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Perfect fluid spacetimes obeying certain restrictions on the energy-momentum tensor

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Abstract. This paper is concerned with the study of a perfect fluid spacetime endowed with the various forms of the energy-momentum tensor T. We establish that a perfect fluid spacetime endowed with covariant constant energy-momentum tensor represents a dark matter era or the matter content is a perfect fluid spacetime with vanishing vorticity; whereas a perfect fluid spacetime endowed with Codazzi type of T represents a dark matter era or the expansion scalar vanishes, provided α_1 remains invariant under the velocity vector field ρ. Also, we show that a perfect fluid spacetime with pseudo-symmetric energy-momentum tensor represents a dark matter era or a phantom era, provided the velocity vector field annihilates the curvature tensor. Moreover, we characterize T-recurrent and weakly-T symmetric perfect fluid spacetimes with Killing velocity vector field and acquired that the perfect fluid spacetimes represent a radiation era in the first case and a stiff matter for the last one.

1. Introduction

This paper deals with 4-dimensional spacetimes (that is, a connected time-directed Lorentz manifolds) (M, g) whose Ricci tensor is of the form

$$S = \alpha_1.g + \alpha_2.A \otimes A, \tag{1}$$

where $\alpha_i \in C^\infty(M)$ and A is the 1-form metrically associated with a fixed unit (say future-directed) timelike vector field. Evidently, if $\alpha_2 \equiv 0$, then the contracted Bianchi identity and the equation (1) together imply that (M, g) is an Einstein manifold in that case, that is, α_1 is constant. In this sense, (1) generalizes the Einstein condition, which justifies spacetimes satisfying it, is sometimes called quasi-Einstein in the literature.

The energy–momentum tensor T performs a significant role as a matter content of the spacetime in general relativity (briefly, GR) and matter is assumed to be fluid having density, pressure and dynamical and kinematic quantities like velocity, acceleration, vorticity, shear and expansion. In GR theory, the fluid

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is termed perfect fluid since it does not have the heat conduction terms [15]. In a perfect fluid spacetime T is given by

$$T = (\nu + p)A \otimes A + p.g \tag{2}$$

where ν denotes the energy density, p indicates the isotropic pressure [21] and the unit timelike vector field ρ of the perfect fluid spacetime is defined by $g(Y, \rho) = A(Y)$, for any Y.

In absence of the cosmological constant in GR theory, the Einstein's field equations (briefly, EFE) is written as

$$S - \frac{r}{2}.g = k^2 T \tag{3}$$

where S is the Ricci tensor, $k = \sqrt{8\pi G}$, G indicates Newton's gravitational constant and the scalar curvature is denoted by r.

Combining the equations (1), (3) and (2), we infer that

$$\alpha_2 = k^2(p + \nu), \ \alpha_1 = -\frac{k^2(p - \nu)}{2}. \tag{4}$$

An important motivation for condition (1) arises from general relativistic cosmology, because it is satisfied for any solution (M, g) of the Einstein field equation for a perfect fluid:

$$S - \frac{r}{2}.g = (\nu + p)A \otimes A + p.g. \tag{5}$$

In this context, A describes the 4-velocity field. Of course, the equation (1) and the equation (5) would actually be equivalent if ν, p were entirely arbitrary, but in physical applications ν and p are not freely prescribed, but must satisfy certain physical constraints. In particular, they are not independent, but (5) must be supplemented by an equation of state $p = p(\nu)$. The most important, best known class of solutions of (5) are the Friedmann-Robertson-Walker spacetimes.

In [1], Alias, Romero and Sanchez introduced the notion of generalized Robertson-Walker (briefly, GRW) spacetimes. A GRW spacetime is a Lorentzian manifold M^n ($n \geq 3$) that may be expressed as $M = -I \times f^2 M^*$, where I being the open interval of the set of real numbers $\mathbb{R}$, $M^{*(n-1)}$ indicates the Riemannian manifold and $f > 0$ is a smooth function, named as scale factor or warping function. The GRW spacetime turns into Robertson-Walker (briefly, RW) spacetime when the dimension of M^* is three and of constant sectional curvature.

In [21], O'Neill established that a Robertson-Walker spacetime is a perfect fluid spacetime. Every 4-dimensional generalized Robertson-Walker spacetime is a perfect fluid spacetime if and only if it is a Robertson-Walker spacetime [14]. We refer([2], [3], [7], [8], [9], [17], [19]) and its references for more insights of the perfect fluid spacetimes.

In modern GR theory and cosmology, it is assumed that the expansion of the universe is accelerating. Moreover, p and ν are interconnected by an equation of the state of the form $p = p(\nu, T_0)$ type, where T_0 indicates the absolute temperature. However, we will spotlight the condition in which T_0 is constant. In this case, the state equation minimizes to $p = p(\nu)$ and the perfect fluid spacetime is termed as isentropic [15]. In addition, if $\nu = p$, the perfect fluid spacetime is named as stiff matter [24]. This era preceded the dark matter era with $p = -\nu$ whereas with $p = 0$, the dust matter era and the radiation era with $p = \frac{\nu}{3}$ [6]. The p and ν of the perfect fluid spacetime obey a simplified condition $p = w\nu$, where w is named the equation of state (briefly, EoS). The evolution of the expansion of the universe and the energy density are notably related to the EoS. We can notice distinguishable phases of the universe making use of EoS. The phase of the universe is accelerating if $w < -\frac{1}{3}$. It incorporates the phantom regime when $w < -1$ and when $-1 < w < 0$, it represents the quintessence phase.

In [25], Tamassy and Binh invented the concept of a Riemannian or semi-Riemannian manifold named as weakly Ricci symmetric manifold if S, the Ricci tensor (non-zero) of the manifold obeys

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + C(Z)S(X, Y), \tag{6}$$

where A, B, C indicate three 1-forms (simultaneously non-zero). The weakly Ricci symmetric manifold turns into a Ricci symmetric ($\nabla S = 0$) manifold if A = B = C = 0 holds and a pseudo Ricci-symmetric manifold if $A = 2A$, B = C = A holds on the manifold.

A Riemannian or semi-Riemannian manifold is named Ricci-recurrent [22] if S obeys

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z), \tag{7}$$

A being a non-zero 1-form.

If the energy-momentum tensor satisfies the equations (6) and (7), then it is named weakly-T symmetric and T-recurrent manifold, respectively, where T indicates energy-momentum tensor.

In 1996, spacetimes equipped with covariant constant energy-momentum tensor have investigated by Chaki and Ray [5]. After this, Sharma and Ghosh [23] have studied perfect fluid spacetime with Killing energy-momentum tensor. Then, De and Velimirović [10] explored spacetimes endowed with semi-symmetric energy-momentum tensor. Recently, Mallick et al.[17] have studied spacetimes with the different types of energy-momentum tensor. To the extent our insight goes, there are numerous outcomes in the literature regarding spacetimes endowed with energy-momentum tensor, however there are a few results in perfect fluid spacetimes. We intend to cover this gap and concentrate to characterize the perfect fluid spacetimes fulfilling certain restrictions on the T.

The content of the article is laid out as: In Section 2, we study perfect fluid spacetimes with covariant constant energy-momentum tensor. Section 3 concerns the investigation of perfect fluid spacetimes with Codazzi type of energy-momentum tensor. The perfect fluid spacetimes with pseudo-symmetric energy-momentum tensor are characterized in Section 4. We study the properties of T-recurrent and weakly-T symmetric perfect fluid spacetimes in the last two consecutive Sections.

2. Perfect fluid spacetimes with Covariant constant energy-momentum tensor

In [5], Chaki and Ray studied general relativistic spacetimes whose energy-momentum tensor is covariant constant. Here, we investigate perfect fluid spacetimes with Covariant constant energy-momentum tensor.

From EFE, we acquire

$$\nabla T = 0 \Leftrightarrow \nabla S = 0, \tag{8}$$

which entails that the scalar curvature r = constant. Contracting the equation (1), we get

$$r = 4\alpha_1 - \alpha_2. \tag{9}$$

Using the foregoing equations, we can say $4\alpha_1 - \alpha_2 = c_1$, a constant. Making use of the equation (4), we infer from the above that

$$k^2(\nu - 3p) = c_1. \tag{10}$$

If $c_1 = 0$, the previous equation yields

$$\omega = \frac{p}{\nu} = \frac{1}{3}, \tag{11}$$

which implies that the perfect fluid spacetime represents radiation era.

Proposition 2.1. *A perfect fluid spacetime with covariant constant energy-momentum tensor obeys the state equation $\nu = 3p+$ constant. In particular, if the constant vanishes, then the perfect fluid spacetime represents a radiation era.*

Since $\nabla S = 0$, then from the equation (1), we acquire

$$d\alpha_1(X)g(Y,Z) + d\alpha_2(X)A(Y)A(Z) \\ +\alpha_2[(\nabla_X A)YA(Z) + A(Y)(\nabla_X A)Z] = 0. \tag{12}$$

Contracting over Y and Z (taking a frame field), we get

$$4d\alpha_1(X) - d\alpha_2(X) = 0, \tag{13}$$

since $A(\rho) = -1$.

Putting $Y = \rho$ in (12) yields

$$d\alpha_1(X)A(Z) - d\alpha_2(X)A(Z) \\ -\alpha_2(\nabla_X A)Z = 0. \tag{14}$$

Again, putting $Z = \rho$ in the foregoing equation, we infer

$$-d\alpha_1(X) + d\alpha_2(X) = 0, \tag{15}$$

Comparing (13) and (15), we get α_1 = constant and hence from (13) α_2 = constant.

Therefore, (14) gives

$$\alpha_2(\nabla_X A)Z = 0, \tag{16}$$

which implies that either $\alpha_2 = 0$, or the 1-form A is closed.

$\alpha_2 = 0$ implies that $p + \nu = 0$ and A is closed entails that the velocity vector field ρ is irrotational. Therefore, the perfect fluid spacetime has zero vorticity. Hence, we write

Theorem 2.2. *A perfect fluid spacetime endowed with covariant constant energy-momentum tensor represents a dark matter era or the matter content is a perfect fluid spacetime with vanishing vorticity.*

The Weyl conformal curvature tensor, in dimension 4, is given by

$$\mathbf{C}(X,Y)Z = R(X,Y)Z - \frac{1}{2}[g(Y,Z)LX - g(X,Z)LY + S(Y,Z)X \\ -S(X,Z)Y - \frac{r}{3}\{g(Y,Z)X - g(X,Z)Y\}],$$

where R indicates the curvature tensor and L, the Ricci operator is defined by $g(LX,Y) = S(X,Y)$.

It is well circulated [12] that in dimension 4,

$$(div C)(X,Y)Z = \frac{1}{2}[\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} \\ +\frac{1}{6}\{dr(X)g(Y,Z) - dr(Y)g(X,Z)\}].$$

Again, we have

$$\nabla S = 0 \Rightarrow div\mathbf{C} = 0, \tag{17}$$

where "div" represents divergence.

The proposition 3.1 of [18] states that in an n-dimensional perfect fluid spacetime satisfying $div\mathbf{C} = 0$ is a generalized Robertson-Walker spacetime and conformal curvature tensor obeys the condition $\mathbf{C}(X,Y)\rho = 0$, provided the flow vector field ρ is irrotational. Also in [16], it is established that $\mathbf{C}(X,Y)\rho = 0$ implies $\mathbf{C} = 0$ for $n = 4$. Hence, the perfect fluid spacetime satisfying covariant constant energy-momentum tensor is a generalized Robertson-Walker spacetime. Again, "Every generalized Robertson-Walker spacetime with $n = 4$ is a perfect fluid spacetime if and only if it is a Robertson-Walker spacetime [14]" and thus the spacetime becomes Robertson-Walker spacetime. Robertson-Walker spacetimes with constant scalar curvature are described in [20]. From the foregoing discussion we conclude the subsequent result.

Theorem 2.3. *A perfect fluid spacetime with covariant constant energy-momentum tensor represents a Robertson-Walker spacetime.*

3. Perfect fluid spacetimes with Codazzi type of energy-momentum tensor

In [17], Mallick et al. established that a spacetime equipped with Codazzi type of energy-momentum tensor represents a Yang Pure Space. Here, we extend this condition in a perfect fluid spacetime and establish that the perfect fluid spacetime represents a Robertson-Walker spacetime with the same condition.

Suppose the energy-momentum tensor is of Codazzi type in a perfect fluid spacetime, that is,

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z). \tag{18}$$

Using the equation (3), we acquire

$$(\nabla_X S)(Y, Z) - \frac{dr(X)}{2} g(Y, Z) = k^2 (\nabla_X T)(Y, Z). \tag{19}$$

Making use of (18), from (19) we infer that

$$(\nabla_X S)(Y, Z) - \frac{dr(X)}{2} g(Y, Z) = (\nabla_Y S)(X, Z) - \frac{dr(Y)}{2} g(X, Z). \tag{20}$$

Contracting over X and Y (taking a frame field), we obtain

$$dr(Z) = 0, \tag{21}$$

which entails that r = constant. Hence, from (20), we get

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z), \tag{22}$$

which implies that S is of Codazzi type.

The converse is also true.

Guilfoyle and Nolan [13] defined "Yang Pure Space" as a Lorentzian manifold (M^4, g) and the metric tensor of the manifold solves Yang's equations:

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0.$$

Also, in the same paper the authors established that a perfect fluid spacetime with $p + \nu \neq 0$ is a Yang Pure Space if and only if the spacetime is a Robertson-Walker spacetime. Hence, we conclude that the perfect fluid spacetime with Codazzi type of energy-momentum tensor is a Robertson-Walker spacetime.

Theorem 3.1. *A perfect fluid spacetime with Codazzi type of energy-momentum tensor represents a Robertson-Walker spacetime.*

Now, using the equation (1), we have

$$\begin{aligned} & d\alpha_1(X) g(Y, Z) + d\alpha_2(X) A(Y) A(Z) \\ & + \alpha_2[(\nabla_X A) Y A(Z) + A(Y)(\nabla_X A) Z] \\ & = d\alpha_1(Y) g(X, Z) + d\alpha_2(Y) A(X) A(Z) \\ & + \alpha_2[(\nabla_Y A) X A(Z) + A(X)(\nabla_Y A) Z]. \end{aligned} \tag{23}$$

Taking a frame field and contracting over Y and Z , we acquire

$$\begin{aligned} & 4 d\alpha_1(X) - d\alpha_2(X) \\ & + \alpha_2[(\nabla_X A)\rho + A(Y)(\nabla_X A)\rho] \\ & = d\alpha_1(X) + d\alpha_2(\rho) A(X) \\ & + \alpha_2[(\nabla_\rho A) X + div\rho A(X)], \end{aligned} \tag{24}$$

where $div\rho$ denotes the divergence of ρ.

Replacing X by ρ, the foregoing equation yields

$$\begin{aligned}&3d\alpha_1(\rho) + \alpha_2[(\nabla_\rho A)\rho + (\nabla_\rho A)\rho]\\ &= \alpha_2[(\nabla_\rho A)\rho - div\rho].\end{aligned} \tag{25}$$

Since $(\nabla_\rho A)\rho = 0$, therefore previous equation gives

$$3d\alpha_1(\rho) = -\alpha_2 div\rho. \tag{26}$$

Let us suppose that the scalar α_1 remains invariant under the velocity vector field ρ. Then from the previous equation, we infer that

either $\alpha_2 = 0$, or, $div\rho = 0$.

Now, using (4) we say that either $p + \nu = 0$, or, the expansion scalar vanishes [21].

Hence, we state the subsequent result:

Theorem 3.2. *A perfect fluid spacetime endowed with Codazzi type of energy-momentum tensor represents dark matter era or the expansion scalar vanishes, provided α_1 remains invariant under the velocity vector field ρ.*

4. Perfect fluid spacetimes with Pseudo-symmetric energy-momentum tensor

The idea of a pseudo-symmetric manifold was presented in [26], which is different from Chaki's notion [4]. We define endomorphism $(X \wedge_g Y)$ by

$$(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{27}$$

where $X, Y, Z \in \chi(M)$.

We define tensors $R.R$, $R.S$, $Q(g, R)$ and $Q(g, S)$ by:

$$\begin{aligned}(R(X, Y).R)(U, V)W &= R(X, Y)R(U, V)W - R(R(X, Y)U, V)W\\ &\quad -R(U, R(X, Y)V)W - R(U, V)R(X, Y)W,\end{aligned} \tag{28}$$

$$(R(X, Y).S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V), \tag{29}$$

$$\begin{aligned}Q(g, R) &= (g(X, Y).R)(U, V)W\\ &= (X \wedge_g Y)R(U, V)W\\ &\quad -R((X \wedge_g Y)U, V)W - R(U, (X \wedge_g Y)V)W\\ &\quad -R(U, V)(X \wedge_g Y)W\end{aligned} \tag{30}$$

and

$$Q(g, S) = -S((X \wedge_g Y)U, V) - S(U, (X \wedge_g Y)V), \tag{31}$$

respectively, where $X, Y, U, V, W \in \chi(M)$.

A semi-Riemannian manifold is said to be pseudo-symmetric [26] if $R \cdot R$ and $Q(g, R)$ are linearly dependent at each point of the manifold. Hence, we have $R \cdot R = f_R Q(g, R)$ for some smooth function f_R. Also, a semi-Riemannian manifold is said to be Ricci pseudo-symmetric [26] if $R \cdot S = f_S Q(g, S)$, which holds on the set $U_S = \{x \in M : S \neq \frac{r}{n} g \ \ at \ \ x\}$, where f_S is some function on U_S. Every pseudo-symmetric manifold is Ricci pseudo-symmetric, but the converse statement is not true.

Recently, Mallick et al.[17] have investigated spacetimes with Pseudo-symmetric energy-momentum tensor, in the sense of Chaki [4]. But, in this section we choose Deszcz's notion of pseudo-symmetry.

First, we prove the subsequent Proposition:

Proposition 4.1. *A Ricci semi-symmetric perfect fluid spacetime represents a dark matter era or a phantom era.*

Proof. A semi-Riemannian manifold is called Ricci semi-symmetric if the Ricci tensor S obeys the relation

$$(R(X,Y))S(U,V) = 0, \tag{32}$$

which implies that

$$-S(R(X,Y)U,V) - S(U,R(X,Y)V) = 0. \tag{33}$$

Utilizing the equation (1) in the above equation gives

$$-\alpha_2[A(R(X,Y)U)A(V) + A(U)A(R(X,Y)V)] = 0. \tag{34}$$

Replacing V by ρ in (34) yields

$$\alpha_2 g(A(R(X,Y)U,\rho) = 0, \tag{35}$$

which implies that either $\alpha_2 = 0$ or $S(X,\rho) = 0$.

Now, using (4) we infer that either $p + \nu = 0$, or, $\frac{p}{\nu} = -\frac{1}{3}$.

Hence, it represents a dark matter era or a phantom era. □

We choose perfect fluid spacetimes with Pseudo-symmetric energy-momentum tensor. Therefore,

$$R \cdot T = fQ(g,T). \tag{36}$$

Operating R on (3), we get

$$R \cdot S = \kappa^2 R \cdot T = \kappa^2 fQ(g,T). \tag{37}$$

Now,

$$\kappa^2 Q(g,T) = -\kappa^2 T(g(Y,U)X - g(X,U)Y,V) - \kappa^2 T(U, g(Y,V)X - g(X,V)Y). \tag{38}$$

Using the equation (3) in the foregoing equation, we have

$$\begin{aligned}\kappa^2 Q(g,T) &= -g(Y,U)S(X,V) + g(X,U)S(Y,V)\\ &\quad -g(Y,V)S(X,U) + g(X,V)S(U,Y)\\ &= S(g(Y,U)X - g(X,U)Y,V) - S(U, g(Y,V)X - g(X,V)Y)\\ &= Q(g,S).\end{aligned} \tag{39}$$

Hence, from (37) and (39), we acquire

$$R \cdot S = fQ(g,S). \tag{40}$$

Making use of equation (1) in the previous equation, we infer

$$\begin{aligned}&-\alpha_2[A(R(X,Y)U)A(V) + A(U)A(R(X,Y)V)]\\ &= -f\{g(Y,U)S(X,V) - g(X,U)S(Y,V)\\ &+g(Y,V)S(X,U) - g(X,V)S(U,Y)\}.\end{aligned} \tag{41}$$

Setting $U = \rho$ in (41) yields

$$\begin{aligned}&\alpha_2 A(R(X,Y)V)\\ &= -f\{A(Y)S(X,V) - A(X)S(Y,V)\\ &+g(Y,V)S(X,\rho) - g(X,V)S(Y,\rho)\}.\end{aligned} \tag{42}$$

Now assume that the velocity vector field ρ is such that each point of the perfect fluid spacetime ρ annihilates the curvature tensor, that is, $R(X,Y)\rho = 0$, which entails that $A(R(X,Y)V) = 0$. If the velocity vector field ρ is parallel, then ρ obeys the foregoing equation.

Therefore, taking $Y = \rho$ in (42), we obtain, either $f = 0$ or, the spacetime is Einstein which is not possible, since it contradicts the definition of proper Ricci pseudo symmetric spacetime.

Hence, $R \cdot T = 0 \Rightarrow R \cdot S = 0$, that is, perfect fluid spacetime is Ricci symmetric.

Thus, using the previous proposition, we can write the subsequent theorem:

Theorem 4.2. *A perfect fluid spacetime with Pseudo-symmetric energy-momentum tensor represents dark matter era or the phantom era, provided the velocity vector field annihilates the curvature tensor.*

Remark 4.3. *The above Theorem is the generalization of a Theorem of De and Velimirović [10].*

5. *T*-recurrent perfect fluid spacetimes

Suppose the energy-momentum tensor is recurrent type in a perfect fluid spacetime, that is,

$$(\nabla_X T)(Y,Z) = A(X)T(Y,Z). \tag{43}$$

Utilizing (19), we get

$$\begin{aligned}&(\nabla_X S)(Y,Z) - \frac{dr(X)}{2}g(Y,Z)\\ &= A(X)S(Y,Z) - \frac{r}{2}A(X)g(Y,Z).\end{aligned} \tag{44}$$

Now, using the equation (1), we obtain

$$\begin{aligned}&d\alpha_1(X)g(Y,Z) + d\alpha_2(X)A(Y)A(Z)\\ &+\alpha_2[(\nabla_X A)YA(Z) + A(Y)(\nabla_X A)Z]\\ &-\frac{dr(X)}{2}g(Y,Z) = A(X)S(Y,Z) - \frac{r}{2}A(X)g(Y,Z).\end{aligned} \tag{45}$$

Contracting over the vector field Y and Z (taking a frame field), we acquire

$$\begin{aligned}&4d\alpha_1(X) - d\alpha_2(X) + 2\alpha_2(\nabla_X A)\rho - 2dr(X)\\ &= -(4\alpha_1 - \alpha_2)A(X).\end{aligned} \tag{46}$$

Also, if ρ is Killing, then we have [11], $\mathfrak{L}_\rho p = 0$ and $\mathfrak{L}_\rho \nu = 0$, where $\mathfrak{L}$ indicates the Lie derivative operator. We know that $\alpha_1 = -\frac{k^2(p-\nu)}{2}$ and $\alpha_2 = k^2(p+\nu)$. Hence, we get

$$d\alpha_1(\rho) = d\alpha_2(\rho) = 0. \tag{47}$$

Also, from (9) we have

$$r = 4\alpha_1 - \alpha_2.$$

Thus, from the above results, we infer $dr(\rho) = 0$.

Putting $X = \rho$ in (46) and utilizing (4) and (47) yields

$$3p - \nu = 0. \tag{48}$$

Hence, we state:

Theorem 5.1. *A T-recurrent perfect fluid spacetime represents radiation era, provided the velocity vector field is Killing.*

6. Weakly-*T* symmetric perfect fluid spacetimes

Suppose the energy-momentum tensor is of weakly symmetric type in a perfect fluid spacetime, that is,

$$(\nabla_X T)(Y,Z) = A(X)T(Y,Z) + \mathrm{B}(Y)T(X,Z) + \mathrm{C}(Z)T(Y,X). \tag{49}$$

Utilizing (19), we infer

$$\begin{aligned}(\nabla_X S)(Y,Z) - \frac{dr(X)}{2}g(Y,Z) &= A(X)S(Y,Z) - \frac{r}{2}A(X)g(Y,Z)\\ &\quad \mathrm{B}(Y)S(X,Z) - \frac{r}{2}\mathrm{B}(Y)g(X,Z)\\ &\quad \mathrm{C}(Z)S(Y,X) - \frac{r}{2}\mathrm{C}(Z)g(Y,X).\end{aligned} \tag{50}$$

Now, using the equation (1), we acquire

$$\begin{aligned}&d\alpha_1(X)g(Y,Z) + d\alpha_2(X)A(Y)A(Z)\\ &+\alpha_2[(\nabla_X A)YA(Z) + A(Y)(\nabla_X A)Z]\\ &-\frac{dr(X)}{2}g(Y,Z) = A(X)S(Y,Z) - \frac{r}{2}A(X)g(Y,Z)\\ &\mathrm{B}(Y)S(X,Z) - \frac{r}{2}\mathrm{B}(Y)g(X,Z)\\ &\mathrm{C}(Z)S(Y,X) - \frac{r}{2}\mathrm{C}(Z)g(Y,X).\end{aligned} \tag{51}$$

Contracting over the vector fields Y and Z (taking a frame field), we obtain

$$\begin{aligned}&4d\alpha_1(X) - d\alpha_2(X) + 2\alpha_2(\nabla_X A)\rho - 2dr(X)\\ &= (4\alpha_1 - \alpha_2)A(X) - 2rA(X) + 2S(X,\rho)\\ &-\frac{r}{2}\mathrm{B}(X) - \frac{r}{2}\mathrm{C}(X).\end{aligned} \tag{52}$$

Also, if ρ is Killing, then we have [11]

$$d\alpha_1(\rho) = d\alpha_2(\rho) = 0 \Rightarrow dr(\rho) = 0. \tag{53}$$

Putting $X = \rho$ in (52) and using (53) gives

$$\alpha_1 = 0 \Rightarrow (p - \nu) = 0. \tag{54}$$

Hence, we state:

Theorem 6.1. *A weakly-T symmetric perfect fluid spacetime represents stiff matter fluid, provided the velocity vector field is Killing.*

7. Discussion

In 1915, Albert Einstein introduced the concept of GR theory. Here, the gravitational field is considered as spacetime curvature and its prime resource is energy-momentum tensor. In GR theory, the matter content of the universe is stated by picking the suitable energy-momentum tensor and is accepted to act like a perfect fluid spacetime in the cosmological models. Here, EFE performs the fundamental part to build the cosmological model. Perfect fluid spacetime models in GR theory are of significant interest in a few spaces of plasma physics, astronomy, atomic physical science and nuclear physics.

This study is an augmentation of past works done on spacetimes by Chaki and Ray [5], Sharma and Ghosh [23], De and Velimirović [10] and Mallick, De and Suh [17]. We sum up this thought in the perfect fluid spacetimes and acquired several interesting results.

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