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Tauberian theorems for Cesàro summability in neutrosophic normed spaces

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Abstract. In this paper, we introduce the concepts of Cesàro summability and Tauberian theorem in neutrosophic normed spaces. We study that Cesàro summability in neutrosophic norm space does not imply ordinary convergence, and we give an example in support of our statement. We define slowly oscillating sequences in neutrosophic normed spaces and prove that Cesàro summability of slowly oscillating sequences implies ordinary convergence in neutrosophic normed spaces. Finally, we define $q-$bounded sequence with respect to the neutrosophic norm and also show how it relates to oscillating sequence in neutrosophic normed spaces.

1. Introduction

In 1965, Iranian mathematician Zadeh [27] created the notion of fuzzy set, which deals with real-world situations. Following that, Attanassov investigated fuzzy set and developed the intuitionistic fuzzy set [2] which expanded fuzzy set theory. Smarandache[21] expanded on intuitionistic fuzzy sets by proposing neutrosophic sets. The generalization of intuitionistic fuzzy set is the neutrosophic set. Murat Kirişki and Necip Şimşek defined neutrosophic metric space[12], neutrosophic normed space[13] and presented the characterization of these concepts. Nowadays, many authors have done phenomenal work on the application of various science and engineering concepts by using the inexactness of neutrosophic norm.

Cesàro summability method [7] assigns values to some infinite sums that are not necessarily convergent in the usual sense. Cesàro summability of a sequence is defined as the limit ($n \to \infty$) of the arithmetic mean of first n partial sums of the sequence. A sequence (ς_n) is called Cesàro summable to ς, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} s_i = \varsigma.$$

Here, s_i's are partial sums of the sequence (ς_n) and ς is called Cesàro sum of the series $\sum_{n=1}^{\infty} \varsigma_n$. For more information on different kind of summability method, one can refer to [1], [6] [7].

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As we know that Tauberian theorems with respect to various summability methods are the most satisfying proof of the converse part of Abel's theorems [25]. Numerous number of authors are constantly working in the field of Tauberian theory to find out the Tauberian theorem with respect to the most extended norm. For example, Talo and Basar[23] establish the necessary and sufficient Tauberian condition for the A^r summability method, Moricz[17] gives some conditions under which a statistically convergent sequence follows from the statistical summability $(C, 1)$ of sequence, Slepenchuk[22] introduces Tauberian conditions for absolute summability method. Jena et al.[10] used the generalized Tauberian condition with the notion of general convergence and statistical convergence of $(L, 1, 1)$ (i.e., logarithmic mean) summability to prove inclusion theorems. Parida et al. [19], [20] used Cesàro summability by de la Vallée Poussin mean to define slow oscillation and obtain a Tauberian theorem for n^{th} real sequence and proved statistical Tauberian theorem via Cesàro integrability mean based on post quantum calculus, respectively. Jena et al. [11],[20] established Tauberian theorems with respect to method of double Cesàro summability and also used the same summability to gave some results on Tauberian theorems for double sequence of fuzzy numbers, respectively. For more information about Tauberian condition for different summabilities, one can refer to [3], [4], [5], [14], [18], [24], [26].

Now we will recall some useful facts related to our main results.

Definition 1.1. *[16] If a binary operation* $\Upsilon : [0,1] \times [0,1] \longrightarrow [0,1]$ *meets the following requirements, it is called a continuous* $t-norm$*:*
(a) $\Upsilon(\iota,\kappa) = \Upsilon(\iota,\kappa)$ *and* $\Upsilon\Big(\iota, \Upsilon(\kappa,u)\Big) = \Upsilon\Big(\Upsilon(\iota,\kappa),u\Big)$*, for all* $\iota,\kappa,u \in [0,1]$*,*
(b) Υ *is continuous,*
(c) $\Upsilon(\iota,1) = \iota$, $\forall\ \iota \in [0,1]$*,*
(d) $\iota \le u$ *and* $\kappa \le q \Longrightarrow \Upsilon(\iota,\kappa) \le \Upsilon(u,q)$*, for each* ι, κ, u, $q \in [0,1]$.

Definition 1.2. *[16] If a binary operation* $\mho : [0,1] \times [0,1] \longrightarrow [0,1]$ *meets the following requirements, it is called continuous* $t-conorm$ *if:*
(a)$\mho(\iota,\kappa) = \mho(\kappa,\iota)$ *and* $\mho\Big(\iota, \mho(\kappa,u)\Big) = \mho\Big(\mho(\iota,\kappa),u\Big)$*, for all* $\iota,\kappa,u \in [0,1]$*,*
(b) $\mho$ *is continuous,*
(c) $\mho(\iota,0) = \iota$, $\forall\ \iota \in [0,1]$*,*
(d) $\iota \le u$ *and* $\kappa \le q \Longrightarrow \mho(\iota,\kappa) \le \mho(u,q)$ *for each* ι, κ, u, $q \in [0,1]$.

Definition 1.3. *[13] Let* S, Υ, *and* $\mho$ *be linear space, continuous* $t-norm$ *and continuous t-conorm respectively. A four tuple of the form* $\{S, \mathbf{F}(\nu,.), \mathbf{G}(\nu,.), \mathbf{H}(\nu,.) : \nu \in S\}$*, is called neutrosophic normed space, where* $\mathbf{F}, \mathbf{G}$*, and* $\mathbf{H}$ *are fuzzy sets on* $S \times \mathbb{R}^+$ *which satisfy the following conditions: For all* $\nu, y \in S$*, and* $p, q \in \mathbb{R}^+$*,*

(i) $0 \le \mathbf{F}(\nu,p) \le 1, 0 \le \mathbf{G}(\nu,p) \le 1, 0 \le \mathbf{H}(\nu,p) \le 1$*, for all* $p \in \mathbb{R}^+$*,*

(ii) $\mathbf{F}(\nu,p) + \mathbf{G}(\nu,p) + \mathbf{H}(\nu,p) \le 3$*, for all* $p \in \mathbb{R}^+$*,*

(iii) $\mathbf{F}(\nu,p) = 1$ *(for* $p > 0$*) if and only if* $\nu = 0$ *,*

(iv) $\mathbf{F}(\alpha\nu,p) = \mathbf{F}(\nu, \frac{p}{|\alpha|})$*, for* $\alpha \neq 0$

(v) $\Upsilon\Big(\mathbf{F}(\nu,p), \mathbf{F}(y,q)\Big) \le \mathbf{F}(\nu+y, p+q)$*,*

(vi) $\mathbf{F}(\nu,.)$ *is continuous non-decreasing function,*

(vii) $\lim_{p\to\infty} \mathbf{F}(\nu,p) = 1$*,*

$(viii)$ $\mathbf{G}(\nu,p) = 0$ *(for* $p > 0$*) if and only if* $\nu = 0$*,*

(ix) $\mathbf{G}(\alpha\nu,p) = \mathbf{G}(\nu, \frac{p}{|\alpha|})$ *for* $\alpha \neq 0$*,*

(x) $\mho\Big(\mathbf{G}(\nu,p), \mathbf{G}(y,q)\Big) \ge \mathbf{G}(\nu+y, p+q)$

(xi) $\mathbf{G}(\nu, .)$ *is continuous non-increasing function,*

(xii) $\lim_{p\to\infty} \mathbf{G}(\nu, p) = 0,$

$(xiii)$ $\mathbf{H}(\nu, p) = 0$ *(for* $p > 0$*) if and only if* $\nu = 0,$

(xiv) $\mathbf{H}(\alpha\nu, p) = \mathbf{H}(\nu, \frac{p}{|\alpha|})$ *if* $\alpha \neq 0,$

(xv) $\mho\Big(\mathbf{H}(\nu, p), \mathbf{H}(y, q)\Big) \geq \mathbf{H}(\nu + y, p + q),$

(xvi) $\mathbf{H}(\nu, .)$ *is continuous non-increasing function,*

$(xvii)$ $\lim_{p\to\infty} \mathbf{H}(\nu, p) = 0,$

$(xviii)$ *If* $p \leq 0$ *then* $\mathbf{F}(\nu, p) = 0, \mathbf{G}(\nu, p) = 1,$ *and* $\mathbf{H}(\nu, \mathbf{p}) = 1.$

Then $\mathcal{N} = \Big(\mathbf{F}, \mathbf{G}, \mathbf{H}\Big)$ is called neutrosophic norm. Throughout the paper we will use usual t-norm and usual t-conorm i.e; $\Upsilon(\tau, \varrho) = min\{\tau, \varrho\}$ and $\mho(\tau, \varrho) = max\{\tau, \varrho\}$.

Example 1.4. *[13] Let* $\Big(S, \|.\|\Big)$ *be a normed space. Let* $\mathbf{F}$*,* $\mathbf{G}$*,* $\mathbf{H}$ *be Fuzyy sets on* $S \times \mathbb{R}^+$ *such that, for* $p \geq \|\nu\|$ *and for all* $\nu \in S$*,*

$$\mathbf{F}(\nu, p) = \begin{cases} 0, & p \leq 0 \\ \frac{p}{p+\|\nu\|}, & p > 0, \end{cases} \quad \mathbf{G}(\nu, p) = \begin{cases} 0, & p \leq 0 \\ \frac{\|\nu\|}{p+\|\nu\|}, & p > 0, \end{cases} \quad \text{and } \mathbf{H}(\nu, p) = \frac{\|\nu\|}{p}.$$

If $p \leq \|\nu\|$ *then* $\mathbf{F}(\nu, p) = 0,$ $\mathbf{G}(\nu, p) = 1$ *and* $\mathbf{H}(\nu, p) = 1.$ *Then* $\Big(S, \mathcal{N}, \Upsilon, \mho\Big)$ *is neutrosophic normed space such that* $\mathcal{N} : S \times \mathbb{R}^+ \longrightarrow [0, 1].$

Theorem 1.5. *[15] Let* $\Big(S, \mathcal{N}, \Upsilon, \mho\Big)$ *be neutrosophic normed space and further assume that* $\mathbf{F}(\nu, p) > 0$ *for all* $p > 0$ *implies that* $\nu = 0.$ *For* $c \in (0, 1)$ *define*

$$\|\nu\|_c = \inf\Big\{p > 0 : \mathbf{F}(\nu, p) > c,\ \mathbf{G}(\nu, p) < 1 - c \text{ and } \mathbf{H}(\nu, p) < 1 - c\Big\}. \tag{1}$$

Then the set $\{\|\nu\|_c : c \in (0, 1)\}$ *called as set of c-norms is an ascending family of norms on neutrrosophic normed space S.*

Now we will discuss about the convergence of the sequence (ς_k) in neutrosophic normed space $\Big(S, \mathcal{N}, \Upsilon, \mho\Big)$.

Definition 1.6. *[13] A sequence* (ς_n) *in neutrosophic normed space* $\Big(S, \mathcal{N}, \Upsilon, \mho\Big)$ *is said to be convergent to* $\varsigma \in S$*, if for each* $t > 0$ *and* $c \in (0,1)$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\mathbf{F}(\varsigma_n - \varsigma, t) > 1 - c,\ \mathbf{G}(\varsigma_n - \varsigma, t) < c \text{ and } \mathbf{H}(\varsigma_n - \varsigma, t) < c, \tag{2}$$

for all $n \geq n_0$*.*

Definition 1.7. *[13] A sequence* (ς_k) *in neutrosophic normed space* $\Big(S, \mathcal{N}, \Upsilon, \mho\Big)$ *is said to be Cauchy, if for each* $t > 0$ *and* $c \in (0,1)$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n, k \geq n_0$*,*

$$\mathbf{F}(\varsigma_k - \varsigma_n, t) > 1 - c,\ \mathbf{G}(\varsigma_k - \varsigma_n, t) < c \text{ and } \mathbf{H}(\varsigma_k - \varsigma_n, t) < c. \tag{3}$$

Definition 1.8. *A sequence* (ς_k) *in a neutrosophic normed space* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$ *is G-Cauchy, if*

$$\lim_{n\to\infty} \mathbf{F}(\varsigma_{n+\jmath} - \varsigma_n, p) = 1,\ \lim_{n\to\infty} \mathbf{H}(\varsigma_{n+\jmath} - \varsigma_n, p) = 0 \text{ and } \lim_{n\to\infty} \mathbf{H}(\varsigma_{n+\jmath} - \varsigma_n, p) = 1,$$

for each $p > 0$ *and* $\jmath \in \mathbb{N}$. *In neutrosophic norm space, any Cauchy sequence is also G-Cauchy.*

Definition 1.9. *The set* $U \subseteq S$ *is called neutrosophic bounded in neutrosophic normed space* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$, *if there exist* $p > 0$ *and* $\theta \in (0, 1)$ *such that*

$$\mathbf{F}(z, p) > 1 - \theta,\ \mathbf{G}(z, p) < \theta \text{ and } \mathbf{H}(z, p) < \theta,$$

for each $z \in U$.

Definition 1.10. *Let* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$ *be an neutrosophic normed space and U be any subset of S then U is said to be* $q-$*bounded, if* $\lim_{p\to\infty} \Phi_U(p) = 1$, $\lim_{p\to\infty} \Psi_U(p) = 0$ *and* $\lim_{p\to\infty} \Pi_U(p) = 0$, *where*

$$\Phi_U(p) = inf\{\mathbf{F}(z, p) : z \in U\},\ \Psi_U(p) = sup\{\mathbf{G}(z, p) : z \in U\},\ \Pi_U(p) = sup\{\mathbf{H}(z, p) : z \in U\}.$$

In neutrosophic normed space $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$ a sequence (ς_n) is bounded if and only if there exists some $p_0 > 0$ and $\theta \in (0, 1)$ such that $\mathbf{F}(\varsigma_n, p_0) > 1 - \theta$, $\mathbf{G}(\varsigma_n, p_0) < \theta$ and $\mathbf{H}(\varsigma_n, p_0) < \theta$ for every positive integer n and $q - bounded$ if and only if

$$\lim_{p\to\infty} \inf_{n\in\mathbb{N}} \mathbf{F}(\varsigma_n,\ p) = 1,\ \lim_{p\to\infty} \sup_{n\in\mathbb{N}} \mathbf{G}(\varsigma_n,\ p) = 0 \text{ and } \lim_{p\to\infty} \sup_{n\in\mathbb{N}} \mathbf{H}(\varsigma_n,\ p) = 0. \tag{4}$$

Definition 1.11. *A sequence* (ς_n) *in a neutrosophic normed space* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$ *is said to be slowly oscillating, if*

$$\sup_{\mu>1} \liminf_{n\to\infty} \min_{n<k\leq\mu_n} \mathbf{F}(\varsigma_k - \varsigma_n,\ t) = 1, \tag{5}$$

$$\inf_{\mu>1} \limsup_{n\to\infty} \max_{n<k\leq\mu_n} \mathbf{G}(\varsigma_k - \varsigma_n,\ t) = 0 \tag{6}$$

and

$$\inf_{\mu>1} \limsup_{n\to\infty} \max_{n<k\leq\mu_n} \mathbf{H}(\varsigma_k - \varsigma_n,\ t) = 0 \tag{7}$$

for all $t > 0$, *where* $\mu_n = \lfloor \mu n \rfloor$ *(Floor function (denoted as* $\lfloor\ \rfloor$*) is a fuction that maps a real number* μ *to a greatest integer less than or equal to* μ *). In neutrosophic normed space, the slow oscillation condition can be restated as;*
A sequence (ς_n) *is slowly oscillating if and only if for* $t > 0$ *and* $0 < \epsilon < 1$, *there exists* $\mu > 1$ *and* $n_0 \in \mathbb{N}$, *depend on* t *and* ϵ, *such that*

$$\mathbf{F}(\varsigma_k - \varsigma_n, t) > 1 - \epsilon, \mathbf{G}(\varsigma_k - \varsigma_n, t) < \epsilon, \text{ and } \mathbf{H}(\varsigma_k - \varsigma_n, t) < \epsilon,$$

where $n_0 \leq n < k \leq \mu_n$.

Lemma 1.12. *[23] For a real number* $\mu > 0$ *define* $\{\mu\}$ *as* $\{\mu\} = \mu - \lfloor \mu \rfloor$. *Then the following assertions are true:*

(*i*) *if* $\mu > 1$, *for each* $k \in \mathbb{N} - \{0\}$ *with* $k \geq \dfrac{1}{\{\mu\}}$ *then we have* $\mu_k > k$.

(*ii*) *if* $0 < \mu < 1$,*then* $\mu_k < k$ *for each* $k \in \mathbb{N} - \{0\}$ *where* $\mu_k = \lfloor \mu k \rfloor$.

Lemma 1.13. *[23] Let μ be a positive real number, define $\{\mu\}$ by $\{\mu\} = \mu - \lfloor\mu\rfloor$. Then the following assertions are true:*

(i) *if $\mu > 1$, for each $k \in \mathbb{N} - \{0\}$ with $k \geq \dfrac{(3\mu - 1)}{\mu(\mu - 1)}$, we have*

$$\frac{\mu}{\mu - 1} < \frac{\mu_k + 1}{\mu_k - k} < \frac{2\mu}{\mu - 1}.$$

(ii) *if $0 < \mu < 1$, for each $k \in \mathbb{N} - \{0\}$ with $k > \dfrac{1}{\mu}$, we have*

$$0 < \frac{\mu_k + 1}{k - \mu_k} < \frac{2\mu}{1 - \mu}.$$

The focus of this article is to provide a basic understanding of Cesàro summability and associated Tauberian theorems in neutrosophic normed spaces. In the next section, we study the sequences which are not convergent in ordinary sense with respect to the neutrosophic norm and then we also work on some conditions, under which the foresaid sequences will converge with respect to the neutrosophic norm.

2. Main Results

In this section, we are going to introduce Cesàro summability methods in neutrosophic normed space and related Tauberian theorems.

Definition 2.1. *Let (ς_n) be a sequence in neutrosophic normed space $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$. Then (ς_n) is said to be Cesàro summable to $\varsigma \in S$ if sequence (σ_n) of arithmatic means is convergent to ζ, i.e.,*

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n}(\varsigma_k) \to \varsigma$$

as n tends to ∞.

Theorem 2.2. *Let (ς_n) be a sequence convergent to ς in neutrosophic normed space $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$. Then the sequence (σ_n) defined as $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n}\varsigma_k$ is Cesàro summable to ς.*

Proof. Let $\varsigma = (\varsigma_n)$ be a convergent sequence and converges to ς, then by the definition 1.6, for fix $t > 0$ and every $\epsilon > 0$ there exists $n_o \in \mathbb{N}$ such that

$$\mathbf{F}\left(\varsigma_n - \varsigma, \frac{t}{2}\right) > 1 - \epsilon, \ \mathbf{G}\left(\varsigma_n - \varsigma, \frac{t}{2}\right) < \epsilon \text{ and } \mathbf{H}\left(\varsigma_n - \varsigma, \frac{t}{2}\right) < \epsilon$$

for $n > n_0$. In light of the facts,

$$\lim_{n\to\infty}\mathbf{F}\left(\sum_{k=0}^{n_0}(\varsigma_k - \varsigma), \frac{(n+1)t}{2}\right) = 1, \ \lim_{n\to\infty}\mathbf{G}\left(\sum_{k=0}^{n_0}(\varsigma_k - \varsigma), \frac{(n+1)t}{2}\right) = 0, \text{ and}$$

$$\lim_{n\to\infty}\mathbf{H}\left(\sum_{k=0}^{n_0}(\varsigma_k - \varsigma), \frac{(n+1)t}{2}\right) = 0,$$

there exists $n_1 \in \mathbb{N}$ such that

$$\mathbf{F}\left(\sum_{k=0}^{n_0}(\varsigma_k-\varsigma),\ \frac{(n+1)t}{2}\right) > 1-\epsilon,\ \ \mathbf{G}\left(\sum_{k=0}^{n_0}(\varsigma_k-\varsigma),\ \frac{(n+1)t}{2}\right) < \epsilon,\ \text{and}$$

$$\mathbf{H}\left(\sum_{k=0}^{n_0}(\varsigma_k-\varsigma),\ \frac{(n+1)t}{2}\right) < \epsilon$$

for $n > n_1$. Then, we have

$$\begin{aligned}
\mathbf{F}\left(\frac{1}{n+1}\sum_{k=0}^{n}\varsigma_k-\varsigma, t\right) &= \mathbf{F}\left(\frac{1}{n+1}\sum_{k=0}^{n}(\varsigma_k-\varsigma), t\right)\\
&= \mathbf{F}\left(\sum_{k=0}^{n}(\varsigma_k-\varsigma), (n+1)t\right)\\
&\geq \min\left\{\mathbf{F}\left(\sum_{k=0}^{n_0}(\varsigma_k-\varsigma),\ \frac{(n+1)t}{2}\right),\ \mathbf{F}\left(\sum_{k=n_0+1}^{n}(\varsigma_k-\varsigma),\ \frac{(n+1)t}{2}\right)\right\}\\
&\geq \min\left\{\mathbf{F}\left(\sum_{k=0}^{n_0}(\varsigma_k-\varsigma),\ \frac{(n+1)t}{2}\right),\ \mathbf{F}\left(\sum_{k=n_0+1}^{n}(\varsigma_k-\varsigma),\ \frac{(n-n_0)t}{2}\right)\right\}\\
&\geq \min\left\{\mathbf{F}\left(\sum_{k=0}^{n_0}(\varsigma_{n_0+1}-\varsigma),\ \frac{(n+1)t}{2}\right),\ \mathbf{F}\left(\varsigma_{n_0+1}-\varsigma,\ \frac{t}{2}\right),\right.\\
&\qquad \left.\mathbf{F}\left(\varsigma_{n_0+2}-\varsigma,\frac{t}{2}\right),........,\mathbf{F}\left(\varsigma_n-\varsigma,\frac{t}{2}\right)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{G}\left(\frac{1}{n+1}\sum_{k=0}^{n}\varsigma_k-\varsigma, t\right) &< \max\left\{\mathbf{G}\left(\sum_{k=0}^{n_0}(\varsigma_{n_0+1}-\varsigma),\ \frac{(n+1)t}{2}\right),\ \mathbf{G}\left(\varsigma_{n_0+1}-\varsigma,\ \frac{t}{2}\right),\right.\\
&\qquad \left.\mathbf{G}\left(\varsigma_{n_0+2}-\varsigma,\frac{t}{2}\right),......,\mathbf{G}\left(\varsigma_n-\varsigma,\frac{t}{2}\right)\right\}\\
&< \epsilon.
\end{aligned}$$

Similarly for $\mathbf{H}$, we have

$$\begin{aligned}
\mathbf{H}\left(\frac{1}{n+1}\sum_{k=0}^{n}\varsigma_k-\varsigma, t\right) &< \max\left\{\mathbf{H}\left(\sum_{k=0}^{n_0}(\varsigma_{n_0+1}-\varsigma),\ \frac{(n+1)t}{2}\right),\ \mathbf{H}\left(\varsigma_{n_0+1}-\varsigma,\ \frac{t}{2}\right),\right.\\
&\qquad \left.\mathbf{H}\left(\varsigma_{n_0+2}-\varsigma,\frac{t}{2}\right),......,\mathbf{H}\left(\varsigma_n-\varsigma,\frac{t}{2}\right)\right\}\\
&< \epsilon.
\end{aligned}$$

□

Converse part of the theorem 2.2 is not true, i.e., as the following example shows, Cesàro summability in neutrosophic normed space does not entail convergence.

Example 2.3. *Let* (ς_n) *be a sequence defined as* $\varsigma_n = (-1)^{(n+1)}$ *in neutrosophic normed space* $\left(\mathbb{R}, \mathcal{N}, \Upsilon, \Omega\right)$*, where* $\mathbb{R}$ *denotes real vector space and* **F**, **G** *and* **H** *are as in Example* 1.4. *By definition* 2.2,

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n}\varsigma_k.$$

Then

$$\sigma_{2n} = \frac{1}{2n+1}\sum_{k=0}^{2n}\varsigma_k \implies \sigma_{2n} = \frac{-1}{2n+1},$$

$\varsigma_{2n} \to 0$ *as* $n \to \infty$ *and*

$$\sigma_{2n+1} = \frac{1}{2n+2}\sum_{k=0}^{2n+1}\varsigma_k \implies \sigma_{2n+1} = 0 \text{ for all } n = 0, 1, 2, 3, 4, ...$$

(σ_{2n}) *and* (σ_{2n+1}) *are two complementary subsequences of the sequence* (σ_n) *such that* (σ_{2n}) *and* (σ_{2n+1}) *converge to 0. This implies that the sequence* (σ_n) *conveges to 0. Now we will check wether sequence* (σ_n) *is convergent or not in neutrosophic normed space*

$$\lim_{n\to\infty}\mathbf{F}(\sigma_{2n}, t) = \lim_{n\to\infty}\mathbf{F}\left(\frac{-1}{2n+1}, t\right) = \lim_{n\to\infty}\frac{t}{t + \left|\frac{-1}{2n+1}\right|} = 1,$$

$$\lim_{n\to\infty}\mathbf{G}(\sigma_{2n}, t) = \lim_{n\to\infty}\mathbf{G}\left(\frac{-1}{2n+1}, t\right) = \lim_{n\to\infty}\frac{\left|\frac{-1}{2n+1}\right|}{t + \left|\frac{-1}{2n+1}\right|} = 0$$

and

$$\lim_{n\to\infty}\mathbf{H}(\sigma_{2n}, t) = \lim_{n\to\infty}\mathbf{H}\left(\frac{-1}{2n+1}, t\right) = \lim_{n\to\infty}\frac{\left|\frac{-1}{2n+1}\right|}{t} = 0.$$

By Definition 1.6 *we get that sequence* (σ_{2n}) *is convergent in neutrosophic normed space. Similarly for the sequence* (σ_{2n+1})

$$\lim_{n\to\infty}\mathbf{F}(\sigma_{2n+1}, t) = \lim_{n\to\infty}\mathbf{F}(0, t) = \lim_{n\to\infty}\frac{t}{t+0} = 1,$$

$$\lim_{n\to\infty}\mathbf{G}(\sigma_{2n+1}, t) = \lim_{n\to\infty}\mathbf{G}(0, t) = \lim_{n\to\infty}\frac{0}{t+0} = 0,$$

and

$$\lim_{n\to\infty}\mathbf{H}(\sigma_{2n+1}, t) = \lim_{n\to\infty}\mathbf{H}(0, t) = \lim_{n\to\infty}\frac{0}{t} = 0,$$

By Definition 1.6 *we get that sequence* (σ_{2n+1}) *is convergent in neutrosophic normed space. Hence sequence* (σ_n) *is convergent in neutrosophic normed space* $\left(\mathbb{R}, \mathcal{N}, \Upsilon, \Omega\right)$*. Hence sequence* (ς_n) *is Cesàro summable to* 0 *in neutrosophic normed space. Now we will check the convergence of sequence* (ς_n) *with respect to neutrosophic norms* **F**, **G**, **H**

$$\lim_{n\to\infty} \mathbf{F}(\varsigma_{2n} - (-1), t) = \lim_{n\to\infty} \mathbf{F}(-1 - (-1), t) = \lim_{n\to\infty} \frac{t}{t+0} = 1,$$

$$\lim_{n\to\infty} \mathbf{G}(\varsigma_{2n} - (-1), t) = \lim_{n\to\infty} \mathbf{G}(-1 - (-1), t) = \lim_{n\to\infty} \frac{0}{t+0} = 0$$

and

$$\lim_{n\to\infty} \mathbf{H}(\varsigma_{2n} - (-1), t) = \lim_{n\to\infty} \mathbf{H}(-1 - (-1), t) = \lim_{n\to\infty} \frac{0}{t} = 0.$$

This implies that sequnce $(\varsigma_{2n}) \to$ -1 in neutrosophic normed space. Similarly for sequence (ς_{2n+1}) we get

$$\lim_{n\to\infty} \mathbf{F}(\varsigma_{2n} - 1, t) = \lim_{n\to\infty} \mathbf{F}(1 - 1, t) = \lim_{n\to\infty} \frac{t}{t+0} = 1,$$

$$\lim_{n\to\infty} \mathbf{G}(\varsigma_{2n} - 1, t) = \lim_{n\to\infty} \mathbf{G}(1 - 1, t) = \lim_{n\to\infty} \frac{0}{t+0} = 0$$

and

$$\lim_{n\to\infty} \mathbf{H}(\varsigma_{2n} - 1, t) = \lim_{n\to\infty} \mathbf{H}(1 - 1, t) = \lim_{n\to\infty} \frac{0}{t} = 0.$$

This implies that sequence $(\varsigma_{2n+1}) \to 1$ in neutrosophic normed space as n tends to ∞. Hence, the sequence (ς_n) is not convergent in neutrosophic normed space.

Now we will find conditions ensure that every Cesàro summable sequence is convergent in neutrosophic normed space. For this we need the above lemmas 1.12, 1.13 *.*

Theorem 2.4. *Let (ς_n) be a sequence in neutrosophic normed space $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$. If (ς_n) is Cesàro summable to $\varsigma \in S$ then it converges to ς if and only if for all $t > 0$*

$$\sup_{\mu>1} \liminf_{n\to\infty} \mathbf{F}\left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\varsigma_k - \varsigma_n),\ t\right) = 1, \tag{8}$$

$$\inf_{\mu>1} \limsup_{n\to\infty} \mathbf{G}\left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\varsigma_k - \varsigma_n),\ t\right) = 0 \tag{9}$$

and

$$\inf_{\mu>1} \limsup_{n\to\infty} \mathbf{H}\left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\varsigma_k - \varsigma_n),\ t\right) = 0. \tag{10}$$

Proof. Let (ς_n) be a sequence in neutrosophic normed space $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$ and Cesàro summable to $\varsigma \in S$.

Necessary part: Let (ς_n) converges to ς. Fix $t > 0$. For any $\mu > 1$ by Lemma 1.12 for each $n \in \mathbb{N} - \{0\}$ with $n \geq \dfrac{1}{\mu}$ we can write

$$\varsigma_n - \sigma_n = \frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n) - \frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n) \tag{11}$$

By Lemma 1.13 , for $n \geq \dfrac{(3\mu - 1)}{\mu(\mu - 1)}$, we have

$$\begin{aligned}\mathbf{F}\left(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ t\right) &= \mathbf{F}\left(\sigma_{\mu_n} - \sigma_n,\ \frac{t}{\frac{\mu_n+1}{\mu_n-n}}\right)\\ &\geq \mathbf{F}\left(\sigma_{\mu_n} - \sigma_n,\ \frac{t}{\frac{2\mu}{\mu_n-1}}\right),\end{aligned}$$

$$\begin{aligned}\mathbf{G}\left(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ t\right) &= \mathbf{G}\left(\sigma_{\mu_n} - \sigma_n,\ \frac{t}{\frac{\mu_n+1}{\mu_n-n}}\right)\\ &\leq \mathbf{G}\left(\sigma_{\mu_n} - \sigma_n,\ \frac{t}{\frac{2\mu}{\mu_n-1}}\right),\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}\left(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ t\right) &= \mathbf{H}\left(\sigma_{\mu_n} - \sigma_n,\ \frac{t}{\frac{\mu_n+1}{\mu_n-n}}\right)\\ &\leq \mathbf{H}\left(\sigma_{\mu_n} - \sigma_n,\ \frac{t}{\frac{2\mu}{\mu_n-1}}\right),\end{aligned}$$

Since (σ_n) is Cauchy,

$$\lim_{n\to\infty}\mathbf{F}\left(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ t\right) = 1$$

$$\lim_{n\to\infty}\mathbf{G}\left(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ t\right) = 0$$

and

$$\lim_{n\to\infty}\mathbf{H}\left(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ t\right) = 0$$

By equation (11) ,we have

$$\lim_{n\to\infty}\mathbf{F}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n),\ t\right) = 1$$

$$\lim_{n\to\infty} \mathbf{G}\bigg(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n),\ t\bigg) = 0,$$

$$\lim_{n\to\infty} \mathbf{H}\bigg(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n),\ t\bigg) = 0.$$

Consequently, equations (8) , (9) , *and* (9) *are proved. Converse part: Suppose that equations* (8) , (9) *and* (10) *are true. Fix $t \geq 0$. Then for given $\epsilon > 0$ there exists $\mu > 1$ and $n_0 \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \mathbf{F}\bigg(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n),\ \frac{t}{3}\bigg) > 1-\epsilon,$$

$$\lim_{n\to\infty} \mathbf{G}\bigg(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n),\ \frac{t}{3}\bigg) < \epsilon$$

and

$$\lim_{n\to\infty} \mathbf{H}\bigg(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n),\ \frac{t}{3}\bigg) < \epsilon.$$

for all $n > n_0$. There exists $n_1 \in \mathbb{N}$ such that $\mathbf{F}\Big(\sigma_n - \varsigma,\ \frac{t}{3}\Big) > 1-\epsilon$, $\mathbf{G}\Big(\sigma_n - \varsigma,\ \frac{t}{3}\Big) < \epsilon$ and $\mathbf{H}\Big(\sigma_n - \varsigma,\ \frac{t}{3}\Big) < \epsilon$ for $n > n_1$. There exists $n_2 \in \mathbb{N}$ such that

$$\mathbf{F}\bigg(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ \frac{t}{3}\bigg) > 1-\epsilon,$$

$$\mathbf{G}\bigg(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ \frac{t}{3}\bigg) < \epsilon$$

and

$$\mathbf{H}\bigg(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ \frac{t}{3}\bigg) < \epsilon$$

In view of the fact that $\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n) \to 0$. Hence, we have

$$\begin{aligned}\mathbf{F}(\varsigma_n - \varsigma,\ t) = \mathbf{F}(\varsigma_n - \sigma_n + \sigma_n - \varsigma,\ t) =& \mathbf{F}\bigg(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n) - \frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n) + \sigma_n - \varsigma,\ t\bigg)\\ &\geq \min\bigg\{\mathbf{F}\bigg(\frac{\mu_n + 1}{\mu_n - n}(\sigma_{\mu_n} - \sigma_n),\ \frac{t}{3}\bigg),\\ &\mathbf{F}\bigg(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\varsigma_k - \varsigma_n),\ \frac{t}{3}\bigg),\ \mathbf{F}\bigg(\sigma_n - \varsigma,\ \frac{t}{3}\bigg)\bigg\}\\ &> 1-\epsilon,\end{aligned}$$

$$\begin{aligned}\mathbf{G}(\varsigma_n-\varsigma,\ t)=\mathbf{G}(\varsigma_n-\sigma_n+\sigma_n-\varsigma,\ t)=&\mathbf{G}\left(\frac{\mu_n+1}{\mu_n-n}(\sigma_{\mu_n}-\sigma_n)-\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n)+\sigma_n-\varsigma,\ t\right)\\ &\leq\max\left\{\mathbf{G}\left(\frac{\mu_n+1}{\mu_n-n}(\sigma_{\mu_n}-\sigma_n),\ \frac{t}{3}\right),\right.\\ &\left.\mathbf{G}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n),\ \frac{t}{3}\right),\ \mathbf{G}\left(\sigma_n-\varsigma,\ \frac{t}{3}\right)\right\}\\ &<\epsilon.\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}(\varsigma_n-\varsigma,\ t)=\mathbf{H}(\varsigma_n-\sigma_n+\sigma_n-\varsigma,\ t)=&\mathbf{H}\left(\frac{\mu_n+1}{\mu_n-n}(\sigma_{\mu_n}-\sigma_n)-\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n)+\sigma_n-\varsigma,\ t\right)\\ &\leq\max\left\{\mathbf{H}\left(\frac{\mu_n+1}{\mu_n-n}(\sigma_{\mu_n}-\sigma_n),\ \frac{t}{3}\right),\right.\\ &\left.\mathbf{H}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n),\ \frac{t}{3}\right),\ \mathbf{H}\left(\sigma_n-\varsigma,\ \frac{t}{3}\right)\right\}\\ &<\epsilon.\end{aligned}$$

For $n > \max\{n_0,\ n_1,\ n_2\}$. This completes the proof. If $0 < \mu < 1$, by Lemma 1.12, we have

$$\varsigma_n-\sigma_n=\frac{\mu_n+1}{n-\mu_n}(\sigma_n-\sigma_{\mu_n})+\frac{1}{n-\mu_n}\sum_{k=\mu_n+1}^{n}(\varsigma_k-\varsigma_n).$$

□

Theorem 2.5. *Let (ς_n) be a sequence in neutrosophic normed space $(S,\mathcal{N},\Upsilon,\Omega)$. For $t > 0$, conditions ((5), (6) , (7)) are equivalent to*

$$\sup_{0<\mu<1}\liminf_{n\to\infty}\min_{\mu_n<k\leq n}\mathbf{F}(\varsigma_k-\varsigma_n,\ t)=1, \tag{12}$$

$$\inf_{0<\mu<1}\limsup_{n\to\infty}\max_{\mu_n<k\leq n}\mathbf{G}(\varsigma_k-\varsigma_n,\ t)=0, \tag{13}$$

$$\inf_{0<\mu<1}\limsup_{n\to\infty}\max_{\mu_n<k\leq n}\mathbf{H}(\varsigma_k-\varsigma_n,\ t)=0, \tag{14}$$

respectively.

Proof. *We will only show that condition 5 is equivalent to condition 12 because other conditions can be done similarly. For $t > 0$ and $\mu > 1$, let*

$$\mathbf{T}(\mu)=\liminf_{n\to\infty}\min_{n<k\leq\lfloor\mu_n\rfloor}\mathbf{F}(\varsigma_k-\varsigma_n,\ t)$$

and

$$\mathbf{W}\Big(\frac{1}{\mu}\Big) = \liminf_{k\to\infty} \min_{\lfloor \frac{k}{\mu}\rfloor < n \le k} \mathbf{F}(\varsigma_k - \varsigma_n,\ t).$$

For each $\mu > 1$ *there exists an increasing sequence* (n_m) *such that*

$$\mathbf{T}(\mu) = \lim_{m\to\infty} \min_{n_m < k \le \lfloor \mu_{n_m} \rfloor} \mathbf{F}(\varsigma_k - \varsigma_{n_m},\ t).$$

There exists a sequence (k_m) *in* $(n_m, \lfloor \mu_{n_m} \rfloor]$ *such that*

$$\min_{n_m < k \le \lfloor \mu_{n_m} \rfloor} \mathbf{F}(\varsigma_k - \varsigma_{n_m},\ t) = \mathbf{F}(\varsigma_{k_m} - \varsigma_{n_m},\ t).$$

Note that $k_m \in (n_m, \lfloor \mu_{n_m} \rfloor]$ *implies* $n_m \in \Big(\lfloor \frac{k_m}{\mu} \rfloor,\ k_m\Big)$ *(see Remark 3 from [17]), we have*

$$\begin{aligned}
\mathbf{W}\Big(\frac{1}{\mu}\Big) &= \liminf_{k\to\infty} \min_{\lfloor \frac{k}{\mu}\rfloor < n \le k} \mathbf{F}(\varsigma_k - \varsigma_n,\ t)\\
&\le \lim_{m\to\infty} \min_{\lfloor \frac{k_m}{\mu}\rfloor < n \le k_m} \mathbf{F}(\varsigma_{k_m} - \varsigma_n,\ t)\\
&\le \lim_{m\to\infty} \mathbf{F}(\varsigma_{k_m} - \varsigma_{n_m},\ t)\\
&= \lim_{m\to\infty} \min_{n_m < k \le \mu_{n_m}} \mathbf{F}(\varsigma_k - \varsigma_{n_m},\ t)\\
&= \mathbf{T}(\mu)\\
\mathbf{W}(\frac{1}{\mu}) &\le \mathbf{T}(\mu)
\end{aligned}$$

On changing the roles of $\mathbf{T}(\mu)$, *and* $\mathbf{W}(\frac{1}{\mu})$ *and follow the above steps we will get* $\mathbf{W}(\frac{1}{\mu}) \ge \mathbf{T}(\mu)$. *Hence, we get* $\mathbf{W}(\frac{1}{\mu}) = \mathbf{T}(\mu)$, *i.e., conditions* 5 *and* 12 *are equivalent. This completes the proof.* □

Example 2.6. *Let* $\mathbb{R}$ *be real vector space and* $\big(\mathbb{R}, \mathcal{N}, \Upsilon, \Omega\big)$ *be neutrosophic normed space. Let* (ς_n) *be a sequnce in neutrosophic normed sapce defined as* $\varsigma_n = \sum_{i=1}^{n} \dfrac{1}{i}$. *Then sequence* (ς_n) *is slowly oscillating with respect to the neutrophic norm defined in example* 1.4 *Fix* $t > 0$. *For any* $r \in (0, 1)$, *choose* $\mu = tr + 1$. *Then for* $1 < n < m < \mu_n$, *in view of the fact that*

$$\begin{aligned}
|\varsigma_m - \varsigma_n| = \left|\sum_{i=1}^{m} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i}\right| &= \left|\sum_{i=1}^{n} \frac{1}{i} + \sum_{i=n+1}^{m} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i}\right|\\
&= \left|\sum_{i=n+1}^{m} \frac{1}{i}\right|\\
&= \left|\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{m}\right|\\
&\le \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} = \frac{m-n}{n}\\
&< \mu - 1 = tr
\end{aligned}$$

we get,

$$\mathbf{F}(\varsigma_m - \varsigma_n,\ t) = \frac{t}{t + |\varsigma_m - \varsigma_n|} > \frac{t}{t + tr} > 1 - r$$

$$\mathbf{G}(\varsigma_m - \varsigma_n,\ t) = \frac{|\varsigma_m - \varsigma_n|}{t + |\varsigma_m - \varsigma_n|} < \frac{tr}{t + |\varsigma_m - \varsigma_n|,} = \frac{r}{1 + \frac{|\varsigma_m - \varsigma_n|}{t}} \tag{15}$$

Since $\frac{|\varsigma_m - \varsigma_n|}{t} \geq 0$, *we get* $\frac{r}{1 + \frac{|\varsigma_m - \varsigma_n|}{t}} \leq r$. *Hence, from the equation 3.8 we have* $\mathbf{G}(\varsigma_m - \varsigma_n,\ t) < r$. *Similarly*

$$\mathbf{H}(\varsigma_m - \varsigma_n,\ t) = \frac{|\varsigma_m - \varsigma_n|}{t} < \frac{tr}{t} = r.$$

Hence the given sequence is slowly oscillating with respect to neutrosophic norm.

Theorem 2.7. *Let* $\left(S, \|.\|\right)$ *be a normed space and* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$ *be neutrosophic normed space as defined in Example* 1.4. *Then a sequence* (ς_n) *is slowly oscillating in* $\left(S, \|.\|\right)$ *if and only if* (ς_n) *is slowly oscillating in* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$.

Proof. Direct part; Let sequence (ς_n) *is slowly oscillating in* $\left(S, \|.\|\right)$. *Fix* $t > 0$ *and for any* $r \in (0, 1)$. *Define* $r_o = tr > 0$. *Then there exists* $\mu > 1$ *and* $n_0 \in \mathbb{N}$ *such that* $\|\varsigma_m - \varsigma_n\| < r_0$ *whenever* $n_0 < n < m \leq \mu_n$. *We get*

$$\mathbf{F}(\varsigma_m - \varsigma_n,\ t) = \frac{t}{t + \|\varsigma_m - \varsigma_n\|} > \frac{t}{t + tr} > 1 - r$$

$$\mathbf{G}(\varsigma_m - \varsigma_n,\ t) = \frac{\|\varsigma_m - \varsigma_n\|}{t + \|\varsigma_m - \varsigma_n\|} < \frac{tr}{t + \|\varsigma_m - \varsigma_n\|,} = \frac{r}{1 + \frac{\|\varsigma_m - \varsigma_n\|}{t}} \leq r$$

and

$$\mathbf{H}(\varsigma_m - \varsigma_n,\ t) = \frac{\|\varsigma_m - \varsigma_n\|}{t} < \frac{tr}{t} = r.$$

Hence, sequence (ς_n) *oscillates slowly in neutrosophic normed space* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$.
Conversely, let sequence (ς_n) *is slowly oscillating in neutrosophic normed space* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$. *Now using the definition of slowly oscillating sequence in neutrosophic normed space, for given* $r \in (0, \frac{1}{2})$, *there exists* $\mu > 1$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\mathbf{F}(\varsigma_m - \varsigma_n,\ 1) = \frac{1}{1 + \|\varsigma_m - \varsigma_n\|} > 1 - r \quad \text{whenever } n_0 < n < m \leq \mu_n.$$

From the above inequality we have

$$\|\varsigma_m - \varsigma_n\| < \frac{r}{1 - r} < 2r$$

This shows that sequence (ς_n) *is slowly oscillating in* $(S, \|.\|)$. *This completes the proof.* □

from Definition 1.7, 1.11 and 1.8 we have the relation *Cauchy* $\Longrightarrow$ *slow oscillation* $\Longrightarrow$ *G* − *Cauchy*. But the converse need not be true

Example 2.8. *Let* $\left(\mathbb{R}, \mathcal{N}, \Upsilon, \Omega\right)$ *be neutrosophic normed space defined as in example* 1.4. *Then the sequence* (ς_n) *defined as* $\varsigma_n = \sum_{j=1}^{n} \frac{1}{\sqrt{j}}$ *is* $G-$*Cauchy but not slowly oscillating and sequence* (π_n) *defined as* $\pi_n = \sum_{i=1}^{n} \frac{1}{i}$ *is slowly oscillating but is not Cauchy.*

Theorem 2.9. *Let $\left(S,\mathcal{N},\Upsilon,\mho\right)$ be a neutrosophic normed space and satisfies equation* (1) *. Let (ς_n) be a sequence in S. Then (ς_n) is slowly oscillating in $\left(S,\mathcal{N},\Upsilon,\mho\right)$ if and only if (ς_n) is slowly oscillating in $(S,\|.\|_c)$ for each $c \in (0,1)$.*

Proof. Let for given $c \in (0,1)$ and $s > 0$, sequence (ς_n) is slowly oscillating in $\left(S,\mathcal{N},\Upsilon,\mho\right)$. Then for $\epsilon = 1-c$ there exists $\mu > 1$ and $n_0 \in \mathbb{N}$ so that for $n_0 \leq n < k < \mu_n$ we get

$$\mathbf{F}(\varsigma_k-\varsigma_n,s) > 1-\epsilon,\ \ \mathbf{G}(\varsigma_k-\varsigma_n,s) < \epsilon \ \text{ and }\ \mathbf{H}(\varsigma_k-\varsigma_n,s) < \epsilon,$$

and

$$\|\varsigma_k-\varsigma_n\|_c = \inf\{t>0 : \mathbf{F}(\varsigma_k-\varsigma_n,t) > c,\ \mathbf{G}(\varsigma_k-\varsigma_n,t) < 1-c \ \text{ and }\ \mathbf{H}(\varsigma_k-\varsigma_n,t) < 1-c\} < s.$$

This implies that sequence (ς_n) is slowly oscillating in $(S,\|.\|_c)$.

Conversely, choose c ∈ (0,1) and let (ς_n) be slowly oscillating sequence in $(S,\|.\|_c)$. Then for $s > 0$ there exist $\mu > 1$ and $n_0 \in (0,1)$ such that

$$\|\varsigma_k-\varsigma_n\|_c = \inf\{t>0 : \mathbf{F}(\varsigma_k-\varsigma_n,t) > c,\ \mathbf{G}(\varsigma_k-\varsigma_n,t) < 1-c \ \text{ and }\ \mathbf{H}(\varsigma_k-\varsigma_n,t) < 1-c\} < s.$$

Whenever $n_0 \leq n < k < \mu_n$. Thus $\mathbf{F}(\varsigma_k-\varsigma_n,s) > c$, $\mathbf{G}(\varsigma_k-\varsigma_n,s) < 1-c$ and $\mathbf{H}(\varsigma_k-\varsigma_n,s) < 1-c$ whenever $n_0 \leq n < k < \mu_n$. Since s and c were arbitrary. Hence, sequence (ς_n) is slowly oscillating in $\left(S,\mathcal{N},\Upsilon,\mho\right)$. □

Theorem 2.10. *Let (ς_n) be a sequence in neutrosophic normed space $\left(S,\mathcal{N},\Upsilon,\mho\right)$. If (ς_n) is slowly oscillating then equations* (2.2)*,* (9) *and* (10) *are satisfied.*

Proof. Let (ς_n) be slowly oscillating sequence in neutrosophic normed space $\left(S,\mathcal{N},\Upsilon,\mho\right)$. For fix $t > 0$ and $\epsilon \in (0,1)$ there exist $\mu > 1$ and $n_0 \in (0,1)$ such that

$$\mathbf{F}(\varsigma_k-\varsigma_n,) > 1-\epsilon,\ \ \mathbf{G}(\varsigma_k-\varsigma_n,t) < \epsilon \ \text{ and }\ \mathbf{H}(\varsigma_k-\varsigma_n,t) < \epsilon,$$

whenever $n_0 \leq n < k < \mu_n$. We have

$$\begin{aligned}\mathbf{F}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n),\ t\right) &= \mathbf{F}\left(\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n),\ (\mu_n-n)t\right)\\ &\geq \min\{\mathbf{F}(\varsigma_{n+1}-\varsigma_n,\ t),\ \mathbf{F}(\varsigma_{n+2}-\varsigma_n,\ t)....,\mathbf{F}(\varsigma_{\mu_n}-\varsigma_n,\ t)\}\\ &> 1-\epsilon,\end{aligned}$$

$$\begin{aligned}\mathbf{G}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n),\ t\right) &= \mathbf{G}\left(\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n),\ (\mu_n-n)t\right)\\ &\leq \max\{\mathbf{G}(\varsigma_{n+1}-\varsigma_n,\ t),\ \mathbf{G}(\varsigma_{n+2}-\varsigma_n,\ t)....,\mathbf{G}(\varsigma_{\mu_n}-\varsigma_n,\ t)\}\\ &< \epsilon\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n),\ t\right) &= \mathbf{H}\left(\sum_{k=n+1}^{\mu_n}(\varsigma_k-\varsigma_n),\ (\mu_n-n)t\right)\\ &\leq \max\{\mathbf{H}(\varsigma_{n+1}-\varsigma_n,\ t),\ \mathbf{H}(\varsigma_{n+2}-\varsigma_n,\ t)....,\mathbf{H}(\varsigma_{\mu_n}-\varsigma_n,\ t)\}\\ &< \epsilon\end{aligned}$$

whenever $n_0 \leq n < k < \mu_n$. This completes the proof. □

From the previous theorems 2.2 and 2.10, we get the following theorem.

Theorem 2.11. *Let* (ς_n) *be a sequence in neutrosophic normed space* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$. *If* (ς_n) *is Cesàro summable to* $\varsigma \in S$ *and slowly oscillating, then* (ς_n) *converges to* ς.

Theorem 2.12. *Let* (ς_n) *be a sequence in neutrosophic normed space* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$. *If* $\{n(\varsigma_n\text{-}\varsigma_{n-1})\}$ *is* $q-$ *bounded, then* (ς_n) *is slowly oscillating.*

Proof. Given $\epsilon \in (0,1)$. By equation (1.6), $\exists$ Ł$_\epsilon > 0$ such that,

$$t > Ł_\epsilon \quad \text{implies that} \quad \inf_{n\in\mathbb{N}} \mathbf{F}(n(\varsigma_n - \varsigma_{n-1}),\ t) > 1-\epsilon,\ \sup_{n\in\mathbb{N}} \mathbf{G}(n(\varsigma_n - \varsigma_{n-1}),\ t) < \epsilon$$

and

$$\sup_{n\in\mathbb{N}} \mathbf{H}(n(\varsigma_n - \varsigma_{n-1}),\ t) < \epsilon.$$

For each $t > 0$, if we take $\mu < 1 + \frac{t}{Ł_\epsilon}$, then $n_0 \leq n < k < \mu_n$

$$\begin{aligned}
\mathbf{F}(\varsigma_k - \varsigma_n,\ t) &= \mathbf{F}\left(\sum_{j=n+1}^{k} (\varsigma_j - \varsigma_{j-1}),\ t\right) \\
&\geq \min_{n+1\leq j\leq k} \mathbf{F}\left((\varsigma_j - \varsigma_{j-1}),\ \frac{t}{k-n}\right) \\
&= \min_{n+1\leq j\leq k} \mathbf{F}\left(j(\varsigma_j - \varsigma_{j-1}),\ \frac{jt}{k-n}\right) \\
&\geq \min_{n+1\leq j\leq k} \mathbf{F}\left(j(\varsigma_j - \varsigma_{j-1}),\ \frac{nt}{k-n}\right) \\
&\geq \min_{n+1\leq j\leq k} \mathbf{F}\left(j(\varsigma_j - \varsigma_{j-1}),\ \frac{t}{\frac{k}{n}-1}\right) \\
&\geq \min_{n+1\leq j\leq k} \mathbf{F}\left(j(\varsigma_j - \varsigma_{j-1}),\ \frac{t}{\mu-1}\right) \\
&\geq \inf_{n\in\mathbb{N}} \mathbf{F}\left(n(\varsigma_n - \varsigma_{n-1}),\ \frac{t}{\mu-1}\right) \\
&> 1-\epsilon
\end{aligned}$$

and similarly

$$\mathbf{G}(\varsigma_k - \varsigma_n,\ t) < \sup_{n\in\mathbb{N}} \mathbf{G}\left(n(\varsigma_n - \varsigma_{n-1}),\ \frac{t}{\mu-1}\right) < \epsilon$$

and

$$\mathbf{H}(\varsigma_k - \varsigma_n,\ t) < \sup_{n\in\mathbb{N}} \mathbf{H}\left(n(\varsigma_n - \varsigma_{n-1}),\ \frac{t}{\mu-1}\right) < \epsilon.$$

Hence sequence (ς_n) is slowly oscillating. □

In view of Theorem 2.11 and 2.12, we give an analogue of classical two sided Tauberian theorem due to Hardy [8].

Theorem 2.13. *Let* (ς_n) *be a sequence in neutrosophic normed space* $\left(S, \mathcal{N}, \Upsilon, \Omega\right)$. *If* (ς_n) *is Cesàro summable to* $\varsigma \in S$ *and* $\{n(\varsigma_n\text{-}\varsigma_{n-1})\}$ *is* $q-$ *bounded, then* (ς_n) *converges to* ς.

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