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A-numerical radius inequalities and A-translatable radii of semi-Hilbert space operators

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Abstract. We develop A-numerical radius inequalities of the product and the commutator of semi-Hilbert space operators using the notion of A-numerical radius distance and A-seminorm distance. Further, we introduce a pair of translatable radii of semi-Hilbert space operators in the direction of another operator and obtain related inequalities which generalize the relevant inequalities studied in the setting of Hilbert space.

1. Introduction and terminologies

Throughout this paper, $\mathcal{H}$ denotes a non trivial complex Hilbert space with inner product $\langle .,. \rangle$ and associated norm $\|\cdot\|$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. Let the symbol I stand for the identity operator on $\mathcal{H}$. For every operator $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ stand for the null space, the range and the closure of the range of T, respectively. The adjoint of T is denoted by T^*. Let $\mathcal{B}(\mathcal{H})^+$ be the cone of positive operators, i.e.,

$$\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}) : \langle Ax, x \rangle \geq 0, \forall x \in \mathcal{H}\}.$$

An operator $A \in \mathcal{B}(\mathcal{H})^+$ defines a positive semi-definite sesquilinear form

$$\langle .,. \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \ \ \langle x, y \rangle_A = \langle Ax, y \rangle, \ \forall x, y \in \mathcal{H}.$$

Naturally, this semi-inner product induces a seminorm $\|\cdot\|_A$ defined by

$$\|x\|_A = \sqrt{\langle x, x \rangle_A} = \left\|A^{\frac{1}{2}} x\right\|, \forall x \in \mathcal{H}.$$

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Observe that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. Then $\|\cdot\|_A$ is a norm on $\mathcal{H}$ if and only if A is an injective operator and the semi-normed space $(\mathcal{B}(\mathcal{H}), \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. Given $T \in \mathcal{B}(\mathcal{H})$, if there exists $c > 0$ satisfying $\|Tx\|_A \leq c\,\|x\|_A$ for all $x \in \overline{\mathcal{R}(A)}$, then

$$\|T\|_A = \sup_{\substack{x\in\overline{\mathcal{R}(A)}\\ x\neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x\in\overline{\mathcal{R}(A)}\\ \|x\|_A=1}} \|Tx\|_A < \infty.$$

From now on, we suppose that $A \neq 0$ is a positive operator in $\mathcal{B}(\mathcal{H})$ and we denote

$$\mathcal{B}^A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\|_A < \infty\}.$$

It can be seen that $\mathcal{B}^A(\mathcal{H})$ is not a subalgebra of $\mathcal{B}(\mathcal{H})$, and $\|T\|_A = 0$ if and only if $T^*AT = 0$. Moreover, for $T \in \mathcal{B}^A(\mathcal{H})$ we have

$$\|T\|_A = \sup\left\{\left|\langle Tx, y\rangle_A\right| : x, y \in \overline{\mathcal{R}(A)} \text{ and } \|x\|_A = \left\|y\right\|_A = 1\right\}.$$

For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called A-adjoint of T if for every $x, y \in \mathcal{H}$

$$\langle Tx, y\rangle_A = \langle x, Sy\rangle_A,$$

that is, $AS = T^*A$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called A-selfadjoint if AT is selfadjoint, i.e., $AT = T^*A$, and it is called A-positive if AT is positive.
The existence of A-adjoint operator is not guaranteed. The set of all operators which admit A-adjoints is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas theorem [10], we get

$$\begin{aligned}\mathcal{B}_A(\mathcal{H}) &= \{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}\\ &= \{T \in \mathcal{B}(\mathcal{H}) : \exists\, c > 0 \textit{ such that } \|ATx\| \leq c\,\|Ax\|, \forall x \in \mathcal{H}\}.\end{aligned}$$

If $T \in \mathcal{B}_A(\mathcal{H})$, then T admits an A-adjoint operator. Moreover, there exists a distinguished A-adjoint operator of T, namely, the reduced solution of the equation $AX = T^*A$, i.e., $T^{\sharp_A} = A^{\dagger}T^*A$, where $A^{\dagger}$ is the Moore-Penrose inverse of A. The A-adjoint operator $T^{\sharp_A}$ satisfies

$$AT^{\sharp_A} = T^*A, \mathcal{R}\left(T^{\sharp_A}\right) \subseteq \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}\left(T^{\sharp_A}\right) = \mathcal{N}(T^*A).$$

Again, by applying Douglas theorem [10], we can see that

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \exists\, c > 0 \textit{ such that } \|Tx\|_A \leq c\,\|x\|_A, \forall x \in \mathcal{H}\}.$$

Any operator in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ is called A-bounded operator. Moreover, it was proved in [2] that if $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, then

$$\|T\|_A := \sup_{x\notin\mathcal{N}(\mathcal{A})} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{x\in\mathcal{H}, \|x\|_A=1} \|Tx\|_A.$$

In addition, if T is A-bounded, then $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ and

$$\|Tx\|_A \leq \|T\|_A \|x\|_A, \forall x \in \mathcal{H}.$$

Note that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$ which are neither closed nor dense in $\mathcal{B}(\mathcal{H})$ (see [2, 3]). Moreover, the following inclusions

$$\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathcal{B}^A(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}),$$

hold with equality if A is injective and has a closed range.
Now, we collect some properties of $T^{\sharp_A}$ and its relationship with the seminorm $\|\cdot\|_A$. Let $T \in \mathcal{B}_A(\mathcal{H})$, then the following statements hold:

(1) If $AT = TA$, then $T^{\sharp_A} = P_{\overline{\mathfrak{R}(A)}}T^*$.

(2) $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$, $\left(T^{\sharp_A}\right)^{\sharp_A} = P_{\overline{\mathfrak{R}(A)}}TP_{\overline{\mathfrak{R}(A)}}$ and $\left(\left(T^{\sharp_A}\right)^{\sharp_A}\right)^{\sharp_A} = T^{\sharp_A}$.

(3) $T^{\sharp_A}T$ and $TT^{\sharp_A}$ are A-selfadjoint and A-positive.

(4) If $S \in \mathcal{B}_A(\mathcal{H})$, then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$.

(5) $\|T\|_A = \left\|T^{\sharp_A}\right\|_A = \left\|T^{\sharp_A}T\right\|_A^{\frac{1}{2}} = \left\|TT^{\sharp_A}\right\|_A^{\frac{1}{2}}$.

Note that $P_{\overline{\mathfrak{R}(A)}}$ stands for the projection onto $\overline{\mathfrak{R}(A)}$ and henceforth we write P instead of $P_{\overline{\mathfrak{R}(A)}}$ for simplicity. The concept of the classical numerical radius was generalized to the A-numerical radius (see in [23]) as follows:

$$\omega_A(T) = \sup\left\{|\langle Tx, x\rangle_A| : x \in \mathcal{H},\ \|x\|_A = 1\right\}.$$

It follows that

$$\omega_A(T) = \omega_A\left(T^{\sharp_A}\right) \text{ for any } T \in \mathcal{B}_A(\mathcal{H}).$$

It is well known that for any $T \in \mathcal{B}_A(\mathcal{H})$

$$T^{\sharp_A}P = PT^{\sharp_A} = T^{\sharp_A},$$

but $PT \neq TP$ and the equality holds if $\mathcal{N}(A)^{\perp}$ is invariant for T (see [22]).

A fundamental inequality for the A-numerical radius is the power inequality (see [14, 19]), which says that for $T \in \mathcal{B}_A(\mathcal{H})$,

$$\omega_A(T^n) \leq \omega_A^n(T),\ n \in \mathbb{N}.$$

Further, A-numerical radius $\omega_A(\cdot)$ is a seminorm on $\mathcal{B}_A(\mathcal{H})$, and it satisfies that

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A, \tag{1}$$

for every $T \in \mathcal{B}_A(\mathcal{H})$. Moreover, it is known that if T is A-selfadjoint, then

$$\omega_A(T) = \|T\|_A. \tag{2}$$

For proofs and more facts about A-numerical radius of operators, we refer the reader to [23, 24]. Any operator $T \in \mathcal{B}_A(\mathcal{H})$ can be represented as

$$T = \mathfrak{R}_A(T) + i\mathfrak{I}_A(T),$$

where

$$\mathfrak{R}_A(T) = \frac{T + T^{\sharp_A}}{2} \text{ and } \mathfrak{I}_A(T) = \frac{T - T^{\sharp_A}}{2i}.$$

Further, $\mathfrak{R}_A(T)$ and $\mathfrak{I}_A(T)$ are A-selfadjoint operators. In addition, we have

$$\left\|\mathfrak{R}_A(T)\right\|_A \leq \omega_A(T)$$

and

$$\left\|\mathfrak{I}_A(T)\right\|_A \leq \omega_A(T).$$

The A-Crawford number of $T \in \mathcal{B}_A(\mathcal{H})$ is defined as

$$c_A(T) = \inf\left\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\right\}.$$

Recently, several improvements of A-numerical radius inequalities are given in [17, 19, 24]. Further generalizations and refinements of A-numerical radius are discussed in [6–9, 13].

The paper is organized as follows. In section 2, we develop new inequalities for the A-numerical radius of the product and the commutator of operators acting on a semi-Hilbert space. In section 3, we introduce a pair of translatable radii of a semi-Hilbert space operator in the direction of another operator, and develop related inequalities.

2. Inequalities involving A-seminorm distance and A -numerical radius distance

In this section, we generalize and refine some inequalities involving A-seminorm distance and A-numerical radius distance of semi-Hilbert space operators. To give our first result, we need the following lemma.

Lemma 2.1. *If $x, y \in \mathcal{H}$ with $y \neq 0$, then*

$$\inf_{\lambda\in\mathbb{C}} \left\|x - \lambda y\right\|_A^2 = \frac{\|x\|_A^2 \left\|y\right\|_A^2 - \left|\langle x, y\rangle_A\right|^2}{\left\|y\right\|_A^2}.$$

Proof. First we note the following identity which can be found in [12]:

$$\inf_{\lambda\in\mathbb{C}} \|a - \lambda b\|^2 = \frac{\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2}{\|b\|^2},$$

for any $a, b \in \mathcal{H}$ with $b \neq 0$.
Choosing $a = A^{\frac{1}{2}} x$ and $b = A^{\frac{1}{2}} y$ in the above identity, we get

$$\inf_{\lambda\in\mathbb{C}} \left\|A^{\frac{1}{2}} (x - \lambda y)\right\|^2 = \frac{\left\|A^{\frac{1}{2}} x\right\|^2 \left\|A^{\frac{1}{2}} y\right\|^2 - \left|\left\langle A^{\frac{1}{2}} x, A^{\frac{1}{2}} y\right\rangle\right|^2}{\left\|A^{\frac{1}{2}} y\right\|^2}.$$

This implies that

$$\inf_{\lambda\in\mathbb{C}} \left\|x - \lambda y\right\|_A^2 = \frac{\|x\|_A^2 \left\|y\right\|_A^2 - \left|\langle x, y\rangle_A\right|^2}{\left\|y\right\|_A^2},$$

as required. □

For our next result we need the notion of A-seminorm distance. For $T \in B_A(\mathcal{H})$, let $D_A(T)$ denote the A-seminorm distance of T from the scalar operators, i.e.,

$$D_A(T) = \inf_{\lambda\in\mathbb{C}} \|T - \lambda I\|_A.$$

By using the A-seminorm distance $D_A(T)$, we prove the following inequalities.

Theorem 2.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\sqrt{D_A^2(T) + c_A^2(T)} \leq \|T\|_A \leq \sqrt{D_A^2(T) + \omega_A^2(T)}.$$

Proof. Let $x \in \mathcal{H}$ be an A-unit vector, i.e, $\|x\|_A = 1$. In view of Lemma 2.1, we observe that

$$\begin{aligned}\inf_{\lambda\in\mathbb{C}} \|Tx - \lambda x\|_A^2 &= \frac{\|Tx\|_A^2 \|\lambda x\|_A^2 - |\langle Tx, \lambda x\rangle_A|^2}{\|\lambda x\|_A^2} \\ &= \|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 \\ &\leq \|T\|_A^2 - c_A^2(T).\end{aligned}$$

Therefore,

$$\inf_{\lambda\in\mathbb{C}} \|Tx - \lambda x\|_A^2 \leq \|T\|_A^2 - c_A^2(T).$$

By taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, it follows that

$$D_A^2(T) + c_A^2(T) \leq \|T\|_A^2,$$

which gives the first inequality of the theorem. Now we prove the second inequality. From Lemma 2.1, we have

$$\|x\|_A^2 \left\|y\right\|_A^2 - \left|\left\langle x, y\right\rangle_A\right|^2 = \left\|y\right\|_A^2 \inf_{\lambda\in\mathbb{C}} \left\|x - \lambda y\right\|_A^2. \tag{3}$$

Now, replacing x by Tx and y by x in the identity (3), we obtain that

$$\|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 = \inf_{\lambda\in\mathbb{C}} \|Tx - \lambda x\|_A^2.$$

This implies that

$$\|Tx\|_A^2 = \inf_{\lambda\in\mathbb{C}} \|Tx - \lambda x\|_A^2 + |\langle Tx, x\rangle_A|^2.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality, we get

$$\|T\|_A^2 \leq \inf_{\lambda\in\mathbb{C}} \|T - \lambda I\|_A^2 + \omega_A^2(T) = D_A^2(T) + \omega_A^2(T).$$

This completes the proof. □

Remark 2.3. *The inequality in [18, Th. 2.2] follows from the first inequality of Theorem 2.2 by considering the identity operator instead of A.*

The following lemma plays a crucial role in our next proof.

Lemma 2.4. *Let $x, y, z \in \mathcal{H}$ and let $\lambda, \mu \in \mathbb{C}$. Then*

$$\left|\langle x, z\rangle_A \langle y, z\rangle_A\right| \leq \left|\langle x, y\rangle_A\right| + \inf_{\lambda\in\mathbb{C}} \|x - \lambda z\|_A \inf_{\mu\in\mathbb{C}} \left\|y - \mu z\right\|_A.$$

Proof. On account of [11], we have

$$|\langle a, c\rangle \langle b, c\rangle| \leq |\langle a, b\rangle| + \inf_{\lambda\in\mathbb{C}} \|a - \lambda c\| \inf_{\mu\in\mathbb{C}} \left\|b - \mu c\right\|.$$

for all $a, b, c \in \mathcal{H}$ and for every $\lambda, \mu \in \mathbb{C}$.
Choosing $a = A^{\frac{1}{2}}x$, $b = A^{\frac{1}{2}}y$ and $c = A^{\frac{1}{2}}z$ in the above inequality, we obtain

$$\begin{aligned} & \left|\left\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}z\right\rangle \left\langle A^{\frac{1}{2}}y, A^{\frac{1}{2}}z\right\rangle\right| \\ \leq\ & \left|\left\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y\right\rangle\right| + \inf_{\lambda\in\mathbb{C}} \left\|A^{\frac{1}{2}}x - \lambda A^{\frac{1}{2}}z\right\| \inf_{\mu\in\mathbb{C}} \left\|A^{\frac{1}{2}}y - \mu A^{\frac{1}{2}}z\right\|. \end{aligned}$$

This implies that

$$\left|\langle Ax, z\rangle \langle Ay, z\rangle\right| \leq \left|\langle Ax, y\rangle\right| + \inf_{\lambda\in\mathbb{C}} \left\|A^{\frac{1}{2}}(x - \lambda z)\right\| \inf_{\mu\in\mathbb{C}} \left\|A^{\frac{1}{2}}(y - \mu z)\right\|.$$

Thus,

$$\left|\langle x, z\rangle_A \langle y, z\rangle_A\right| \leq \left|\langle x, y\rangle_A\right| + \inf_{\lambda\in\mathbb{C}} \|x - \lambda z\|_A \inf_{\mu\in\mathbb{C}} \left\|y - \mu z\right\|_A,$$

as required. □

In [11], Dragomir proved that if $T \in \mathcal{B}(\mathcal{H})$, then

$$\omega^2(T) \leq \omega(T^2) + \inf_{\lambda \in \mathbb{C}} \|T - \lambda I\|. \tag{4}$$

The following theorem generalizes the inequality (4).

Theorem 2.5. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A^{2r}(T) \leq 2^{r-1} \left(\omega_A^r(T^2) + D_A^{2r}(T) \right),$$

for any $r \geq 1$.

Proof. Let $x \in \mathcal{H}$ be an A-unit vector in $\mathcal{H}$. Replacing x by Tx, y by $T^{\sharp_A}x$ and z by x in Lemma 2.4, we get

$$\left| \langle Tx, x \rangle_A \left\langle T^{\sharp_A} x, x \right\rangle_A \right| \leq \left| \left\langle Tx, T^{\sharp_A} x \right\rangle_A \right| + \inf_{\lambda \in \mathbb{C}} \|Tx - \lambda x\|_A \inf_{\mu \in \mathbb{C}} \left\| T^{\sharp_A} x - \mu x \right\|_A.$$

This implies that

$$|\langle Tx, x \rangle_A|^2 \leq \left| \left\langle T^2 x, x \right\rangle_A \right| + \|(T - \lambda I)x\|_A \left\| \left(T^{\sharp_A} - \mu I \right) x \right\|_A.$$

Now, by the elementary inequality $\left(\frac{\alpha+\beta}{2}\right)^r \leq \frac{\alpha^r+\beta^r}{2}$, $\alpha, \beta > 0$ and $r \geq 1$, we obtain

$$|\langle Tx, x \rangle_A|^{2r} \leq 2^{r-1} \left(\left| \left\langle T^2 x, x \right\rangle_A \right|^r + \|(T - \lambda I)x\|_A^r \left\| \left(T^{\sharp_A} - \mu I \right) x \right\|_A^r \right).$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality, we obtain

$$\omega_A^{2r}(T) \leq 2^{r-1} \left(\omega_A^r(T^2) + \|T - \lambda I\|_A^r \left\| T^{\sharp_A} - \mu I \right\|_A^r \right).$$

Finally, by taking the infimum over $\lambda, \mu \in \mathbb{C}$, we get

$$\omega_A^{2r}(T) \leq 2^{r-1} \left(\omega_A^r(T^2) + D_A^r(T) D_A^r\left(T^{\sharp_A} \right) \right).$$

Further, for every $T \in \mathcal{B}_A(\mathcal{H})$ and for every $\lambda \in \mathbb{C}$ one can observe that

$$\begin{aligned} \|T - \lambda I\|_A &= \left\| (T - \lambda I)^{\sharp_A} \right\|_A \\ &= \left\| T^{\sharp_A} - \overline{\lambda} P \right\|_A = \left\| (T - \lambda P)^{\sharp_A} \right\|_A \\ &= \|T - \lambda P\|_A. \end{aligned}$$

Therefore, we have

$$\begin{aligned} D_A\left(T^{\sharp_A} \right) &= \inf_{\lambda \in \mathbb{C}} \left\| T^{\sharp_A} - \lambda I \right\|_A = \inf_{\lambda \in \mathbb{C}} \left\| T^{\sharp_A} - \lambda P \right\|_A \\ &= \inf_{\lambda \in \mathbb{C}} \left\| \left(T - \overline{\lambda} I \right)^{\sharp_A} \right\|_A \\ &= \inf_{\lambda \in \mathbb{C}} \left\| T - \overline{\lambda} I \right\|_A \\ &= D_A(T). \end{aligned}$$

Thus,

$$\omega_A^{2r}(T) \leq 2^{r-1} \left(\omega_A^r(T^2) + D_A^{2r}(T) \right).$$

This completes the proof. □

In particular, considering $r = 1$ in Theorem 2.5, we get the following corollary.

Corollary 2.6. *If $T \in \mathcal{B}_A(\mathcal{H})$, then*

$$\omega_A(T) \leq \sqrt{\omega_A(T^2) + D_A^2(T)}.$$

Now, we consider the A-seminorm distance $D_A(T, B)$ defined as follows: For $T, B \in \mathcal{B}_A(\mathcal{H})$,

$$D_A(T, B) = \inf_{\lambda \in \mathbb{C}} \|T - \lambda B\|_A.$$

Applying compactness argument it is easy to observe that there exists $\lambda_0 \in \mathbb{C}$ such that $D_A(T, B) = \|T - \lambda_0 B\|_A$. Using this generalized distance $D_A(T, B)$, and proceeding similarly as in Theorem 2.2, we have the following inequalities.

Theorem 2.7. *Let $T, B \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\frac{\sqrt{m_A^2(B) D_A^2(T, B) + c_A^2(B^{\sharp_A} T)}}{\|B\|_A} \leq \|T\|_A \leq \frac{\sqrt{\|B\|_A^2 D_A^2(T, B) + w_A^2(B^{\sharp_A} T)}}{m_A(B)},$$

where $m_A(B) = \inf_{\|x\|_A = 1} \|Bx\|_A$.

Next, we need the following two inequalities.

Lemma 2.8. *[22] Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A\left(TS^{\sharp_A} \pm ST\right) \leq 2\|S\|_A \,\omega_A(T).$$

Theorem 2.9. *[4, Th. 2.2] Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS) \leq \|T\|_A \omega_A(S) + \frac{1}{2} \min\left\{\omega_A(TS + ST^{\sharp_A}), \omega_A(TS - ST^{\sharp_A})\right\}.$$

We are now in a position to prove the following result.

Theorem 2.10. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS) \leq \min\left\{\Big(\|T\|_A + D_A(T)\Big)\omega_A(S), \Big(\|S\|_A + D_A(S)\Big)\omega_A(T)\right\}.$$

Proof. There exists $\lambda_0 \in \mathbb{C}$ such that $D_A(T) = \|T - \lambda_0 I\|_A$. If $\lambda_0 = 0$, then by the inequalities in (1), we get

$$\omega_A(TS) \leq \|TS\|_A \leq \|T\|_A \|S\|_A \leq 2\|T\|_A \,\omega_A(S) = (\|T\|_A + D_A(T))\,\omega_A(S).$$

Next consider $\lambda_0 \neq 0$, and let $\mu = \frac{\overline{\lambda_0}}{|\lambda_0|}$. Then, from Theorem 2.9 we have

$$\begin{aligned}
\omega_A(TS) &= \omega_A(\mu TS) \leq \|T\|_A \,\omega_A(S) + \frac{1}{2}\omega_A\left(\mu TS - \overline{\mu} ST^{\sharp_A}\right) \\
&= \|T\|_A \,\omega_A(S) + \frac{1}{2}\omega_A\left(\overline{\mu} S^{\sharp_A} T^{\sharp_A} - \mu T^{\sharp_A^{\sharp_A}} S^{\sharp_A}\right) \\
&= \|T\|_A \,\omega_A(S) + \frac{1}{2}\omega_A\left(\mu T^{\sharp_A^{\sharp_A}} S^{\sharp_A^{\sharp_A}} - \overline{\mu} S^{\sharp_A^{\sharp_A}} T^{\sharp_A}\right) \\
&= \|T\|_A \,\omega_A(S) + \frac{1}{2}\omega_A\left(\mu\left(T^{\sharp_A^{\sharp_A}} - \lambda_0 I\right) S^{\sharp_A^{\sharp_A}} - \overline{\mu} S^{\sharp_A^{\sharp_A}}\left(T^{\sharp_A^{\sharp_A}} - \lambda_0 I\right)^{\sharp_A}\right) \\
&\leq \|T\|_A \,\omega_A(S) + \left\|T^{\sharp_A^{\sharp_A}} - \lambda_0 I\right\|_A \omega_A\left(S^{\sharp_A^{\sharp_A}}\right) \quad \textit{(by using Lemma 2.8)} \\
&= \|T\|_A \,\omega_A(S) + \left\|T^{\sharp_A^{\sharp_A}} - \lambda_0 I\right\|_A \omega_A(S).
\end{aligned}$$

Now, by using the fact $\left\|X^{\sharp_A}\right\|_A = \|X\|_A$ for all $X \in \mathcal{B}_A(\mathcal{H})$ we can observe that

$$\left\|T^{\sharp_A^{\sharp_A}} - \lambda_0 I\right\|_A = \left\|T^{\sharp_A} - \overline{\lambda_0}P\right\|_A = \left\|(T-\lambda_0 I)^{\sharp_A}\right\|_A = \|T-\lambda_0 I\|_A .$$

Therefore,

$$\omega_A(TS) \leq \|T\|_A\,\omega_A(S) + \|T-\lambda_0 I\|_A\,\omega_A(S) = \Big(\|T\|_A + D_A(T)\Big)\omega_A(S). \tag{5}$$

Replacing T by $S^{\sharp_A}$ and S by $T^{\sharp_A}$ in the above inequality and since $D_A(S^{\sharp_A}) = D_A(S)$, we get

$$\omega_A(TS) \leq \Big(\|S\|_A + D_A(S)\Big)\omega_A(T). \tag{6}$$

Combining the inequalities in (5) and (6), we obtain the desired inequality. □

Remark 2.11. *Clearly, $D_A(T) \leq \|T\|_A$ and $D_A(S) \leq \|S\|_A$. Therefore, we have*

$$\Big(\|T\|_A + D_A(T)\Big)\omega_A(S) \leq 2\|T\|_A w_A(S) \ \text{ and } \Big(\|S\|_A + D_A(S)\Big)\omega_A(T) \leq 2\|S\|_A w_A(T).$$

Thus, the inequality in Theorem 2.10 is better than the well-known existing inequality

$$\omega_A(TS) \leq \min\{2\|T\|_A\omega_A(S), 2\|S\|_A\omega_A(T)\},$$

see in [24].

For our next result we need the notion of A-numerical radius distance. For $T \in \mathcal{B}_A(\mathcal{H})$, let $d_A(T)$ denote the A-numerical radius distance of T from the scalar operators, i.e.,

$$d_A(T) = \inf_{\lambda\in\mathbb{C}} \omega_A(T-\lambda I).$$

Applying compactness argument we observe that there exists $\lambda_0 \in \mathbb{C}$ such that $d_A(T) = \omega_A(T-\lambda_0 I)$. Next, using the A-numerical distance $d_A(T)$, we obtain the following inequalities.

Theorem 2.12. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\|T\|_A \leq \omega_A(T) + d_A(T) \leq 2\omega_A(T).$$

Proof. There exists $\lambda_0 \in \mathbb{C}$ such that $d_A(T) = \omega_A(T-\lambda_0 I)$. If $\lambda_0 = 0$, then $\|T\|_A \leq 2\omega_A(T) = \omega_A(T) + d_A(T)$. Now, we take $\lambda_0 \neq 0$, and let $\mu = \frac{\overline{\lambda_0}}{|\lambda_0|}$. Therefore,

$$\begin{aligned}
\|T\|_A = \|\mu T\|_A &= \left\|\mathfrak{R}_A(\mu T) + i\mathfrak{I}_A(\mu T)\right\|_A \\
&\leq \left\|\mathfrak{R}_A(\mu T)\right\|_A + \left\|\mathfrak{I}_A(\mu T)\right\|_A \\
&= \left\|\mathfrak{R}_A(\mu T)\right\|_A + \left\|\mathfrak{I}_A(\mu(T-\lambda_0 I))\right\|_A \\
&\leq \omega_A(T) + \omega_A(T-\lambda_0 I).
\end{aligned}$$

Hence, $\|T\|_A \leq \omega_A(T) + d_A(T)$. The second inequality follows from the fact that $d_A(T) \leq \omega_A(T)$. □

The following corollary reads as follows.

Corollary 2.13. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS) \leq \Big(\omega_A(T) + d_A(T)\Big)\Big(\omega_A(S) + d_A(S)\Big) \leq 4\omega_A(T)\omega_A(S).$$

Proof. The proof follows from the fact that $\omega_A(TS) \leq \|TS\|_A \leq \|T\|_A\|S\|_A$ and using the inequalities in Theorem 2.12. □

The inequality in Corollary 2.13 also have been obtained in [24, Th. 3.5].
To obtain next result we need the following lemma.

Lemma 2.14. *[24] Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS \pm ST) \leq 4\omega_A(T)\,\omega_A(S).$$

Now, we are in a position to prove the following result.

Theorem 2.15. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS - ST) \leq 4d_A(T)\,d_A(S) \leq 4\omega_A(T)\,\omega_A(S).$$

Proof. Let $\lambda_0, \zeta_0 \in \mathbb{C}$ such that $\omega_A(T - \lambda_0 I) = d_A(T)$ and $\omega_A(S - \zeta_0 I) = d_A(S)$. Then, we have

$$\begin{aligned}
\omega_A(TS - ST) &= \omega_A((T - \lambda_0 I)(S - \zeta_0 I) - (S - \zeta_0 I)(T - \lambda_0 I)) \\
&\leq 4\omega_A((T - \lambda_0 I)\,\omega_A(S - \zeta_0 I)) \text{ (by using Lemma 2.14)} \\
&= 4d_A(T)\,d_A(S).
\end{aligned}$$

Thus,

$$\omega_A(TS - ST) \leq 4d_A(T)\,d_A(S).$$

The second desired inequality follows from the fact that $d_A(T) \leq \omega_A(T)$ and $d_A(S) \leq \omega_A(S)$. □

Remark 2.16. *By taking $A = I$ in Theorem 2.15 we get a recent result proved by Abu-Omar and Kittaneh in [1].*

Again, we need the following lemma to prove the next refinement.

Lemma 2.17. *[5, Th. 2.4] Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS - ST) \leq 2\sqrt{2}\|T\|_A\omega_A(S).$$

Theorem 2.18. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS - ST) \leq 2\sqrt{2}\min\left\{D_A(T)d_A(S), D_A(S)d_A(T)\right\} \leq 2\sqrt{2}\|T\|_A\omega_A(S).$$

Proof. Let $\lambda_0, \zeta_0 \in \mathbb{C}$ such that $\|T - \lambda_0 I\| = D_A(T)$ and $\omega_A(S - \zeta_0 I) = d_A(S)$. Then, we have

$$\begin{aligned}
\omega_A(TS - ST) &= \omega_A((T - \lambda_0 I)(S - \zeta_0 I) - (S - \zeta_0 I)(T - \lambda_0 I)) \\
&\leq 2\sqrt{2}\,\|T - \lambda_0 I\|_A\,\omega_A(S - \zeta_0 I) \text{ (by using Lemma 2.17)} \\
&= 2\sqrt{2}D_A(T)d_A(S).
\end{aligned}$$

Thus, $\omega_A(TS - ST) \leq 2\sqrt{2}D_A(T)d_A(S)$.
Replacing T by S and S by T in the above inequality, we have

$$\omega_A(TS - ST) \leq 2\sqrt{2}D_A(S)d_A(T).$$

Combining the above two inequalities we get the first inequality. The second inequality follows from the fact $D_A(T) \leq \|T\|_A$ and $d_A(S) \leq w_A(S)$. □

Now, we generalize the A-numerical distance $d_A(T)$ as in the following from: For $T, B \in \mathcal{B}_A(\mathcal{H})$,

$$d_A(T, B) = \inf_{\lambda \in \mathbb{C}} \omega_A(T - \lambda B).$$

Using this generalized A-numerical distance $d_A(T, B)$, we obtain the following inequalities.

Theorem 2.19. *Let $T, S, B \in \mathcal{B}_A(\mathcal{H})$ be such that B commutes with both T and S. Then*

$$\omega_A(TS - ST) \leq 4d_A(T, B)\, d_A(S, B) \leq 4\omega_A(T)\,\omega_A(S).$$

We skip the detail of the proof of the above theorem as it follows analogously as Theorem 2.15. Similarly, proceeding as in Theorem 2.18 we get the following theorem.

Theorem 2.20. *Let $T, S, B \in \mathcal{B}_A(\mathcal{H})$ be such that B commutes with both T and S. Then*

$$\omega_A(TS - ST) \leq 2\sqrt{2}\min\left\{D_A(T,B)d_A(S,B), D_A(S,B)d_A(T,B)\right\} \leq 2\sqrt{2}\|T\|_A\omega_A(S).$$

Our next result reads as follows:

Theorem 2.21. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}\omega_A(TS + ST) &\leq 2\min\left\{\omega_A(T)\left(\omega_A(S) + d_A(S)\right), \omega_A(S)\left(\omega_A(T) + d_A(T)\right)\right\}\\ &\leq 4\omega_A(T)\,\omega_A(S).\end{aligned}$$

Proof. Let $\lambda_0 \in \mathbb{C}$ such that $\omega_A(S - \lambda_0 I) = d_A(S)$. If $\lambda_0 = 0$, then we get

$$\omega_A(TS + ST) \leq 2\omega_A(T)\left(\omega_A(S) + d_A(S)\right) = 4\omega_A(T)\,\omega_A(S).$$

As in the proof of Theorem 2.10, we may assume that $\lambda_0 \neq 0$, and let $\mu = \frac{\overline{\lambda_0}}{|\lambda_0|}$. Then,

$$\begin{aligned}\omega_A(TS + ST) &= \omega_A(T(\mu S) + (\mu S)T)\\ &= \omega_A(T\mathfrak{R}_A(\mu S) + iT\mathfrak{I}_A(\mu S) + \mathfrak{R}_A(\mu S)T + i\mathfrak{I}_A(\mu S)T)\\ &\leq \omega_A(T\mathfrak{R}_A(\mu S) + \mathfrak{R}_A(\mu S)T) + \omega_A(T\mathfrak{I}_A(\mu S) + \mathfrak{I}_A(\mu S)T).\end{aligned}$$

It is easy to verify that

$$\mathfrak{R}_A^{\sharp_A}(\mu S) = \mathfrak{R}_A^{\sharp_A^{\sharp_A}}(\mu S) \text{ and } \mathfrak{I}_A^{\sharp_A}(\mu S) = \mathfrak{I}_A^{\sharp_A^{\sharp_A}}(\mu S).$$

Therefore, it follows from Lemma 2.8 that

$$\begin{aligned}\omega_A(T\mathfrak{R}_A(\mu S) + \mathfrak{R}_A(\mu S)T) &= \omega_A\left(\mathfrak{R}_A^{\sharp_A}(\mu S)T^{\sharp_A} + T^{\sharp_A}\mathfrak{R}_A^{\sharp_A}(\mu S)\right)\\ &= \omega_A\left(T^{\sharp_A}\mathfrak{R}_A^{\sharp_A^{\sharp_A}}(\mu S) + \mathfrak{R}_A^{\sharp_A}(\mu S)T^{\sharp_A}\right)\\ &\leq 2\left\|\mathfrak{R}_A^{\sharp_A}(\mu S)\right\|_A \omega_A\left(T^{\sharp_A}\right)\\ &= 2\left\|\mathfrak{R}_A(\mu S)\right\|_A \omega_A(T).\end{aligned}$$

Similarly,

$$\omega_A(T\mathfrak{I}_A(\mu S) + \mathfrak{I}_A(\mu S)T) \leq 2\left\|\mathfrak{I}_A(\mu S)\right\|_A \omega_A(T).$$

Hence,

$$\begin{aligned}\omega_A(TS + ST) &\leq 2\omega_A(T)\left(\left\|\mathfrak{R}_A(\mu S)\right\|_A + \left\|\mathfrak{I}_A(\mu S)\right\|_A\right)\\ &= 2\omega_A(T)\left(\left\|\mathfrak{R}_A(\mu S)\right\|_A + \left\|\mathfrak{I}_A(\mu(S - \lambda_0 I))\right\|_A\right).\end{aligned}$$

Since $\left\|\mathfrak{R}_A(\mu S)\right\|_A \leq \omega_A((\mu S)) = \omega_A(S)$ and $\left\|\mathfrak{I}_A(\mu(S - \lambda_0 I))\right\|_A \leq \omega_A(S - \lambda_0 I)$, we get

$$\omega_A(TS + ST) \leq 2\omega_A(T)(\omega_A(S) + \omega_A(S - \lambda_0 I)) = 2\omega_A(T)(\omega_A(S) + d_A(S)).$$

Now, replacing T by S and S by T in the above inequality, we get

$$\omega_A(TS + ST) \leq 2\omega_A(S)(\omega_A(T) + d_A(T)).$$

Combining the above two inequalities we obtain the first inequality. The second inequality follows from $d_A(T) \leq \omega_A(T)$ and $d_A(S) \leq \omega_A(S)$. □

3. Translatable radii of an operator in semi-Hilbert space

In [20, 21] authors introduced and studied a couple of translatable radii of a bounded linear operator T on a Hilbert space in the direction of another bounded linear operator S as follows: If $0 \notin \sigma_{app}(S)$ ($\sigma_{app}(S)$ denotes the approximate point spectrum of S), let

$$\begin{aligned} M_S(T) &= \sup_{\|x\|=1} \left\| Tx - \frac{\langle Tx, Sx\rangle}{\langle Sx, Sx\rangle} Sx \right\| \\ \text{i.e., } M_S(T) &= \sup_{\|x\|=1} \left\{ \|Tx\|^2 - \frac{|\langle Tx, Sx\rangle|^2}{\langle Sx, Sx\rangle} \right\}^{1/2} \end{aligned}$$

and if $0 \notin \overline{W(S)}$, let

$$\widetilde{M}_S(T) = \sup_{\|x\|=1} \left\| Tx - \frac{\langle Tx, x\rangle}{\langle Sx, x\rangle} Sx \right\|.$$

$M_S(T)$ and $\widetilde{M}_S(T)$ are defined as translatable radii of the operator T in the direction of S. Author [20] further proved that if $0 \notin \overline{W(S)}$ (the closure of the numerical range of S) then

$$\widetilde{M}_S(T) \geq M_S(T) \geq m_S(T)/\|S^{-1}\|,$$

where $m_S(T)$ is the radius of the smallest circle containing the set

$$W_S(T) = \left\{ \frac{\langle Tx, Sx\rangle}{\langle Sx, Sx\rangle} : \|x\| = 1 \right\}.$$

Here, we introduce the translatable radius of T in the direction of S with respect to seminorm $\|\cdot\|_A$ as follows : Let $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. If $0 \notin \sigma_{app}(A^{1/2}S)$, then let

$$M_S(T)_A = \sup_{\|x\|_A=1} \left\| Tx - \frac{\langle Tx, Sx\rangle_A}{\langle Sx, Sx\rangle_A} Sx \right\|_A$$

and if $0 \notin \overline{W_A(S)}$, then let

$$\widetilde{M}_S(T)_A = \sup_{\|x\|_A=1} \left\| Tx - \frac{\langle Tx, x\rangle_A}{\langle Sx, x\rangle_A} Sx \right\|_A.$$

It is easy to observe that

$$M_S(T)_A = \sup_{\|x\|_A=1} \left\{ \|Tx\|_A^2 - \frac{|\langle Tx, Sx\rangle_A|^2}{\langle Sx, Sx\rangle_A} \right\}^{1/2},$$

and $M_S(T)_A = M_S(T + \mu S)_A$ for all $\mu \in \mathbb{C}$, that is, $M_S(T)_A$ is translation invariant in the direction of S. For $S = A = I$ we get the transcendental radius studied in [15, 16]. We also observe that $\widetilde{M}_S(T)_A = \widetilde{M}_S(T + \mu S)_A$ for all $\mu \in \mathbb{C}$. $M_S(T)_A$ and $\widetilde{M}_S(T)_A$ are defined as the translatable radius of T in the direction of S with respect to seminorm $\|\cdot\|_A$. Now, we consider the set

$$W_S(T)_A = \left\{ \frac{\langle Tx, Sx\rangle_A}{\langle Sx, Sx\rangle_A} : x \in \mathcal{H}, \|x\|_A = 1 \right\},$$

if $0 \notin \sigma_{app}(A^{1/2}S)$ and

$$\widetilde{W}_S(T)_A = \left\{ \frac{\langle Tx, x\rangle_A}{\langle Sx, x\rangle_A} : x \in \mathcal{H}, \|x\|_A = 1 \right\},$$

if $0 \notin \overline{W_A(S)}$. Clearly, $W_S(T)_A = W_S(T)$ and $\widetilde{W}_S(T)_A = \widetilde{W}_S(T)$ if $A = I$. Let $m_S(T)_A$ (resp. $\widetilde{m}_S(T)_A$) be the radius of the smallest circle containing the set $W_S(T)_A$ (resp. $\widetilde{W}_S(T)_A$) and let $|W_S(T)_A| = \sup\{|\lambda| : \lambda \in W_S(T)_A\}$ and $|\widetilde{W}_S(T)_A| = \sup\{|\lambda| : \lambda \in \widetilde{W}_S(T)_A\}$. Then it is easy to observe that

$$m_S(T)_A = \inf_{\mu\in\mathbb{C}} |W_S(T - \mu S)_A|$$

and

$$\widetilde{m}_S(T)_A = \inf_{\mu\in\mathbb{C}} |\widetilde{W}_S(T - \mu S)_A|.$$

Next we prove a nice relation between the translatable radius $M_S(T)_A$ and $D_A(T, S)$. To do so we need the following lemma, which follows from [25, Th. 2.2].

Lemma 3.1. *Let $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Then the following are equivalent:*
(i) There exists a sequence $\{x_n\}$ in $\mathcal{H}$ with $\|x_n\|_A = 1$ such that $\langle Tx_n, Sx_n\rangle_A \to 0$ and $\|Tx_n\|_A \to \|T\|_A$.
(ii) $\|T - \mu S\|_A \geq \|T\|_A$ for all $\mu \in \mathbb{C}$.

Theorem 3.2. *Let $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ be such that $0 \notin \sigma_{app}(A^{1/2}S)$. Then*

$$M_S(T)_A = D_A(T, S) = \inf_{\mu\in\mathbb{C}} \|T - \mu S\|_A.$$

Proof. There exists $\mu_0 \in \mathbb{C}$ such that $D_A(T, S) = \|T - \mu_0 S\|_A$. Therefore, for all $\mu \in \mathbb{C}$ we get,

$$\|T - \mu_0 S\|_A \leq \|T - \mu S\|_A = \|(T - \mu_0 S) + (\mu_0 - \mu)S\|_A.$$

Since $M_S(T)_A = M_S(T - \mu S)_A$ for all scalars μ, so without loss of generality we assume that $\|T\|_A \leq \|T - \mu S\|_A$ for all scalars μ. Therefore, it follows from Lemma 3.1 that there exists a sequence $\{x_n\}$ in $\mathcal{H}$ with $\|x_n\|_A = 1$ such that $\langle Tx_n, Sx_n\rangle_A \to 0$ and $\|Tx_n\|_A \to \|T\|_A$. Now,

$$\begin{aligned}
\|T\|_A &= \lim_{n\to\infty} \|Tx_n\|_A \\
&= \lim_{n\to\infty} \left\{ \|Tx_n\|_A^2 - \frac{|\langle Tx_n, Sx_n\rangle_A|^2}{\langle Sx_n, Sx_n\rangle_A} \right\}^{1/2} \\
&\leq M_S(T)_A.
\end{aligned}$$

Also, for any $x \in \mathcal{H}$ with $\|x\|_A = 1$, we have

$$\begin{aligned}
\|T\|_A &\geq \|Tx\|_A \\
&\geq \left\{ \|Tx\|_A^2 - \frac{|\langle Tx, Sx\rangle_A|^2}{\langle Sx, Sx\rangle_A} \right\}^{1/2}.
\end{aligned}$$

This implies that $\|T\|_A \geq M_S(T)_A$. Therefore, $\|T\|_A = M_S(T)_A$, that is, $\|T - \mu_0 S\|_A = M_S(T - \mu_0 S)_A = M_S(T)_A$. This completes the proof. □

Applying Theorem 3.2 we obtain the following corollary.

Corollary 3.3. *Let $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ be such that $0 \notin \sigma_{app}(A^{1/2}S)$. Then*

$$\widetilde{M}_S(T)_A \geq M_S(T)_A = D_A(T, S) \geq m_S(T)_A/\|S^{-1}\|_A.$$

Proof. For any $x \in \mathcal{H}$ with $\|x\|_A = 1$, we have $\frac{|\langle Tx,Sx\rangle_A|}{\langle Sx,Sx\rangle_A} \leq \frac{\|T\|_A}{\|Sx\|_A} \leq \|T\|_A\|S^{-1}\|_A$ for all $T \in \mathcal{B}_{A^{1/2}}(A)$. Therefore, $|W_S(T)_A| \leq \|T\|_A\|S^{-1}\|_A$ for all $T \in \mathcal{B}_{A^{1/2}}(A)$. Thus, $|W_S(T-\mu S)_A| \leq \|T-\mu S\|_A\|S^{-1}\|_A$ for all $\mu \in \mathbb{C}$. Taking the infimum over $\mu \in \mathbb{C}$, we get $m_S(T)_A \leq D_A(T,S)\|S^{-1}\|_A$, which gives the last inequality. Now, let $Tx = \frac{\langle Tx,Sx\rangle_A}{\langle Sx,Sx\rangle_A}Sx + h$ and $Tx = \frac{\langle Tx,x\rangle_A}{\langle Sx,x\rangle_A}Sx + \tilde{h}$, where $\langle h, Sx\rangle_A = 0$ and $\langle \tilde{h}, x\rangle_A = 0$. Then we have,

$$\tilde{h} = h + \left\{\frac{\langle Tx,Sx\rangle_A}{\langle Sx,Sx\rangle_A} - \frac{\langle Tx,x\rangle_A}{\langle Sx,x\rangle_A}\right\}Sx.$$

This implies that

$$\|\tilde{h}\|_A^2 = \|h\|_A^2 + \left|\frac{\langle Tx,Sx\rangle_A}{\langle Sx,Sx\rangle_A} - \frac{\langle Tx,x\rangle_A}{\langle Sx,x\rangle_A}\right|^2\|Sx\|_A^2.$$

Thus, $\|\tilde{h}\|_A \geq \|h\|_A$, that is, $\left\|Tx - \frac{\langle Tx,x\rangle_A}{\langle Sx,x\rangle_A}Sx\right\|_A \geq \left\|Tx - \frac{\langle Tx,Sx\rangle_A}{\langle Sx,Sx\rangle_A}Sx\right\|_A$, which implies $\widetilde{M}_S(T)_A \geq M_S(T)_A$. Also, it follows from Theorem 3.2 that $M_S(T)_A = D_A(T,S)$, so we complete the proof. □

Remark 3.4. *For $A = I$, we get the inequality developed in [20] and for $S = A = I$, we get the inequality developed in [16].*

Finally, we obtain the following inequality which generalizes the inequality given in [20, Th. 2].

Proposition 3.5. *Let $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ be such that $c_A(S) \geq r > 0$. Then*

$$M_S(T)_A = D_A(T,S) \geq r\,\widetilde{m}_S(T)_A.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have, $\left|\frac{\langle Tx,x\rangle_A}{\langle Sx,x\rangle_A}\right| \leq \|T\|_A/r$, that is, $|\widetilde{W}_S(T)_A| \leq \|T\|_A/r$ for all operators $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Therefore, $r|\widetilde{W}_S(T-\mu S)_A| \leq \|T-\mu S\|_A$ for all scalars μ. Taking the infimum over $\mu \in \mathbb{C}$ we obtain that $r\,\tilde{m}_S(T)_A \leq D_A(T,S)$. Also, from Theorem 3.2 we have $M_S(T)_A = D_A(T,S)$. This completes the proof. □

Declarations.
The authors have no competing interests to declare that are relevant to the content of this article.

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