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On some Newton's type inequalities for differentiable convex functions via Riemann-Liouville fractional integrals

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Abstract. In this paper, we establish a new integral identity involving Riemann-Liouville fractional integrals and differentiable functions. Then, we use the newly established identity and prove several Newton's type inequalities for differentiable convex functions and functions of bounded variation. Moreover, we give a mathematical example and graphical analysis of newly established inequalities to show their validity.

1. Introduction

Fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has grown in popularity and relevance over the last three decades, owing to its demonstrated applications in a wide range of seemingly disparate domains of science and engineering. It does, in fact, give a number of potentially valuable tools for solving differential and integral equations, as well as a variety of other problems involving mathematical physics special functions, as well as their extensions and generalizations in one or more variables.

The concept of fractional calculus is widely thought to have originated with a question posed to Gottfried Wilhelm Leibniz (1646-1716) by Marquis de L'Hôpital (1661-1704) in 1695, in which he tried to understand the meaning of Leibniz's notation $\frac{d^n y}{dx^n}$ for the derivative of order $n = \{0, 1, 2, \cdots\}$ when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$?). Leibniz replied to L'Hôpital on September 30, 1695, with the following message: "... This is an apparent paradox from which, one day, useful consequences will be drawn. ..."

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The theories of differential, integral, and integro-differential equations, and special functions of mathematical physics as well as their extensions and generalizations in one and more variables, some of the areas of presently applications of fractional calculus include Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on.

Because of the importance of the Fractional Calculus, researchers have utilized it to establish various fractional integral inequalities that have been shown to be quite useful in approximation theory. Inequalities such as Hermite-Hadamard, Simpson's, midpoint, Ostrowski's and trapezoidal inequalities are examples and by using these inequalities, we can obtain the bounds of formulas used in numerical integration. In [22], Sarikaya et al. proved some Hermite-Hadamard type inequalities and trapezoidal type inequalities for the first time using the Riemann-Liouville fractional integrals. Set [25] proved a Riemann-Liouville fractional version of the Ostrowski's inequalities for differentiable functions. İşcan and Wu used harmonic convexity and proved Hermite-Hadamard type inequalities in [13]. Sarikaya and Yildrim [23] used Riemann-Liouville fractional integrals to prove some new Hermite-Hadamard type inequalities and midpoint type inequalities for differentiable convex functions. Sarikaya et al. [21] proved general version of Simpson's type inequalities for differentiable s-convex functions. In [20], the authors used Riemann-Liouville frcational integrals and proved some Simpson's type inequalities for general convex functions. Another version of Simpson's type inequalities for differentiable s-convex functions was provided by the Chen and Huang in [6]. Recently, Sarikaya and Ertugral [24] defined a new class of fractional integrals, called generalized fractional and they used these integrals to prove general version of Hermite-Hadamard type inequalities for convex functions. In [33], the authors used generalized fractional integrals and proved some trapezoidal type inequalities for harmonic convex functions. Budak et al. [5] proved several variants of Ostrowski's and Simpson's type for differentiable convex functions via generalized fractional integrals. For more inequalities via fractional integrals, one can consult [1, 4, 14–16, 18, 30, 31, 34] and references therein. On the other hand several papers focused on the functions of bounded vatiation to prove some important inequalities such as Ostrowski type [11], Simpson type [7, 10], tapezoid type [3, 8], midpoint type [9].

Inspired by the ongoing studies, we establish some Newton's formula type inequalities for differentiable convex functions and functions of bounded variations via Riemann-Liouville fractional integrals and give some graphical analysis of the newly established inequalities.

This paper is summarized as follows: Section 2 provides a brief overview of the fundamentals of fractional calculus as well as other related studies in this field. In Section 3, we establish an integral identity that plays a major role in establishing the main outcomes of this paper. Some new inequalities of Newton's type for differentiable convex functions via Riemann-Liouville fractional integrals are presented in Section 4. Some fractional Newton type inequalities for functions of bounded variation are given in Section 5. Section 6 concludes with some suggestions for future research.

2. Fractional Integrals and Related Inequalities

In this section, we recall some basic notations and notions of the fractional integrals. We also recall some inequalities via different fractional integrals.

Definition 2.1. *[12, 17, 29] Let $\Upsilon \in L_1[\lambda_1, \lambda_2]$. The Riemann-Liouville fractional integrals (RLFIs) $J^{\alpha}_{\lambda_1+}\Upsilon$ and $J^{\alpha}_{\lambda_2-}\Upsilon$ of order $\alpha > 0$ with $\lambda_1 \geq 0$ are defined as follows:*

$$J^{\alpha}_{\lambda_1+}\Upsilon(x) = \frac{1}{\Gamma(\lambda_1)} \int_{\lambda_1}^{x} (x-\mu)^{\alpha-1} \Upsilon(\mu)\, d\mu,\ x > \lambda_1$$

and

$$J^{\alpha}_{\lambda_2-}\Upsilon(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\lambda_2} (\mu-x)^{\alpha-1} \Upsilon(\mu)\, d\mu, x < \lambda_2,$$

respectively, where Γ is the well-known Gamma function. For more applications and generalizations of fractional calculus, one can consult [26–28].

In 2013, Sarikaya et al. proved the following fractional Hermite-Hadmard type inequality for the first time:

Theorem 2.2. *[22] For a positive convex function $\Upsilon : I \subset \mathbb{R} \to \mathbb{R}$ with $\Upsilon \in L_1[\lambda_1, \lambda_2]$ and $0 \le \lambda_1 < \lambda_2$, the following inequality holds:*

$$\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\lambda_1+}\Upsilon(\lambda_2)+J^{\alpha}_{\lambda_2-}\Upsilon(\lambda_1)\right] \le \frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}. \tag{1}$$

After that Sarikaya and Yildırım proved the following new version of fractional Hermite-Hadamard inequality:

Theorem 2.3. *[23] For a positive convex function $\Upsilon : I \subset \mathbb{R} \to \mathbb{R}$ with $\Upsilon \in L_1[\lambda_1, \lambda_2]$, $0 \le \lambda_1 < \lambda_2$ and $\lambda_1, \lambda_2 \in I$, the following inequality holds:*

$$\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right) \le \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\lambda_1+\lambda_2}{2}\right)+}\Upsilon(\lambda_2)+J^{\alpha}_{\left(\frac{\lambda_1+\lambda_2}{2}\right)-}\Upsilon(\lambda_1)\right] \le \frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}. \tag{2}$$

Remark 2.4. *If we set $\alpha = 1$ in inequalities (1) and (2), then we obtain the classical Hermite-Hadamard inequality (see, [19]):*

$$\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right) \le \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\,dx \le \frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}.$$

The following fractional version of Simpson's type inequalities for differentiable s-convex functions was given by Chen and Huang:

Theorem 2.5. *[6] Suppose that a differentiable function $\Upsilon : I \subset [0,\infty) \to \mathbb{R}$ with $\Upsilon \in L_1[\lambda_1, \lambda_2]$, $0 \le \lambda_1 < \lambda_2$ and $\lambda_1, \lambda_2 \in I^{\circ}$ (interior of I). If $|\Upsilon'|$ is a s-convex function, then following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\Upsilon(\lambda_1)+4\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+\Upsilon(\lambda_2)\right]\right. \\ &\left.-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\lambda_1+}\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right]\right| \\ \le\ &\frac{\lambda_2-\lambda_1}{s+1}\left[|\Upsilon'(\lambda_1)|+|\Upsilon'(\lambda_2)|\right]\mathcal{I}(s,\alpha),\end{aligned} \tag{3}$$

where

$$\mathcal{I}(s,\alpha) = \int_0^1\left|\frac{\mu^{\alpha}}{2}-\frac{1}{3}\right|\left[(1+\mu)^s+(1-\mu)^s\right]d\mu.$$

Remark 2.6. *From inequality (3), we have*

(i) If we set $\alpha = 1$, then we obtain the classical Simpson's inequality for s-convex functions (see, [21, Theorem 7]):

$$\begin{aligned}&\left|\frac{1}{6}\left[\Upsilon(\lambda_1)+4\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+\Upsilon(\lambda_2)\right]-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\,dx\right| \\ \le\ &\frac{(\lambda_2-\lambda_1)(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}\left[|\Upsilon'(\lambda_1)|+|\Upsilon'(\lambda_2)|\right].\end{aligned}$$

(ii) If we set $s = \alpha = 1$, then we obtain the classical Simpson's inequality for convex functions (see, [21, Corollary 1]):

$$\begin{aligned}&\left|\frac{1}{6}\left[\Upsilon(\lambda_1)+4\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+\Upsilon(\lambda_2)\right]-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\,dx\right|\\ \leq\ &\frac{5(\lambda_2-\lambda_1)}{72}\left[|\Upsilon'(\lambda_1)|+|\Upsilon'(\lambda_2)|\right].\end{aligned}$$

3. An Identity

In this section, we prove an integral identity to prove the main results.

Lemma 3.1. *Let $\Upsilon : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° with $\Upsilon \in L_1[\lambda_1, \lambda_2]$, then the following RLFIs identity holds*:

$$\begin{aligned}&\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\\&-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^\alpha}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\\=\ &\frac{\lambda_2-\lambda_1}{9}\left[I_1+I_2+I_3\right],\end{aligned}\tag{4}$$

where

$$\begin{aligned}I_1&=\int_0^1\left(\mu^\alpha-\frac{3}{8}\right)\Upsilon'\left(\mu\frac{2\lambda_1+\lambda_2}{3}+(1-\mu)\lambda_1\right)d\mu,\\ I_2&=\int_0^1\left(\mu^\alpha-\frac{1}{2}\right)\Upsilon'\left(\mu\frac{\lambda_1+2\lambda_2}{3}+(1-\mu)\frac{2\lambda_1+\lambda_2}{3}\right)d\mu\end{aligned}$$

and

$$I_3=\int_0^1\left(\mu^\alpha-\frac{5}{8}\right)\Upsilon'\left(\mu\lambda_2+(1-\mu)\frac{\lambda_1+2\lambda_2}{3}\right)d\mu.$$

Proof. Using integration by parts and change of variables, we have

$$\begin{aligned}I_1&=\int_0^1\left(\mu^\alpha-\frac{3}{8}\right)\Upsilon'\left(\mu\frac{2\lambda_1+\lambda_2}{3}+(1-\mu)\lambda_1\right)d\mu\\&=\frac{15}{8(\lambda_2-\lambda_1)}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+\frac{9}{8(\lambda_2-\lambda_1)}\Upsilon(\lambda_1)\\&\quad-\frac{3\alpha}{(\lambda_2-\lambda_1)}\int_0^1\mu^{\alpha-1}\Upsilon\left(\mu\frac{2\lambda_1+\lambda_2}{3}+(1-\mu)\lambda_1\right)d\mu\\&=\frac{15}{8(\lambda_2-\lambda_1)}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+\frac{9}{8(\lambda_2-\lambda_1)}\Upsilon(\lambda_1)-\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha+1}}J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1).\end{aligned}\tag{5}$$

Similarly, we have

$$\begin{aligned}I_2&=\int_0^1\left(\mu^\alpha-\frac{1}{2}\right)\Upsilon'\left(\mu\frac{\lambda_1+2\lambda_2}{3}+(1-\mu)\frac{2\lambda_1+\lambda_2}{3}\right)d\mu\\&=\frac{3}{2(\lambda_2-\lambda_1)}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\frac{3}{2(\lambda_2-\lambda_1)}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)\end{aligned}\tag{6}$$

$$-\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha+1}} J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)$$

and

$$\begin{aligned} I_3 &= \int_0^1 \left(\mu^{\alpha}-\frac{5}{8}\right)\Upsilon'\left(\mu\lambda_2+(1-\mu)\frac{\lambda_1+2\lambda_2}{3}\right)d\mu \\ &= \frac{9}{8(\lambda_2-\lambda_1)}\Upsilon(\lambda_2)+\frac{15}{8(\lambda_2-\lambda_1)}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)-\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha+1}} J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right). \end{aligned} \tag{7}$$

Thus, we obtain the required equality by adding (5)-(7) and multiplying the resultant one by $\frac{\lambda_2-\lambda_1}{9}$. □

4. Fractional Newton's Inequalities for Differentiable Convex Functions

In this section, we prove some new Newton's inequalities for differentiable convex function via Riemann-Liouville fractional integrals. For sake of brevity, we use the following notations:

$$\begin{aligned} A_1(\alpha) &= \int_0^1 \mu\left|\mu^{\alpha}-\frac{3}{8}\right|d\mu \\ &= \left(\frac{3}{8}\right)^{\frac{2}{\alpha}+1}-\frac{2}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{\alpha+2}{\alpha}}+\frac{1}{\alpha+2}-\frac{3}{16}, \\ A_2(\alpha) &= \int_0^1 \left|\mu^{\alpha}-\frac{3}{8}\right|d\mu \\ &= 2\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}-\frac{2}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}-\frac{3}{8}, \\ A_3(\alpha) &= \int_0^1 \mu\left|\mu^{\alpha}-\frac{1}{2}\right|d\mu \\ &= \left(\frac{1}{2}\right)^{\frac{2}{\alpha}+1}-\frac{2}{\alpha+2}\left(\frac{1}{2}\right)^{\frac{\alpha+2}{\alpha}}+\frac{1}{\alpha+2}-\frac{1}{4}, \\ A_4(\alpha) &= \int_0^1 \left|\mu^{\alpha}-\frac{1}{2}\right|d\mu \\ &= 2\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{2}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}-\frac{1}{2}, \\ A_5(\alpha) &= \int_0^1 \mu\left|\mu^{\alpha}-\frac{5}{8}\right|d\mu \\ &= \left(\frac{5}{8}\right)^{\frac{2}{\alpha}+1}-\frac{2}{\alpha+2}\left(\frac{5}{8}\right)^{\frac{\alpha+2}{\alpha}}+\frac{1}{\alpha+2}-\frac{5}{16}, \\ A_6(\alpha) &= \int_0^1 \left|\mu^{\alpha}-\frac{5}{8}\right|d\mu \\ &= 2\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}-\frac{2}{\alpha+1}\left(\frac{5}{8}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}-\frac{5}{8}. \end{aligned}$$

Theorem 4.1. *We assume that the conditions of Lemma 3.1 hold. If $|\Upsilon'|$ is convex function, then we have the following Newton's type inequality:*

$$\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right. \tag{8}$$

$$
\begin{aligned}
&-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\Bigg| \\
\leq\ & \frac{\lambda_2-\lambda_1}{27}\left[\left|\Upsilon'(\lambda_1)\right|\left(3A_2(\alpha)-A_1(\alpha)+2A_4(\alpha)-A_3(\alpha)+A_6(\alpha)-A_5(\alpha)\right)\right. \\
&\left.+\left|\Upsilon'(\lambda_2)\right|\left(A_1(\alpha)+A_4(\alpha)+A_3(\alpha)+2A_6(\alpha)+A_5(\alpha)\right)\right].
\end{aligned}
$$

Proof. Taking modulus in (4) and applying convexity of $|\Upsilon'|$, we have

$$
\begin{aligned}
&\Bigg|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right] \\
&-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\Bigg| \\
\leq\ & \frac{\lambda_2-\lambda_1}{9}\left[\int_0^1\left|\mu^{\alpha}-\frac{3}{8}\right|\left|\Upsilon'\left(\mu\frac{2\lambda_1+\lambda_2}{3}+(1-\mu)\lambda_1\right)\right|d\mu\right. \\
&+\int_0^1\left|\mu^{\alpha}-\frac{1}{2}\right|\left|\Upsilon'\left(\mu\frac{\lambda_1+2\lambda_2}{3}+(1-\mu)\frac{2\lambda_1+\lambda_2}{3}\right)\right|d\mu \\
&\left.+\int_0^1\left|\mu^{\alpha}-\frac{5}{8}\right|\left|\Upsilon'\left(\mu\lambda_2+(1-\mu)\frac{\lambda_1+2\lambda_2}{3}\right)\right|d\mu\right] \\
=\ & \frac{\lambda_2-\lambda_1}{9}\left[\int_0^1\left|\mu^{\alpha}-\frac{3}{8}\right|\left|\Upsilon'\left(\frac{3-\mu}{3}\lambda_1+\frac{\mu}{3}\lambda_2\right)\right|d\mu\right. \\
&+\int_0^1\left|\mu^{\alpha}-\frac{1}{2}\right|\left|\Upsilon'\left(\frac{2-\mu}{3}\lambda_1+\frac{1+\mu}{3}\lambda_2\right)\right|d\mu \\
&\left.+\int_0^1\left|\mu^{\alpha}-\frac{5}{8}\right|\left|\Upsilon'\left(\frac{1-\mu}{3}\lambda_1+\frac{2+\mu}{3}\lambda_2\right)\right|d\mu\right] \\
\leq\ & \frac{\lambda_2-\lambda_1}{9}\left[\left|\Upsilon'(\lambda_1)\right|\int_0^1\frac{3-\mu}{3}\left|\mu^{\alpha}-\frac{3}{8}\right|d\mu+\left|\Upsilon'(\lambda_2)\right|\int_0^1\frac{\mu}{3}\left|\mu^{\alpha}-\frac{3}{8}\right|d\mu\right. \\
&+\left|\Upsilon'(\lambda_1)\right|\int_0^1\frac{2-\mu}{3}\left|\mu^{\alpha}-\frac{1}{2}\right|d\mu+\left|\Upsilon'(\lambda_2)\right|\int_0^1\frac{1+\mu}{3}\left|\mu^{\alpha}-\frac{1}{2}\right|d\mu \\
&\left.+\left|\Upsilon'(\lambda_1)\right|\int_0^1\frac{1-\mu}{3}\left|\mu^{\alpha}-\frac{5}{8}\right|d\mu+\left|\Upsilon'(\lambda_2)\right|\int_0^1\frac{2+\mu}{3}\left|\mu^{\alpha}-\frac{5}{8}\right|d\mu\right] \\
=\ & \frac{\lambda_2-\lambda_1}{27}\left[\left|\Upsilon'(\lambda_1)\right|\left(3A_2(\alpha)-A_1(\alpha)+2A_4(\alpha)-A_3(\alpha)+A_6(\alpha)-A_5(\alpha)\right)\right. \\
&\left.+\left|\Upsilon'(\lambda_2)\right|\left(A_1(\alpha)+A_4(\alpha)+A_3(\alpha)+2A_6(\alpha)+A_5(\alpha)\right)\right].
\end{aligned}
$$

This completes the proof. □

Example 4.2. *Let* $[\lambda_1,\lambda_2]=[0,1]$ *and define the function* $\Upsilon:[0,1]\to\mathbb{R}$, $\Upsilon(\mu)=\frac{\mu^3}{3}$ *such that* $\Upsilon'(\mu)=\mu^2$ *and* $|\Upsilon'|$ *is convex on* $[0,1]$ *.Under these assumptions we have*

$$
\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]=\frac{1}{12}.
$$

By definition of Riemann Liouville fractional integrals, we obtain

$$
J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)=J^{\alpha}_{\frac{1}{3}-}\Upsilon(0)=\frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{3}}x^{\alpha-1}\frac{x^3}{3}dx=\frac{1}{3\Gamma(\alpha)(\alpha+3)3^{\alpha+3}},
$$

$$\begin{aligned} J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right) &= J^{\alpha}_{\frac{2}{3}-}\Upsilon\left(\frac{1}{3}\right) \\ &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(x-\frac{1}{3}\right)^{\alpha-1}\frac{x^3}{3}dx \\ &= \frac{8\alpha^3+34\alpha^2+39\alpha+9}{\Gamma(\alpha)\,\alpha(\alpha+1)(\alpha+2)(\alpha+3)\,3^{\alpha+4}} \end{aligned}$$

and

$$\begin{aligned} J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right) &= J^{\alpha}_{1-}\Upsilon\left(\frac{2}{3}\right) \\ &= \frac{1}{\Gamma(\alpha)}\int_{\frac{2}{3}}^{1}\left(x-\frac{2}{3}\right)^{\alpha-1}\frac{x^3}{3}dx \\ &= \frac{33\alpha^3+154\alpha^2+183\alpha+50}{\Gamma(\alpha)\,\alpha(\alpha+1)(\alpha+2)(\alpha+3)\,3^{\alpha+4}}. \end{aligned}$$

By these equalities we have

$$\begin{aligned} &\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right] \\ = \;&\frac{42\alpha^3+190\alpha^2+223\alpha+59}{3^5(\alpha+1)(\alpha+2)(\alpha+3)}. \end{aligned}$$

The left hand side of the inequality (8) reduce to

$$\begin{aligned} &\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right. \\ &\left.-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\right| \\ = \;&\left|\frac{42\alpha^3+190\alpha^2+223\alpha+59}{3^5(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{1}{12}\right| \\ : \;&= LHS. \end{aligned}$$

On the other hand, since $|\Upsilon'(\lambda_1)| = 0$ *and* $|\Upsilon'(\lambda_2)| = 1$, *we have the right hand side of the inequality (8) as follows:*

$$\begin{aligned} &\frac{\lambda_2-\lambda_1}{27}\left[|\Upsilon'(\lambda_1)|\left(3A_2(\alpha)-A_1(\alpha)+2A_4(\alpha)-A_3(\alpha)+A_6(\alpha)-A_5(\alpha)\right)\right. \\ &\left.+|\Upsilon'(\lambda_2)|\left(A_1(\alpha)+A_4(\alpha)+A_3(\alpha)+2A_6(\alpha)+A_5(\alpha)\right)\right] \\ = \;&\frac{1}{27}\left[A_1(\alpha)+A_4(\alpha)+A_3(\alpha)+2A_6(\alpha)+A_5(\alpha)\right] \\ : \;&= RHS. \end{aligned}$$

It is clear from Figure 1 that LHS ≤ RHS for all $\alpha > 0$.

Remark 4.3. *In Theorem 4.1, if we set* $\alpha = 1$, *then we have the following inequality:*

$$\begin{aligned} &\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\,dx\right| \\ \le \;&\frac{75(\lambda_2-\lambda_1)}{1728}\left[|\Upsilon'(\lambda_1)|+|\Upsilon'(\lambda_2)|\right]. \end{aligned}$$

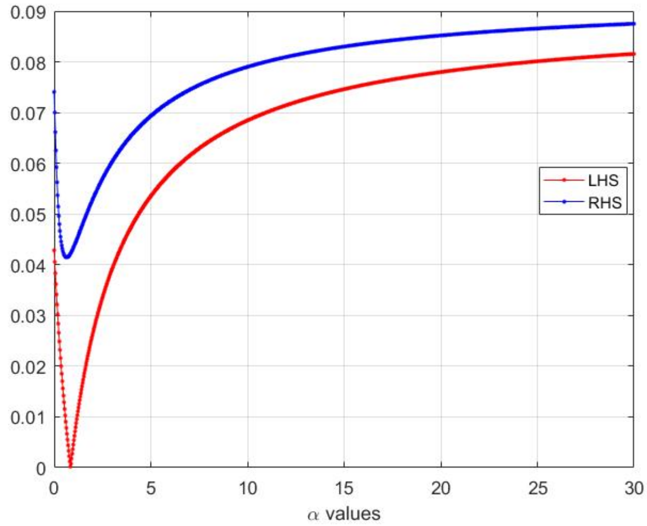

Figure 1: An example to Theorem 4.1

Theorem 4.4. *We assume that the conditions of Lemma 3.1 hold. If* $|\Upsilon'|^q$ *,* $q \geq 1$ *is convex function, then we have the following Newton's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right.\\
&\left.-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\right|\\
\leq\ &\frac{\lambda_2-\lambda_1}{9}\left[A_2^{1-\frac{1}{q}}(\alpha)\left(|\Upsilon'(\lambda_1)|^q\frac{3A_2(\alpha)-A_1(\alpha)}{3}+|\Upsilon'(\lambda_2)|^q\frac{A_1(\alpha)}{3}\right)^{\frac{1}{q}}\right.\\
&+A_4^{1-\frac{1}{q}}(\alpha)\left(|\Upsilon'(\lambda_1)|^q\frac{2A_4(\alpha)-A_3(\alpha)}{3}+|\Upsilon'(\lambda_2)|^q\frac{A_4(\alpha)+A_3(\alpha)}{3}\right)^{\frac{1}{q}}\\
&\left.+A_6^{1-\frac{1}{q}}(\alpha)\left(|\Upsilon'(\lambda_1)|\frac{A_6(\alpha)-A_5(\alpha)}{3}+|\Upsilon'(\lambda_2)|^q\frac{2A_6(\alpha)+A_5(\alpha)}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. By taking modulus in (4) and applying power mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right.\\
&\left.-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\right|\\
=\ &\frac{\lambda_2-\lambda_1}{9}\left[\int_0^1\left|\mu^{\alpha}-\frac{3}{8}\right|\left|\Upsilon'\left(\frac{3-\mu}{3}\lambda_1+\frac{\mu}{3}\lambda_2\right)\right|d\mu\right.\\
&+\int_0^1\left|\mu^{\alpha}-\frac{1}{2}\right|\left|\Upsilon'\left(\frac{2-\mu}{3}\lambda_1+\frac{1+\mu}{3}\lambda_2\right)\right|d\mu\\
&\left.+\int_0^1\left|\mu^{\alpha}-\frac{5}{8}\right|\left|\Upsilon'\left(\frac{1-\mu}{3}\lambda_1+\frac{2+\mu}{3}\lambda_2\right)\right|d\mu\right]
\end{aligned}$$

$$
\begin{aligned}
\leq\quad & \frac{\lambda_2-\lambda_1}{9}\left[\left(\int_0^1\left|\mu^\alpha-\frac{3}{8}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu^\alpha-\frac{3}{8}\right|\left|\Upsilon'\left(\frac{3-\mu}{3}\lambda_1+\frac{\mu}{3}\lambda_2\right)\right|^q d\mu\right)^{\frac{1}{q}}\right.\\
& +\left(\int_0^1\left|\mu^\alpha-\frac{1}{2}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu^\alpha-\frac{1}{2}\right|\left|\Upsilon'\left(\frac{2-\mu}{3}\lambda_1+\frac{1+\mu}{3}\lambda_2\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
& \left.+\left(\int_0^1\left|\mu^\alpha-\frac{5}{8}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu^\alpha-\frac{5}{8}\right|\left|\Upsilon'\left(\frac{1-\mu}{3}\lambda_1+\frac{2+\mu}{3}\lambda_2\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Using convexity of $|\Upsilon'|^q$, we have

$$
\begin{aligned}
& \left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right.\\
& \left.-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^\alpha}\left[J^\alpha_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^\alpha_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^\alpha_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\right|\\
\leq\quad & \frac{\lambda_2-\lambda_1}{9}\left[\left(\int_0^1\left|\mu^\alpha-\frac{3}{8}\right|d\mu\right)^{1-\frac{1}{q}}\right.\\
& \times\left(|\Upsilon'(\lambda_1)|^q\int_0^1\frac{3-\mu}{3}\left|\mu^\alpha-\frac{3}{8}\right|d\mu+|\Upsilon'(\lambda_2)|^q\int_0^1\frac{\mu}{3}\left|\mu^\alpha-\frac{3}{8}\right|d\mu\right)^{\frac{1}{q}}\\
& +\left(\int_0^1\left|\mu^\alpha-\frac{1}{2}\right|d\mu\right)^{1-\frac{1}{q}}\\
& \times\left(|\Upsilon'(\lambda_1)|^q\int_0^1\frac{2-\mu}{3}\left|\mu^\alpha-\frac{1}{2}\right|d\mu+|\Upsilon'(\lambda_2)|^q\int_0^1\frac{1+\mu}{3}\left|\mu^\alpha-\frac{1}{2}\right|d\mu\right)^{\frac{1}{q}}\\
& +\left(\int_0^1\left|\mu^\alpha-\frac{5}{8}\right|d\mu\right)^{1-\frac{1}{q}}\\
& \left.\times\left(|\Upsilon'(\lambda_1)|^q\int_0^1\frac{1-\mu}{3}\left|\mu^\alpha-\frac{5}{8}\right|d\mu+|\Upsilon'(\lambda_2)|^q\int_0^1\frac{2+\mu}{3}\left|\mu^\alpha-\frac{5}{8}\right|d\mu\right)^{\frac{1}{q}}\right]\\
=\quad & \frac{\lambda_2-\lambda_1}{9}\left[A_2^{1-\frac{1}{q}}(\alpha)\left(|\Upsilon'(\lambda_1)|^q\frac{3A_2(\alpha)-A_1(\alpha)}{3}+|\Upsilon'(\lambda_2)|^q\frac{A_1(\alpha)}{3}\right)^{\frac{1}{q}}\right.\\
& +A_4^{1-\frac{1}{q}}(\alpha)\left(|\Upsilon'(\lambda_1)|^q\frac{2A_4(\alpha)-A_3(\alpha)}{3}+|\Upsilon'(\lambda_2)|^q\frac{A_4(\alpha)+A_3(\alpha)}{3}\right)^{\frac{1}{q}}\\
& \left.A_6^{1-\frac{1}{q}}(\alpha)\left(|\Upsilon'(\lambda_1)|\frac{A_6(\alpha)-A_5(\alpha)}{3}+|\Upsilon'(\lambda_2)|^q\frac{2A_6(\alpha)+A_5(\alpha)}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Thus, the proof is completed. □

Remark 4.5. *In Theorem 4.4, if we set $\alpha=1$, then we have the following inequality:*

$$
\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\,dx\right|
$$

$$
\begin{aligned}
\leq\ & \frac{\lambda_2-\lambda_1}{36}\left[\left(\frac{17}{16}\right)^{1-\frac{1}{q}}\left(\frac{973\left|\Upsilon'(\lambda_1)\right|^q+251\left|\Upsilon'(\lambda_2)\right|^q}{1152}\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{1}{4}\right)\left(\frac{\left|\Upsilon'(\lambda_1)\right|^q+\left|\Upsilon'(\lambda_2)\right|^q}{2}\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{17}{16}\right)^{1-\frac{1}{q}}\left(\frac{251\left|\Upsilon'(\lambda_1)\right|+973\left|\Upsilon'(\lambda_2)\right|^q}{1152}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Theorem 4.6. *We assume that the conditions of Lemma 3.1 hold. If $|\Upsilon'|^q$, $q>1$ is convex function, then we have the following Newton's type inequality:*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right. \qquad (9)\\
&\left.-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\right|\\
\leq\ & \frac{\lambda_2-\lambda_1}{9}\left[A_7^{\frac{1}{p}}(\alpha,p)\left(\frac{5\left|\Upsilon'(\lambda_1)\right|^q+\left|\Upsilon'(\lambda_2)\right|^q}{6}\right)^{\frac{1}{q}}\right.\\
&\left.+A_8^{\frac{1}{p}}(\alpha,p)\left(\frac{\left|\Upsilon'(\lambda_1)\right|^q+\left|\Upsilon'(\lambda_2)\right|^q}{2}\right)^{\frac{1}{q}}+A_9^{\frac{1}{p}}(\alpha,p)\left(\frac{\left|\Upsilon'(\lambda_1)\right|^q+5\left|\Upsilon'(\lambda_2)\right|^q}{6}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

where $q^{-1}+p^{-1}=1$ and

$$
\begin{aligned}
A_7(\alpha,p) &= \int_0^1\left|\mu^{\alpha}-\frac{3}{8}\right|^p d\mu,\\
A_8(\alpha,p) &= \int_0^1\left|\mu^{\alpha}-\frac{1}{2}\right|^p d\mu
\end{aligned}
$$

and

$$
A_9(\alpha,p)=\int_0^1\left|\mu^{\alpha}-\frac{5}{8}\right|^p d\mu.
$$

Proof. Taking modulus in (4) and applyin Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right.\\
&\left.-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\right|\\
=\ & \frac{\lambda_2-\lambda_1}{9}\left[\int_0^1\left|\mu^{\alpha}-\frac{3}{8}\right|\left|\Upsilon'\left(\frac{3-\mu}{3}\lambda_1+\frac{\mu}{3}\lambda_2\right)\right|d\mu\right.\\
&+\int_0^1\left|\mu^{\alpha}-\frac{1}{2}\right|\left|\Upsilon'\left(\frac{2-\mu}{3}\lambda_1+\frac{1+\mu}{3}\lambda_2\right)\right|d\mu\\
&\left.+\int_0^1\left|\mu^{\alpha}-\frac{5}{8}\right|\left|\Upsilon'\left(\frac{1-\mu}{3}\lambda_1+\frac{2+\mu}{3}\lambda_2\right)\right|d\mu\right]\\
\leq\ & \frac{\lambda_2-\lambda_1}{9}\left[\left(\int_0^1\left|\mu^{\alpha}-\frac{3}{8}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Upsilon'\left(\frac{3-\mu}{3}\lambda_1+\frac{\mu}{3}\lambda_2\right)\right|^q d\mu\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$+\left(\int_0^1\left|\mu^\alpha-\frac{1}{2}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Upsilon'\left(\frac{2-\mu}{3}\lambda_1+\frac{1+\mu}{3}\lambda_2\right)\right|^q d\mu\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\mu^\alpha-\frac{5}{8}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Upsilon'\left(\frac{1-\mu}{3}\lambda_1+\frac{2+\mu}{3}\lambda_2\right)\right|^q d\mu\right)^{\frac{1}{q}}\Bigg].$$

By using convexity of $|\Upsilon'|^q$, $q>1$,we obtain

$$\begin{aligned}
&\Bigg|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\\
&-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^\alpha}\left[J^\alpha_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^\alpha_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^\alpha_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\Bigg|\\
\leq\;&\frac{\lambda_2-\lambda_1}{9}\Bigg[\left(\int_0^1\left|\mu^\alpha-\frac{3}{8}\right|^p d\mu\right)^{\frac{1}{p}}\left(|\Upsilon'(\lambda_1)|^q\int_0^1\frac{3-\mu}{3}d\mu+|\Upsilon'(\lambda_2)|^q\int_0^1\frac{\mu}{3}d\mu\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\mu^\alpha-\frac{1}{2}\right|^p d\mu\right)^{\frac{1}{p}}\left(|\Upsilon'(\lambda_1)|^q\int_0^1\frac{2-\mu}{3}d\mu+|\Upsilon'(\lambda_2)|^q\int_0^1\frac{1+\mu}{3}d\mu\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\mu^\alpha-\frac{5}{8}\right|^p d\mu\right)^{\frac{1}{p}}\left(|\Upsilon'(\lambda_1)|^q\int_0^1\frac{1-\mu}{3}d\mu+|\Upsilon'(\lambda_2)|^q\int_0^1\frac{2+\mu}{3}d\mu\right)^{\frac{1}{q}}\Bigg]\\
=\;&\frac{\lambda_2-\lambda_1}{9}\Bigg[A_7^{\frac{1}{p}}(\alpha,p)\left(\frac{5|\Upsilon'(\lambda_1)|^q+|\Upsilon'(\lambda_2)|^q}{6}\right)^{\frac{1}{q}}\\
&+A_8^{\frac{1}{p}}(\alpha,p)\left(\frac{|\Upsilon'(\lambda_1)|^q+|\Upsilon'(\lambda_2)|^q}{2}\right)^{\frac{1}{q}}+A_9^{\frac{1}{p}}(\alpha,p)\left(\frac{|\Upsilon'(\lambda_1)|^q+5|\Upsilon'(\lambda_2)|^q}{6}\right)^{\frac{1}{q}}\Bigg]
\end{aligned}$$

Thus, the proof is completed. □

Remark 4.7. *In Theorem 4.6, if we set $\alpha=1$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\,dx\right|\\
\leq\;&\frac{\lambda_2-\lambda_1}{9}\Bigg[\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|\Upsilon'(\lambda_1)|^q+|\Upsilon'(\lambda_2)|^q}{6}\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Upsilon'(\lambda_1)|^q+|\Upsilon'(\lambda_2)|^q}{2}\right)^{\frac{1}{q}}\\
&+\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Upsilon'(\lambda_1)|^q+5|\Upsilon'(\lambda_2)|^q}{6}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

5. Fractional Newton Type Inequality for Functions of Bounded Variation

In this section, we prove a fractional Newton type inequality for function of bounded variation.

Theorem 5.1. *Let $\Upsilon:[\lambda_1,\lambda_2]\to\mathbb{R}$ be a function of bounded variation on $[\lambda_1,\lambda_2]$. Then we have the following Newton type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right.$$

$$-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^\alpha}\left[J^\alpha_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^\alpha_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^\alpha_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\Bigg|$$
$$\le \frac{5}{24}\bigvee_{\lambda_1}^{\lambda_2}(\Upsilon).$$

where $\bigvee_c^d(\Upsilon)$ *denotes the total variation of* Υ *on* $[c,d]$.

Proof. Define the mapping $K_\alpha(x)$ by,

$$K_\alpha(x)=\begin{cases}(x-\lambda_1)^\alpha-\frac{(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^{\alpha-1}}, & \lambda_1\le x\le\frac{2\lambda_1+\lambda_2}{3}\\ \left(x-\frac{2\lambda_1+\lambda_2}{3}\right)^\alpha-\frac{(\lambda_2-\lambda_1)^\alpha}{2\cdot 3^\alpha} & \frac{2\lambda_1+\lambda_2}{3}<x\le\frac{\lambda_1+2\lambda_2}{3}\\ \left(x-\frac{\lambda_1+2\lambda_2}{3}\right)^\alpha-\frac{5(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^\alpha} & \frac{\lambda_1+2\lambda_2}{3}<x\le\lambda_2.\end{cases}$$

It follows from that

$$\int_{\lambda_1}^{\lambda_2}K_\alpha(x)d\Upsilon(x) \tag{10}$$
$$=\int_{\lambda_1}^{\frac{2\lambda_1+\lambda_2}{3}}\left((x-\lambda_1)^\alpha-\frac{(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^{\alpha-1}}\right)d\Upsilon(x)$$
$$+\int_{\frac{2\lambda_1+\lambda_2}{3}}^{\frac{\lambda_1+2\lambda_2}{3}}\left(\left(x-\frac{2\lambda_1+\lambda_2}{3}\right)^\alpha-\frac{(\lambda_2-\lambda_1)^\alpha}{2\cdot 3^\alpha}\right)d\Upsilon(x)$$
$$+\int_{\frac{\lambda_1 2+\lambda_2}{3}}^{\lambda_2}\left(\left(x-\frac{\lambda_1+2\lambda_2}{3}\right)^\alpha-\frac{5(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^\alpha}\right)d\Upsilon(x).$$

Integrating by parts, we get

$$\int_{\lambda_1}^{\frac{2\lambda_1+\lambda_2}{3}}\left((x-\lambda_1)^\alpha-\frac{(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^{\alpha-1}}\right)d\Upsilon(x) \tag{11}$$
$$=\left((x-\lambda_1)^\alpha-\frac{(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^{\alpha-1}}\right)\Upsilon(x)\Big|_{\lambda_1}^{\frac{2\lambda_1+\lambda_2}{3}}-\alpha\int_{\lambda_1}^{\frac{2\lambda_1+\lambda_2}{3}}(x-\lambda_1)^{\alpha-1}\Upsilon(x)dx$$
$$=\frac{5(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^\alpha}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{2}\right)+\frac{(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^{\alpha-1}}\Upsilon(\lambda_1)-\Gamma(\alpha+1)J^\alpha_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1).$$

Similarly, we have

$$\int_{\frac{2\lambda_1+\lambda_2}{3}}^{\frac{\lambda_1+2\lambda_2}{3}}\left(\left(x-\frac{2\lambda_1+\lambda_2}{3}\right)^\alpha-\frac{(\lambda_2-\lambda_1)^\alpha}{2\cdot 3^\alpha}\right)d\Upsilon(x) \tag{12}$$
$$=\frac{(\lambda_2-\lambda_1)^\alpha}{2\cdot 3^\alpha}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{2}\right)+\frac{(\lambda_2-\lambda_1)^\alpha}{2\cdot 3^\alpha}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{2}\right)-\Gamma(\alpha+1)J^\alpha_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)$$

and

$$\int_{\frac{\lambda_1+2\lambda_2}{3}}^{\lambda_2}\left(\left(x-\frac{\lambda_1+2\lambda_2}{3}\right)^\alpha-\frac{5(\lambda_2-\lambda_1)^\alpha}{8\cdot 3^\alpha}\right)d\Upsilon(x) \tag{13}$$

$$= \frac{(\lambda_2-\lambda_1)^{\alpha}}{8\cdot 3^{\alpha-1}}\Upsilon(\lambda_2)+\frac{5(\lambda_2-\lambda_1)^{\alpha}}{8\cdot 3^{\alpha}}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{2}\right)-\Gamma(\alpha+1)J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right).$$

By putting the equalities (11)-(13) in (10), we have

$$\begin{aligned}
&\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\\
&-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\\
= &\frac{3^{\alpha-1}}{(\lambda_2-\lambda_1)^{\alpha}}\int_{\lambda_1}^{\lambda_2}K_{\alpha}(x)d\Upsilon(x).
\end{aligned}$$

It is well known that if $g,\Upsilon:[\lambda_1,\lambda_2]\to\mathbb{R}$ are such that g is continuous on $[\lambda_1,\lambda_2]$ and Υ is of bounded variation on $[\lambda_1,\lambda_2]$, then $\int_{\lambda_1}^{\lambda_2}g(\mu)d\Upsilon(\mu)$ exist and

$$\left|\int_{\lambda_1}^{\lambda_2}g(\mu)d\Upsilon(\mu)\right|\leq\sup_{\mu\in[\lambda_1,\lambda_2]}\left|g(\mu)\right|\bigvee_{\lambda_1}^{\lambda_2}(\Upsilon). \tag{14}$$

On the other hand, using (14), we get

$$\begin{aligned}
&\left|\frac{1}{8}\left[\Upsilon(\lambda_1)+3\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+3\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)+\Upsilon(\lambda_2)\right]\right.\\
&\left.-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\left[J^{\alpha}_{\frac{2\lambda_1+\lambda_2}{3}-}\Upsilon(\lambda_1)+J^{\alpha}_{\frac{\lambda_1+2\lambda_2}{3}-}\Upsilon\left(\frac{2\lambda_1+\lambda_2}{3}\right)+J^{\alpha}_{\lambda_2-}\Upsilon\left(\frac{\lambda_1+2\lambda_2}{3}\right)\right]\right|\\
\leq &\frac{2^{\alpha-1}}{(\lambda_2-\lambda_1)^{\alpha}}\left|\int_{\lambda_1}^{\lambda_2}K_{\alpha}(x)d\Upsilon(x)\right|\\
\leq &\frac{3^{\alpha-1}}{(\lambda_2-\lambda_1)^{\alpha}}\left[\left|\int_{\lambda_1}^{\frac{2\lambda_1+\lambda_2}{3}}\left((x-\lambda_1)^{\alpha}-\frac{(\lambda_2-\lambda_1)^{\alpha}}{8\cdot 3^{\alpha-1}}\right)d\Upsilon(x)\right|\right.\\
&+\left|\int_{\frac{2\lambda_1+\lambda_2}{3}}^{\frac{\lambda_1+2\lambda_2}{3}}\left(\left(x-\frac{2\lambda_1+\lambda_2}{3}\right)^{\alpha}-\frac{(\lambda_2-\lambda_1)^{\alpha}}{2\cdot 3^{\alpha}}\right)d\Upsilon(x)\right|\\
&\left.+\left|\int_{\frac{\lambda_1+2\lambda_2}{3}}^{\lambda_2}\left(\left(x-\frac{\lambda_1+2\lambda_2}{3}\right)^{\alpha}-\frac{5(\lambda_2-\lambda_1)^{\alpha}}{8\cdot 3^{\alpha}}\right)d\Upsilon(x)\right|\right]\\
\leq &\frac{3^{\alpha-1}}{(\lambda_2-\lambda_1)^{\alpha}}\left[\sup_{x\in\left[\lambda_1,\frac{2\lambda_1+\lambda_2}{3}\right]}\left|(x-\lambda_1)^{\alpha}-\frac{(\lambda_2-\lambda_1)^{\alpha}}{8\cdot 3^{\alpha-1}}\right|\bigvee_{\lambda_1}^{\frac{2\lambda_1+\lambda_2}{3}}(\Upsilon)+\right.\\
&\sup_{x\in\left[\frac{2\lambda_1+\lambda_2}{3},\frac{\lambda_1+2\lambda_2}{3}\right]}\left|\left(x-\frac{2\lambda_1+\lambda_2}{3}\right)^{\alpha}-\frac{(\lambda_2-\lambda_1)^{\alpha}}{2\cdot 3^{\alpha}}\right|\bigvee_{\frac{2\lambda_1+\lambda_2}{3}}^{\frac{\lambda_1+2\lambda_2}{3}}(\Upsilon)\\
&\left.+\sup_{x\in\left[\frac{\lambda_1+2\lambda_2}{3},\lambda_2\right]}\left|\left(x-\frac{\lambda_1+2\lambda_2}{3}\right)^{\alpha}-\frac{5(\lambda_2-\lambda_1)^{\alpha}}{8\cdot 3^{\alpha}}\right|\bigvee_{\frac{\lambda_1+2\lambda_2}{3}}^{\lambda_2}(\Upsilon)\right]\\
= &\frac{3^{\alpha-1}}{(\lambda_2-\lambda_1)^{\alpha}}\left[\frac{5(\lambda_2-\lambda_1)^{\alpha}}{8\cdot 3^{\alpha}}\bigvee_{\lambda_1}^{\frac{2\lambda_1+\lambda_2}{3}}(\Upsilon)+\frac{(\lambda_2-\lambda_1)^{\alpha}}{2\cdot 3^{\alpha}}\bigvee_{\frac{2\lambda_1+\lambda_2}{3}}^{\frac{\lambda_1+2\lambda_2}{3}}(\Upsilon)+\frac{5(\lambda_2-\lambda_1)^{\alpha}}{8\cdot 3^{\alpha}}\bigvee_{\frac{\lambda_1+2\lambda_2}{3}}^{\lambda_2}(\Upsilon)\right]
\end{aligned}$$

$$\leq \frac{5}{24} \bigvee_{\lambda_1}^{\lambda_2} (\Upsilon).$$

This completes the proof. □

Remark 5.2. *If we take $\alpha = 1$ in Theorem 5.1, then we get the inequality*

$$\left| \frac{1}{8} \left[\Upsilon(\lambda_1) + 3\Upsilon\left(\frac{2\lambda_1 + \lambda_2}{3}\right) + 3\Upsilon\left(\frac{\lambda_1 + 2\lambda_2}{3}\right) + \Upsilon(\lambda_2) \right] - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \Upsilon(\mu) d\mu \right| \leq \frac{5}{24} \bigvee_{\lambda_1}^{\lambda_2} (\Upsilon)$$

which is given by Alomari in [2].

6. Conclusion

In this work, we used Riemann-Liouville fractional integrals and proved some new Simpson's second type inequalities for differentiable convex functions. We also gave a mathematical example and graphical analysis to show the validity of newly established results. Moreover, we established fractional Newton type inequalities for functions of bounded variation. It is an interesting and new problem that the upcoming researchers can obtain the similar inequalities for other kinds of convexity and co-ordinated convexity in their future work.

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