} \backslash \sigma^S_{e1}(A)\right) = \emptyset.$$

Suppose that, $\sigma^S_{eap}(A) \cap \left(\mathbb{H} \backslash \sigma^S_{e1}(A)\right) \neq \emptyset$, then there exists $\mathbf{q}_0 \in \mathbb{H}$ such that $\mathbf{q}_0 \in \sigma^S_{eap}(A) \cap \left(\mathbb{H} \backslash \sigma^S_{e1}(A)\right)$. Since $\rho^S(A) \neq \emptyset$, then there exists $\mathbf{q}_1 \neq \emptyset$ such that $N(R_{\mathbf{q}_1}(A)) = \{0\}$ and $R(R_{\mathbf{q}_1}(A)) = V^R_{\mathbb{H}}$, therefore $R_{\mathbf{q}_1}(A) \in \Phi(V^R_{\mathbb{H}})$

and $i(R_{\mathbf{q}_1}(A)) = 0$. Since $\sigma^S_{e1}(A)$ is connected, hence $i\Big(R_{\mathbf{q}_1}(A)\Big)$ is constant on any component in $\sigma^S_{e1}(A)$. Therefore, $i\Big(R_{\mathbf{q}_1}(A)\Big) = i\Big(R_{\mathbf{q}_0}(A)\Big) = 0$. Hence, it follows $\mathbf{q}_0 \notin \sigma^S_{eap}(A)$. Thus, we obtain from the above that $\sigma^S_{eap}(A) = \sigma^S_{e1}(A)$.

The other two equalities follow in the same way. □

**Lemma 2.15.** *Let $A \in \mathcal{B}(V^R_{\mathbb{H}})$. If $0 \in \rho^S(A)$ and $\mathbf{q} \neq 0$, then $R_{\mathbf{q}}(A) \in \Phi(V^R_{\mathbb{H}})$ if and only if $R_{\mathbf{q}^{-1}}(A^{-1}) \in \Phi(V^R_{\mathbb{H}})$ and*

$$i(R_{\mathbf{q}}(A)) = i(R_{\mathbf{q}^{-1}}(A^{-1}))$$

*Proof.* (*a*) For all $\mathbf{q} \neq 0$, we note that

$$R_{\mathbf{q}}(A) = |\mathbf{q}|^2\Big(R_{\mathbf{q}^{-1}}(A^{-1})\Big)A^2. \tag{1}$$

Now, let us suppose that $|\mathbf{q}|^2\Big(R_{\mathbf{q}^{-1}}(A^{-1})\Big)A^2 \in \Phi(V^R_{\mathbb{H}})$. Since $0 \in \rho^S(A)$ Now, by applying Corollary 2.10, we infer that $R_{\mathbf{q}}(A) \in \Phi(V^R_{\mathbb{H}})$. Conversely, assume that $R_{\mathbf{q}}(A) \in \Phi(V^R_{\mathbb{H}})$. Then, the product on the right-hand side of Eq. (1) is in $\Phi(V^R_{\mathbb{H}})$. Besides, $0 \in \rho^S(A)$ implies that $A^2 \in \Phi(V^R_{\mathbb{H}})$, then $|\mathbf{q}|^2\Big(R_{\mathbf{q}^{-1}}(A^{-1})\Big)A^2 \in \Phi(V^R_{\mathbb{H}})$. Then, $R_{\mathbf{q}^{-1}}(A^{-1}) \in \Phi(V^R_{\mathbb{H}})$.

By using Corollary 2.10 and Eq. (1), we get the following result

$$\begin{aligned} i\Big(R_{\mathbf{q}}(A)\Big) &= i\Big(|\mathbf{q}|^2\Big(R_{\mathbf{q}^{-1}}(A^{-1})\Big)A^2\Big) \\ &= i\Big((R_{\mathbf{q}^{-1}}(A^{-1}))\Big) + i\Big(A^2\Big) \\ &= i\Big((R_{\mathbf{q}^{-1}}(A^{-1}))\Big). \end{aligned}$$

□

**Theorem 2.16.** *Let $T_1, T_2 \in \mathcal{B}(V^R_{\mathbb{H}})$. If $\mathbf{q} \in \rho^S(T_1) \cap \rho^S(T_2)\backslash\Big\{0\Big\}$ and $R_{\mathbf{q}^{-1}}(T_2^{-1}) - R_{\mathbf{q}^{-1}}(T_1^{-1}) \in \mathcal{B}_0(V^R_{\mathbb{H}})$, then*

(*i*) *$\sigma^S_{eap}(T_1) = \sigma^S_{eap}(T_2)$.*

(*ii*) *$\sigma^S_{e\delta}(T_1) = \sigma^S_{e\delta}(T_2)$.*

*Proof.* (*i*) Let $0 \in \rho^S(T_1)$, then $R_{\mathbf{q}}(T_1) = |\mathbf{q}|^2\Big(R_{\mathbf{q}^{-1}}(T_1^{-1})\Big)T_1^2$. Implies that,

$$\begin{cases} N(R_{\mathbf{q}}(T_1)) = N(R_{\mathbf{q}^{-1}}(T_1^{-1})) \\ R(R_{\mathbf{q}}(T_1)) = R(R_{\mathbf{q}^{-1}}(T_1^{-1})) \end{cases}$$

hence ensuring that $\mathbf{q} \in \sigma^S_{e1}(T_1)$ equivalent to $R_{\mathbf{q}}(T_1)$ is upper semi-Fredholm if, and only if, $R_{\mathbf{q}^{-1}}(T_1^{-1})$ is upper semi-Fredholm if, and only if, $\mathbf{q}^{-1} \in \sigma^S_{e1}(T_1^{-1})$. Since $R_{\mathbf{q}^{-1}}(T_2^{-1}) - R_{\mathbf{q}^{-1}}(T_1^{-1}) \in \mathcal{B}_0(V^R_{\mathbb{H}})$ and applying Theorem 2.7 we have $R_{\mathbf{q}^{-1}}(T_1^{-1}) + R_{\mathbf{q}^{-1}}(T_2^{-1}) - R_{\mathbf{q}^{-1}}(T_1^{-1}) = R_{\mathbf{q}^{-1}}(T_2^{-1})$ is upper semi-Fredholm. We obtain that $\mathbf{q}^{-1} \in \sigma^S_{e1}(T_2^{-1})$ which is equivalent to say that $\mathbf{q} \in \sigma^S_{e1}(T_2)$.

Now, in order to show that $i\Big(R_{\mathbf{q}}(T_1)\Big) = i\Big(R_{\mathbf{q}}(T_2)\Big)$. Since $R_{\mathbf{q}^{-1}}(T_2^{-1}) - R_{\mathbf{q}^{-1}}(T_1^{-1}) \in \mathcal{B}_0(V^R_{\mathbb{H}})$ and by using Lemma 2.15 and Theorem 2.7 we get

$$\begin{aligned} i\Big(R_{\mathbf{q}}(T_1)\Big) &= i\Big(R_{\mathbf{q}^{-1}}(T_1^{-1})\Big) \\ &= i\Big(R_{\mathbf{q}^{-1}}(T_1^{-1}) + R_{\mathbf{q}^{-1}}(T_2^{-1}) - R_{\mathbf{q}^{-1}}(T_1^{-1})\Big) \\ &= i\Big(R_{\mathbf{q}^{-1}}(T_2^{-1})\Big) \\ &= i\Big(R_{\mathbf{q}}(T_2)\Big). \end{aligned}$$

(*ii*) This assertion follows, immediately, from (*i*). □

## 3. Stability of Essential S-Spectra

The reader interested in the results of this section may also refer to [10], which constitutes the real basis of our work. The purpose of this section, is to present the following useful stability of essential spectra.

**Theorem 3.1.** *Let $A, B \in \mathcal{B}(V_{\mathbb{H}}^R)$ where $A$ commutes with $B$. If every $R_{\mathbf{q}}(A)$ is Fredholm, there exists $A_{\lambda l}$ (resp. $A_{\lambda r}$) a left (resp. right) inverse modulo compact operator of $R_{\mathbf{q}}(A)$ such that If $R_{\mathbf{q}}(B)A_{\lambda l}, AB + BA - |\mathbf{q}|^2 \in \mathcal{B}_0(V_{\mathbb{H}}^R)$ (resp. $A_{\lambda r}R_{\mathbf{q}}(B), AB + BA - |\mathbf{q}|^2 \in \mathcal{B}_0(V_{\mathbb{H}}^R)$), then*

$$\sigma_{ei}^S(A + B) \subseteq \sigma_{ei}^S(A) \text{ where } i \in \{1, 2, eap, e\delta\}.$$

*Proof.* Let $\mathbf{q} \in \mathbb{H}$, if $A_{\lambda l}$ is a left inverse modulo compact operators of $R_{\mathbf{q}}(A)$, then there exists $K \in \mathcal{B}_0(V_{\mathbb{H}}^R)$ such that $A_{\lambda l}R_{\mathbf{q}}(A) = I - K$, thus, we can write

$$R_{\mathbf{q}}(A + B) = (I + R_{\mathbf{q}}(B)A_{\lambda l})R_{\mathbf{q}}(A) + R_{\mathbf{q}}(B)K + AB + BA - |q|^2.$$

In the same way, if there exists $A_{\lambda r}$), a right inverse modulo compact operators of $R_{\mathbf{q}}(A)$, we can write

$$R_{\mathbf{q}}(A + B) = R_{\mathbf{q}}(A)(I + A_{\lambda r}R_{\mathbf{q}}(B)) + K' R_{\mathbf{q}}(B) + BA + AB - |q|^2,$$

where $K' \in \mathcal{B}_0(V_{\mathbb{H}}^R)$. Let $\mathbf{q} \notin \sigma_{eap}^S(A)$, then $R_{\mathbf{q}}(A)$ is upper semi-Fredholm and $i(R_{\mathbf{q}}(A)) \leq 0$. Set $A_{\lambda l}$ (resp. $A_{\lambda r}$) be a left (resp. right) inverse modulo compact operators of $R_{\mathbf{q}}(A)$. Since $R_{\mathbf{q}}(B)A_{\lambda l} \in \mathcal{B}_0(V_{\mathbb{H}}^R)$ (resp. $A_{\lambda r}R_{\mathbf{q}}(B) \in \mathcal{B}_0(V_{\mathbb{H}}^R)$), [11, Remark 6.9.] implies that $(I + R_{\mathbf{q}}(B)A_{\lambda l})$ is upper semi-Fredholm (resp. $(I + A_{\lambda r}R_{\mathbf{q}}(B))$ is upper semi-Fredholm) thus applying Theorem 2.8 , we get $(I + R_{\mathbf{q}}(B)A_{\lambda l})R_{\mathbf{q}}(A)$ is upper semi-Fredholm (resp. $R_{\mathbf{q}}(A)(I + A_{\lambda r}R_{\mathbf{q}}(B))$ is upper semi-Fredholm). Since $R_{\mathbf{q}}(B)K + AB + BA - |q|^2 \in \mathcal{B}_0(V_{\mathbb{H}}^R)$ (resp. $K' R_{\mathbf{q}}(B) + AB + BA - |q|^2 \in \mathcal{B}_0(V_{\mathbb{H}}^R)$), hence $R_{\mathbf{q}}(A + B)$ is upper semi-Fredholm. We conclude that $\sigma_{e1}^S(A + B) \subseteq \sigma_{e1}^S(A)$. Furthermore, by using Theorem 2.8 and Theorem 2.7 we have

$$i(R_{\mathbf{q}}(A + B)) = i(R_{\mathbf{q}}(A)).$$

Take the same approach to find

$$\sigma_{ei}^S(A + B) \subseteq \sigma_{ei}^S(A) \text{ where } i \in \{2, eap, e\delta\}. \quad \square$$

The following theorem shows the relation between the essential spectra of the sum of the two bounded linear operators and the essential spectra, where their products are Fredholm or semi-Fredholm perturbations

**Theorem 3.2.** *Let $A, B \in \mathcal{B}(V_{\mathbb{H}}^R)$.*

*(i) If $AB, BA \in \mathcal{B}_0(V_{\mathbb{H}}^R)$, then $\sigma_{e1}^S(A + B)\backslash\{0\} = \left[\sigma_{e1}^S(A) \cup \sigma_{e1}^S(B)\right]\backslash\{0\}$. Moreover, if $\mathbb{H}\backslash\sigma_{e1}^S(A)$ is connected, then*

$$\sigma_{eap}^S(A + B)\backslash\{0\} = \left[\sigma_{eap}^S(A) \cup \sigma_{eap}^S(B)\right]\backslash\{0\}.$$

*(ii) If $AB, BA \in \mathcal{B}_0(V_{\mathbb{H}}^R)$, then $\sigma_{e2}^S(A + B)\backslash\{0\} = \left[\sigma_{e2}^S(A) \cup \sigma_{e2}^S(B)\right]\backslash\{0\}$. Moreover, if $\mathbb{H}\backslash\sigma_{e2}^S(A)$ is connected, then*

$$\sigma_{e\delta}^S(A + B)\backslash\{0\} = \left[\sigma_{e\delta}^S(A) \cup \sigma_{e\delta}^S(B)\right]\backslash\{0\}.$$

*Proof.* (*i*) For $\mathbf{q} \in \mathbb{H}$, we can write

$$R_{\mathbf{q}}(A + B) = A^2 + AB + BA + B^2 - Re(\mathbf{q})A - Re(\mathbf{q})B + |q|^2. \tag{2}$$

Now, we have the equation:

$$\begin{aligned} R_{\mathbf{q}}(A)R_{\mathbf{q}}(B) &= \left(A^2 - Re(\mathbf{q})A + |q|^2\right)\left(B^2 - Re(\mathbf{q})B + |q|^2\right) \\ &= A^2B^2 - Re(\mathbf{q})A^2B + A^2|q|^2 - Re(\mathbf{q})AB^2 + \left(Re(\mathbf{q})\right)^2 AB \\ &\quad -Re(\mathbf{q})A|q|^2 + B^2|q|^2 - Re(\mathbf{q})B|q|^2 + |q|^4 \\ &= A^2|q|^2 + B^2|q|^2 - Re(\mathbf{q})A|q|^2 - Re(\mathbf{q})B|q|^2 + |q|^4 \\ &\quad +A^2B^2 - Re(\mathbf{q})A^2B - Re(\mathbf{q})AB^2 + \left(Re(\mathbf{q})\right)^2 AB. \end{aligned}$$

From equation (2), it is easy to see,

$$R_{\mathbf{q}}(A)R_{\mathbf{q}}(B) = R_{\mathbf{q}}(A+B)|q|^2 - AB - BA + A^2B^2 - Re(\mathbf{q})A^2B - Re(\mathbf{q})AB^2 + \left(Re(\mathbf{q})\right)^2 AB, \tag{3}$$

and

$$R_{\mathbf{q}}(B)R_{\mathbf{q}}(A) = R_{\mathbf{q}}(B+A)|q|^2 - BA - AB + B^2A^2 - Re(\mathbf{q})B^2BA - Re(\mathbf{q})BA^2 + \left(Re(\mathbf{q})\right)^2 BA. \tag{4}$$

Let $\mathbf{q} \notin \left[\sigma^S_{ei}(A) \cup \sigma^S_{ei}(B)\right]\backslash\left\{0\right\}$. Then, $R_{\mathbf{q}}(A)$ is Fredholm and $R_{\mathbf{q}}(B)$ is Fredholm. Theorem 2.8 ensures that $R_{\mathbf{q}}(A)R_{\mathbf{q}}(A)$ is Fredholm. Since $AB, BA \in \mathcal{B}_0(V^R_{\mathbb{H}})$, then by using Proposition 2.9 we get $-AB - BA + A^2B^2 - Re(\mathbf{q})A^2B - Re(\mathbf{q})AB^2 + \left(Re(\mathbf{q})\right)^2 AB \in \mathcal{B}_0(V^R_{\mathbb{H}})$, and applying Eq. (3), we have $R_{\mathbf{q}}(A+B)$ is Fredholm. Hence $\mathbf{q} \notin \sigma^S_{e4}(A+B)$, and we obtain

$$\sigma^S_{e1}(A+B)\backslash\left\{0\right\} \subset \left[\sigma^S_{e1}(A) \cup \sigma^S_{e1}(B)\right]\backslash\left\{0\right\}.$$

Let $\mathbf{q} \notin \left[\sigma^S_{eap}(A) \cup \sigma^S_{eap}(B)\right]\backslash\left\{0\right\}$, then by Corollary 2.13 we have $R_{\mathbf{q}}(A), R_{\mathbf{q}}(B)$ are upper semi-Fredholm and $i\left(R_{\mathbf{q}}(A)\right) = \left(R_{\mathbf{q}}(B)\right) \leq 0$. Therefore, Theorem 2.8 gives $R_{\mathbf{q}}(A)R_{\mathbf{q}}(B)$ is upper semi-Fredholm and $i\left(R_{\mathbf{q}}(A)R_{\mathbf{q}}(B)\right) \leq 0$. Moreover, since $AB, BA \in \mathcal{B}_0(V^R_{\mathbb{H}})$, we can apply both Eq. (3) and Proposition 2.9, hence ensuring that $R_{\mathbf{q}}(A+B)$ is upper semi-Fredholm and $i\left(R_{\mathbf{q}}(A+B)\right) \leq 0$. Again, by applying Theorem 2.8, we infer that $\mathbf{q} \notin \sigma^S_{eap}(A+B)$ and, then

$$\sigma^S_{eap}(A+B)\backslash\left\{0\right\} \subset \left[\sigma^S_{eap}(A) \cup \sigma^S_{eap}(B)\right]\backslash\left\{0\right\}.$$

Let us suppose that $\mathbf{q} \notin \sigma^S_{e1}(A+B)$. Then, $R_{\mathbf{q}}(A+B)$ is upper semi-Fredholm. Since $AB \in \mathcal{B}_0(V^R_{\mathbb{H}})$ and $BA \in \mathcal{B}_0(V^R_{\mathbb{H}})$, then by using Eqs. (3) and (4), we have

$$R_{\mathbf{q}}(A)R_{\mathbf{q}}(B) \text{ and } R_{\mathbf{q}}(B)R_{\mathbf{q}}(A) \text{are upper semi-Fredholm.}$$

Corollary 2.10 show clearly that $R_{\mathbf{q}}(A)$ is upper semi-Fredholm and $R_{\mathbf{q}}(B)$ is upper semi-Fredholm. We conclude that $\mathbf{q} \notin \sigma^S_{e1}(A) \cup \sigma^S_{e1}(B)$. Hence,

$$\sigma^S_{e1}(A+B)\backslash\left\{0\right\} = \left[\sigma^S_{e1}(A) \cup \sigma^S_{e1}(B)\right]\backslash\left\{0\right\}.$$

Then, from Proposition 2.14 we deduce that

$$\sigma^S_{eap}(A+B)\backslash\left\{0\right\} = \left[\sigma^S_{eap}(A) \cup \sigma^S_{eap}(B)\right]\backslash\left\{0\right\}.$$

Statement (*ii*) can be checked in the same way as (*i*). □

## 4. Quaternionic Quasi-Compact Operators

A systematic study of quaternionic quasi-compact operators has not appeared in the literature. In this regard, in this section, as needed for our purpose, we provide certain significant results about compact right linear operators on $V_{\mathbb{H}}^{R}$.
This first definition was given by Yosida (1939) he called them operators almost completely continuous [16].

**Definition 4.1.** *An operator $T \in V_{\mathbb{H}}^{R}$ is said to be quasi-compact operator if there exists a compact operator $K$ and an integer $m$ such that $\|T^m - K\| < 1$.*

This definition is equivalent to the following: there exists a sequence $(K_n)_n$ of compact operators on $V_{\mathbb{H}}^{R}$ such that $\lim \|T^n - K_n\| = 0$.
It is obvious that any compact operator is almost compact. We note the set of quasi-compact operators by $QK(V_{\mathbb{H}}^{R})$. In what follows we show the equivalence of the definitions of quasi-compactness known in the mathematical literature. We refer the reader to [9] for a detailed presentation of the quasi-compactness.

**Remark 4.2.** *If $T_1$ and $T_2$ are quasi-compact, $T_1 + T_2$ is not generally quasi-compact. Likewise if $T$ is quasi-compact and $A$ is a bounded operator, we do not necessarily have $TA$ or $AT$ quasi-compact. To illustrate this, consider the following operators in $l^2(\mathbb{H})$: the space of complex sequences with summable square.*

$$T_1(x_1, x_2, x_3, \cdots) = (x_2, 0, x_4, 0, x_6, \cdots)$$

*and*

$$T_2(x_1, x_2, x_3, \cdots) = (0, x_3, 0, x_5, 0, x_7, 0, \cdots)$$

*$T_1$ and $T_2$ are almost compact since $T_1^2 = T_2^2 = 0$. But $T_1 + T_2 = (x_2, x_3, x_4, \cdots)$ and $T_1T_2 = (x_3, 0, x_5, 0, x_7, \cdots)$ are not quasi-compact since their point spectrum is the set $\{\mathbf{q} \in \mathbb{H} : |\mathbf{q}| < 1\}$. Still in the same space $l^2(\mathbb{H})$ consider the operators*

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, \cdots)$$

*and*

$$A(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots)$$

*$T$ is almost compact since $T^2 = 0$ but $AT$ is not almost compact ($AT = I$). Consequently the $QK_{\mathbb{H}}^{R}$ set is not closed for the addition and multiplication of operators.*

**Theorem 4.3.** *If $A \in QK(V_{\mathbb{H}}^{R})$, then for all quaternion number $\mathbf{q}$ such that $|\mathbf{q}| \geq 1$, then $(\mathbf{q} - A)$ is a Weyl operator.*

*Proof.* The proof is obtained in the same way of the proof of Theorem [2, Theorem I.6]. □

We will give a result on the stability of the essential S-spectra of a bounded linear operator under a quasi-compact perturbation. Let $A \in \mathcal{B}(V_{\mathbb{H}}^{R})$ define the sets:

$$\begin{aligned} O_A(V_{\mathbb{H}}^{R}) &= \left\{ K \in \mathcal{B}(V_{\mathbb{H}}^{R}) : \begin{array}{c} (AK + KA + K^2 - 2Re(\mathbf{q})K)R_{\mathbf{q}}(A+K)^{-1} \in QK(V_{\mathbb{H}}^{R}) \\ \text{for all } \mathbf{q} \notin \sigma^S(A+K) \end{array} \right\} \\ \mathcal{V}_A(V_{\mathbb{H}}^{R}) &= \left\{ K \in \mathcal{B}(V_{\mathbb{H}}^{R}) : \begin{array}{c} R_{\mathbf{q}}(A+K)^{-1}(AK + KA + K^2 - 2Re(\mathbf{q})K) \in QK(V_{\mathbb{H}}^{R}) \\ \text{for all } \mathbf{q} \notin \sigma^S(A+K) \end{array} \right\}. \end{aligned}$$

**Theorem 4.4.** *Let $A \in \mathcal{B}(V_{\mathbb{H}}^{R})$ with $\rho^S(A) \neq \emptyset$. Then,*

*(i)* $\sigma_{eap}^S(A) = \bigcap_{K \in O_A(V_{\mathbb{H}}^{R})} \sigma_{ap}^S(A+K) = \bigcap_{K \in \mathcal{V}_A(V_{\mathbb{H}}^{R})} \sigma_{ap}^S(A+K).$

*(ii)* $\sigma_{e\delta}^S(A) = \bigcap_{K \in O_A(V_{\mathbb{H}}^{R})} \sigma_{\delta}^S(A+K) = \bigcap_{K \in \mathcal{V}_A(V_{\mathbb{H}}^{R})} \sigma_{\delta}^S(A+K).$

*Proof.* (*i*) Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, show that $\sigma_{eap}^S(A) \subset \bigcap_{K \in \mathcal{O}_A(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K)$. Indeed, let $\mathbf{q} \notin \bigcap_{K \in \mathcal{O}_A(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K)$, then there exists $K \in \mathcal{O}_A(V_{\mathbb{H}}^R)$ such that $\mathbf{q} \in \rho_{ap}^S(A)$ and $(AK + KA + K^2 - 2Re(\mathbf{q})K)R_{\mathbf{q}}(A+K)^{-1} \in QK(V_{\mathbb{H}}^R)$. By using Theorem 4.3, we get $I - (AK + KA + K^2 - 2Re(\mathbf{q})K)R_{\mathbf{q}}(A+K)^{-1} \in \Phi_+(V_{\mathbb{H}}^R)$ and

$$i(I - (AK + KA + K^2 - 2Re(\mathbf{q})K)R_{\mathbf{q}}((A+K)^{-1}) \leq 0.$$

Using the following relation

$$R_{\mathbf{q}}(A) = \left(I - (AK + KA + K^2 - 2Re(\mathbf{q})K)R_{\mathbf{q}}(A+K)^{-1}\right)R_{\mathbf{q}}(A+K).$$

Hence, applying Theorem 2.8, we get $R_{\mathbf{q}}(A) \in \Phi_+(V_{\mathbb{H}}^R)$ and $i(R_{\mathbf{q}}(A)) \leq 0$. We then conclude that, $\mathbf{q} \notin \bigcap_{K \in \mathcal{O}_A(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K)$.

The other inclusion is a direct result of $\mathcal{B}_0(V_{\mathbb{H}}^R) \subset \mathcal{O}_A(V_{\mathbb{H}}^R)$. Note that since $\mathcal{B}_0(V_{\mathbb{H}}^R) \subset \mathcal{O}_A(V_{\mathbb{H}}^R)$ is the minimal subspace (in the sense of inclusion) for which the theorem 4.3 remains true.

Continuing in the same way, we can find $\sigma_{eap}^S(A) = \bigcap_{K \in \mathcal{V}_A(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K)$.

Statement (*ii*) can be checked in the same way as (*i*). □

**Corollary 4.5.** *Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $U(V_{\mathbb{H}}^R)$ and $Z(V_{\mathbb{H}}^R)$ be subspaces of $\mathcal{B}(V_{\mathbb{H}}^R)$ (not necessarily ideals). If $\mathcal{B}_0(V_{\mathbb{H}}^R) \subseteq Z(V_{\mathbb{H}}^R) \subseteq \mathcal{O}_A(V_{\mathbb{H}}^R)$, $\mathcal{B}_0(V_{\mathbb{H}}^R) \subseteq U(V_{\mathbb{H}}^R) \subseteq \mathcal{O}_A(V_{\mathbb{H}}^R)$, then*

(*i*) $e_{eap}^S(A) = \bigcap_{K \in Z(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K) = \bigcap_{K \in U(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K)$.

(*ii*) $e_{e\delta}^S(A) = \bigcap_{K \in Z(V_{\mathbb{H}}^R)} \sigma_{\delta}^S(A+K) = \bigcap_{K \in U(V_{\mathbb{H}}^R)} \sigma_{\delta}^S(A+K)$.

*Proof.* (*i*) Since $\mathcal{B}_0(V_{\mathbb{H}}^R) \subseteq Z(V_{\mathbb{H}}^R) \subseteq \mathcal{O}_A(V_{\mathbb{H}}^R)(X)$ we obtain

$$\bigcap_{K \in \mathcal{O}_A(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K) \subseteq \bigcap_{K \in Z(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K) \subseteq e_{eap}^S(A).$$

Applying Theorem 4.3, we get

$$e_{eap}^S(A) \subseteq \bigcap_{K \in \mathcal{O}_A(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K) \subseteq \bigcap_{K \in Z(V_{\mathbb{H}}^R)} \sigma_{ap}^S(A+K).$$

which completes the proof of the first assertion. For the second, it suffices to replace $Z(V_{\mathbb{H}}^R)$ by $U(V_{\mathbb{H}}^R)$.

Statement (*ii*) can be checked in the same way as (*i*). □

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