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A regularized trace of an even-order differential operator with bounded operator coefficient given in a finite interval

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Abstract. In this study, we obtain a regularized trace formula of an even-order differential operator with a bounded operator coefficient given in a finite interval.

1. Introduction

Let H be separable Hilbert Space with infinite dimension. In Hilbert space $H_1 = L_2(0, \pi, H)$, we consider the operators L_0 and L which are defined by the differential expressions

$$L_0(y) = (-1)^n y^{(2n)}(x)$$

$$L(y) = (-1)^n y^{(2n)}(x) + Q(x)y(x)$$

respectively and same boundary conditions

$$y'(0) = y'''(0) = ... = y^{(2n-1)}(0) = y(\pi) = y''(\pi) = ... = y^{(2n-2)}(\pi) = 0.$$

Let the operator function $Q(x)$ satisfies the conditions given below.
1.For all $x \in [0, \pi]$, $Q(x) : H \to H$ is a self adjoint kernel operator. $Q(x)$ has a second order continuous derivative in the interval $[0, \pi]$ according to the norm in space $\sigma_1(H)$. Here $\sigma_1(H) : H \to H$ is the space of kernel operators [7],
2.$\|Q\| < \frac{3^{2n}-1}{2^{2n+1}}$,
3.H has an orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ where $\Sigma_{k=1}^{\infty} \|Q(x)\varphi_k\| < \infty$.
The inner products in the spaces H_1 and H will be denoted by $(.,.)$ and $(.,.)_H$, respectively. Further, the sum of eigenvalues of a kernel operator A will be denoted by trA and the norm in H will be denoted by $\|.\|_H$
The spectrum of the operator L_0 is the set

$$\sigma(L_0) = \left\{(m + \frac{1}{2})^{2n}\right\}_{m=0}^{\infty}$$

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Each point of this set is an eigenvalue of the operator L_0 with infinite multiplicity. Orthonormal eigenvectors corresponding to the eigenvalue $(m+\frac{1}{2})^{2n}$ are

$$\Psi_{mk}(x) = \sqrt{\frac{1}{\pi}}\cos{(m+\frac{1}{2})x}\varphi_k, \quad (k=1,2,...). \tag{1}$$

Let the resolvents of the operators L_0 and L be R_λ^0 and R_λ, respectively. Let the operator $Q : H_1 \to H_1$ satisfies the conditions 1-3 given above, then the following can be proved:
i)$QR_\lambda^0 \in \sigma_1(H_1)$, for all $\lambda \neq \sigma(L_0)$.
ii)The spectrum $\sigma(L)$ of the operator L is a subset of the combination of discrete intervals

$$F_m = \left[(m+\frac{1}{2})^{2n} - \|Q\|, (m+\frac{1}{2})^{2n} + \|Q\|\right] \quad (m=0,1,2,...),$$

i.e. $\sigma(L) \subset \cup_{m=0}^{\infty} F_m$.
iii)Each point of the spectrum of the operator L different from $(m+\frac{1}{2})^{2n}$ belonging to the interval F_m is a discrete eigenvalue of finite multiplicity.
iv)The series

$$\sum_{k=1}^{\infty}\left[\lambda_{mk} - (m+\frac{1}{2})^{2n}\right] \quad (m=0,1,2,...) \tag{2}$$

are absolutely convergent where $\{\lambda_{mk}\}_{k=1}^{\infty}$ are the eigenvalues of operator L belonging to the interval F_m.
First, [9] obtained the theory of regularized trace of ordinary differential operators. After this study, several mathematicians developed regular trace formulas for several differential operators with scalar coefficients(See [5], [8]-[12], [16]). The lists of the studies related to this subject are given in [13] and [14]. In addition, regularized trace formulas of the differential operators with operator coefficients are investigated in several studies. Some of these studies are [1]-[4],[6] and [15].
This article aims to calculate the regularized trace of the differential operator L with operator coefficient $Q(x)$.

2. Relations between Resolvents and Eigenvalues

For every $\lambda \in \rho(L) = \mathbb{R}\backslash\sigma(L)$,

$$QR_\lambda^0 \in \sigma_1(H_1).$$

Therefore $(R_\lambda - R_\lambda^0) \in \sigma_1(H_1)$ can be seen from the formula $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$.
On the other hand, considering that the series

$$\sum_{k=1}^{\infty}[\lambda_{mk} - (m+\frac{1}{2})^{2n}] \quad (m=0,1,2,...)$$

is absolutely convergent, we can obtain

$$tr(R_\lambda - R_\lambda^0) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\left[\frac{1}{\lambda_{mk}-\lambda} - \frac{1}{(m+\frac{1}{2})^{2n}-\lambda}\right]$$

[6].If this equation is multiplied by $\frac{\lambda}{2\pi i}$ and integrated over the circle

$$|\lambda| = b_p = \frac{1}{2}\left[(p+\frac{1}{2})^{2n} + (p+\frac{3}{2})^{2n}\right] \quad (p \in \mathbb{N}, p \geq 1)$$

then

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda tr(R_\lambda - R_\lambda^0)d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=0}^{p}\sum_{k=1}^{\infty}\left[\frac{1}{\lambda_{mk}-\lambda}-\frac{1}{(m+\frac{1}{2})^{2n}-\lambda}\right]d\lambda$$

$$+\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=p+1}^{\infty}\sum_{k=1}^{\infty}\left[\frac{1}{\lambda_{mk}-\lambda}-\frac{1}{(m+\frac{1}{2})^{2n}-\lambda}\right]d\lambda \tag{3}$$

is obtained. For $m \le p$ and $p \ge 1$,

$$(m+\frac{1}{2})^{2n} - \|Q\| \le \lambda_{mk} \le (m+\frac{1}{2})^{2n} + \|Q\| < (p+\frac{1}{2})^{2n} + \frac{3^{2n}-1}{2^{2n+1}} < \frac{1}{2}\left[(p+\frac{1}{2})^{2n} + (p+\frac{3}{2})^{2n}\right] = b_p.$$

Since,

$$|\lambda_{mk}| < b_p; m \le p; \quad p \ge 1; k = 1, 2, ..., \tag{4}$$

then for $m > p \ge 1$,

$$\lambda_{mk} \ge (m+\frac{1}{2})^{2n} - \|Q\| > (p+\frac{3}{2})^{2n} - \frac{3^{2n}-1}{2^{2n+1}} > \frac{1}{2}\left[(p+\frac{1}{2})^{2n} + (p+\frac{3}{2})^{2n}\right] = b_p.$$

Therefore

$$\lambda_{mk} > b_p; m > p \ge 1; k = 1, 2, ... \tag{5}$$

From (3), (4) and (5),

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda tr(R_\lambda - R_\lambda^0)d\lambda = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-(m+\frac{1}{2})^{2n}}d\lambda - \frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-\lambda_{mk}}d\lambda\right]$$

$$+\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-(m+\frac{1}{2})^{2n}}d\lambda - \frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-\lambda_{mk}}d\lambda\right] = \sum_{m=0}^{p}\sum_{k=1}^{\infty}\left[(m+\frac{1}{2})^{2n}-\lambda_{mk}\right] \tag{6}$$

is obtained. By using the formula $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$, we get

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{4}(-1)^j R_\lambda^0 (QR_\lambda^0)^j - R_\lambda (QR_\lambda^0)^5.$$

If this expression is substituted in equation (6), we find

$$\sum_{m=0}^{p}\sum_{k=1}^{\infty}\left[(m+\frac{1}{2})^{2n}-\lambda_{mk}\right] = \sum_{j=1}^{4}\frac{(-1)^i}{2\pi i}\int_{|\lambda|=b_p}\lambda tr[R_\lambda^0(QR_\lambda^0)^j]d\lambda - \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda tr[R_\lambda(QR_\lambda^0)^5]d\lambda. \tag{7}$$

Let

$$M_{pj} = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda tr[R_\lambda^0(QR_\lambda^0)^j]d\lambda, \tag{8}$$

and

$$M_p = \frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \lambda tr[R_\lambda (QR_\lambda^0)^5] d\lambda. \tag{9}$$

Then the equation (7) can be written as

$$\sum_{m=0}^{p} \sum_{k=1}^{\infty} \left[\lambda_{mk} - (m + \frac{1}{2})^{2n} \right] = \sum_{j=1}^{4} M_{pj} + M_p. \tag{10}$$

Theorem 2.1. *If the operator function $Q(x)$ satisfies the 3rd condition, then*

$$M_{pj} = \frac{(-1)^j}{2\pi i j} \int\limits_{|\lambda|=b_p} tr[(QR_\lambda^0)^j] d\lambda.$$

Proof. It can be shown that the operator function QR_λ^0 in the $\rho(L_0)$ region is analytical with respect to the norm in the space $\sigma_1(H_1)$ and

$$tr\left\{ \left[(QR_\lambda^0)^j \right]' \right\} = j tr\left[(QR_\lambda^0)' (QR_\lambda^0)^{j-1} \right]. \tag{11}$$

Considering that $(QR_\lambda^0)' = Q(R_\lambda^0)^2$, the formula (11) can be written as

$$tr\left\{ \left[(QR_\lambda^0)^j \right]' \right\} = j tr\left[R_\lambda^0 (QR_\lambda^0)^j \right]. \tag{12}$$

From (8) and (12),we have

$$M_{pj} = \frac{(-1)^{j+1}}{2\pi i j} \int\limits_{|\lambda|=b_p} \lambda tr\{[(QR_\lambda^0)^j]'\} d\lambda.$$

Hence,

$$M_{pj} = \frac{(-1)^{j+1}}{2\pi i j} \int\limits_{|\lambda|=b_p} tr\{[\lambda (QR_\lambda^0)^j]' - (QR_\lambda^0)^j\} d\lambda = \frac{(-1)^j}{2\pi i j} \int\limits_{|\lambda|=b_p} tr[(QR_\lambda^0)^j] d\lambda + \frac{(-1)^{j+1}}{2\pi i j} \int\limits_{|\lambda|=b_p} tr\{[\lambda (QR_\lambda^0)^j]'\} d\lambda. \tag{13}$$

It can easily be shown that

$$tr\{[\lambda (QR_\lambda^0)^j]'\} = \{tr[\lambda (QR_\lambda^0)^j]\}'.$$

Thus,

$$\int\limits_{|\lambda|=b_p} tr\{[\lambda (QR_\lambda^0)^j]'\} d\lambda = \int\limits_{|\lambda|=b_p} \{tr[\lambda (QR_\lambda^0)^j]\}' d\lambda. \tag{14}$$

The integral on the right side of this equation can be written as

$$\int\limits_{|\lambda|=b_p} \{tr[\lambda (QR_\lambda^0)^j]\}' d\lambda = \int\limits_{|\lambda|=b_p, Im\lambda \geq 0} \{tr[\lambda (QR_\lambda^0)^j]\}' d\lambda + \int\limits_{|\lambda|=b_p, Im\lambda \leq 0} \{tr[\lambda (QR_\lambda^0)^j]\}' d\lambda. \tag{15}$$

Let ε_0 be a constant that satisfies the condition $0 < \varepsilon_0 < b_p - (p + \frac{1}{2})^{2n}$. Considering that the function $tr[\lambda (QR_\lambda^0)^j]$ is analytical in its simply connected regions

$$G_1 = \{\lambda \in \mathbb{C} : b_p - \varepsilon_0 < |\lambda| < b_p + \varepsilon_0, Im\lambda > -\varepsilon_0\},$$

$$G_2 = \{\lambda \in \mathbb{C} : b_p - \varepsilon_0 < |\lambda| < b_p + \varepsilon_0, Im\lambda < \varepsilon_0\},$$

and

$$\{\lambda \in \mathbb{C} : \quad |\lambda| = b_p, \quad Im\lambda \geq 0\} \subset G_1,$$

$$\{\lambda \in \mathbb{C} : \quad |\lambda| = b_p, \quad Im\lambda \leq 0\} \subset G_2,$$

from (15), we obtain

$$\int\limits_{|\lambda|=b_p} \{tr[\lambda(QR_\lambda^0)^j]\}' d\lambda = tr[-b_p(QR_{-b_p}^0)^j] - tr[b_p(QR_{b_p}^0)^j] + tr[b_p(QR_{b_p}^0)^j] - tr[-b_p(QR_{-b_p}^0)^j] = 0. \tag{16}$$

Finally, from (13), (14) and (16), we have

$$M_{pj} = \frac{(-1)^j}{2\pi i j} \int\limits_{|\lambda|=b_p} tr[(QR_\lambda^0)^j] d\lambda.$$

□

3. Regularized Trace Formula

In this section, we will find the formula for the sum of the following series

$$\sum_{m=0}^{\infty} \left\{ \sum_{k=1}^{\infty} \left[\lambda_{mk} - (m + \frac{1}{2})^{2n} \right] - \frac{1}{\pi} \int\limits_0^{\pi} trQ(x)dx \right\}.$$

The sum of this series is called the regular trace of the L operator.
According to Theorem (2.1)

$$M_{p1} = -\frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} tr(QR_\lambda^0) d\lambda. \tag{17}$$

Let $\{\Psi_{mk}\}_{m=0,k=1}^{\infty\ \infty}$ be the eigenvectors system of the L_0 operator and orthonormal basis of the space H_1, from (17)

$$M_{p1} = -\frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} (QR_\lambda^0 \Psi_{mn}, \Psi_{mn}) d\lambda = -\frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{(Q\Psi_{mk}, \Psi_{mk})}{(m + \frac{1}{2})^{2n} - \lambda} d\lambda$$

$$= \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} (Q\Psi_{mk}, \Psi_{mk}) \frac{1}{2\pi i} \int\limits_{|\lambda|=b_p} \frac{1}{\lambda - (m + \frac{1}{2})^{2n}} d\lambda \tag{18}$$

is obtained. Since, for $m \leq p$,

$$(m + \frac{1}{2})^{2n} < b_p = \frac{1}{2} \left[(p + \frac{1}{2})^{2n} + (p + \frac{3}{2})^{2n} \right],$$

and for $m > p$,

$$(m + \frac{1}{2})^{2n} > b_p = \frac{1}{2} \left[(p + \frac{1}{2})^{2n} + \left(p + \frac{3}{2}\right)^{2n} \right]$$

from equation (18) we have

$$M_{p1} = \sum_{m=0}^{p}\sum_{k=1}^{\infty}(Q\Psi_{mk}, \Psi_{mk})\frac{1}{2\pi i}\int_{|\lambda|=b_p} \frac{1}{\lambda - (m+\frac{1}{2})^{2n}}d\lambda = \sum_{m=0}^{p}\sum_{k=1}^{\infty}(Q\Psi_{mk}, \Psi_{mk}). \tag{19}$$

By using (1) and (19), we find

$$M_{p1} = \sum_{m=0}^{p}\sum_{k=1}^{\infty}\int_0^{\pi}(Q(x)\sqrt{\frac{2}{\pi}}\cos(m+\frac{1}{2})x\varphi_k, \sqrt{\frac{2}{\pi}}\cos(m+\frac{1}{2})x\varphi_k)_H dx$$

$$= \frac{2}{\pi}\sum_{m=0}^{p}\sum_{k=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_k,\varphi_k)_H\cos^2(m+\frac{1}{2})xdx = \frac{1}{\pi}\sum_{m=0}^{p}\sum_{k=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_k,\varphi_k)_H(1+\cos(2m+1)x)dx \tag{20}$$

Since $\sum_{n=1}^{\infty}(Q(x)\varphi_k,\varphi_k)_H = trQ(x)$, from (20), we write

$$M_{p1} = \frac{p+1}{\pi}\int_0^{\pi} trQ(x)dx + \frac{1}{\pi}\sum_{m=0}^{p}\int_0^{\pi} trQ(x)\cos(2m+1)xdx. \tag{21}$$

Lemma 3.1. *If the $Q(x)$ operator function satisfies the 2nd and 3rd conditions, then*

$$\|R_\lambda\| < const p^{(-2n+1)}$$

on the circle $|\lambda| = b_p$.

When 2nd and 3rd conditions are satisfied,

$$\{\lambda_{mk}\}_{k=1}^{\infty} \subset \left[(m+\frac{1}{2})^{2n} - \|Q\|, (m+\frac{1}{2})^{2n} + \|Q\|\right] \quad (m = 0,1,2...)$$

and

$$|\lambda_{mk} - (m+\frac{1}{2})^{2n}| \leq \|Q\| < \frac{3^{2n}-1}{2^{2n+1}} \quad (m = 0,1,2,...; k = 0,1,2...).$$

Considering these relations, for $m \leq p$, we have

$$|\lambda_{mk} - \lambda| = |\lambda - (m+\frac{1}{2})^{2n} - (\lambda_{mk} - (m+\frac{1}{2})^{2n})| \geq |\lambda - (m+\frac{1}{2})^{2n}| - |\lambda_{mk} - \left(m+\frac{1}{2}\right)^{2n}|$$

$$> |\lambda| - (m+\frac{1}{2})^{2n} - \frac{3^{2n}-1}{2^{2n+1}} \geq \frac{1}{2}\left[(p+\frac{1}{2})^{2n} + (p+\frac{3}{2})^{2n}\right] - (p+\frac{1}{2})^{2n} - \frac{3^{2n}-1}{2^{2n+1}}$$

$$= \frac{1}{2}\left[p+\frac{3}{2})^{2n} - (p+\frac{1}{2})^{2n}\right] - \frac{3^{2n}-1}{2^{2n+1}} > n(p+\frac{1}{2})^{2n-1} - \frac{3^{2n}-1}{2^{2n+1}}$$

$$> const(p^{2n-1}), \tag{22}$$

and for $m \geq p+1$, we have

$$|\lambda_{mk} - \lambda| = |(m+\frac{1}{2})^{2n} - \lambda - ((m+\frac{1}{2})^{2n} - \lambda_{mk})| \geq |(m+\frac{1}{2})^{2n} - \lambda| - |(m+\frac{1}{2})^{2n} - \lambda_{mk}| \geq (m+\frac{1}{2})^{2n} - |\lambda| - \frac{3^{2n}-1}{2^{2n+1}}$$

$$\geq (p+\frac{3}{2})^{2n} - \frac{1}{2}\left[(p+\frac{1}{2})^{2n} + (p+\frac{3}{2})^{2n}\right] - \frac{3^{2n}-1}{2^{2n+1}} = \frac{1}{2}\left[(p+\frac{3}{2})^{2n} - (p+\frac{1}{2})^{2n}\right] - \frac{3^{2n}-1}{2^{2n+1}} > const(p^{2n-1}). \tag{23}$$

Moreover,

$$\|R_\lambda\| = \max_{\substack{m=0,1,...\\ k=1,2,...}} \{|\lambda_{mk} - \lambda|^{-1}\} \tag{24}$$

and from (22), (23) and (24), we obtain

$$\|R_\lambda\| < const(p^{-2n+1}).$$

Lemma 3.2. *If the function $Q(x)$ satisfies the 2nd and 3rd conditions,*

$$\|QR_\lambda^0\|_{\sigma_1(H_1)} < const(p^{2-2n})$$

on the circle $|\lambda| = b_p$.

Proof. For $\lambda \notin \sigma(L_0)$

$$\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \|QR_\lambda^0 \Psi_{mk}\| = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} |(m+\frac{1}{2})^{2n} - \lambda|^{-1} \|Q\Psi_{mk}\|$$

$$= \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} |(m+\frac{1}{2})^{2n} - \lambda|^{-1} [\int_0^\pi \|Q(x)\sqrt{\frac{2}{\pi}}\cos(m+\frac{1}{2})x\varphi_k\|_H^2 dx]^{\frac{1}{2}}$$

$$= \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} |(m+\frac{1}{2})^{2n} - \lambda|^{-1} [\int_0^\pi \|Q(x)\varphi_k\|_H^2 \cos^2(m+\frac{1}{2})x dx]^{\frac{1}{2}}$$

$$< \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} |(m+\frac{1}{2})^{2n} - \lambda|^{-1} [\int_0^\pi \|Q(x)\varphi_k\|_H^2 dx]^{\frac{1}{2}} = \sum_{m=0}^{\infty} |(m+\frac{1}{2})^{2n} - \lambda|^{-1} \sum_{k=1}^{\infty} \|Q(x)\varphi_k\| \tag{25}$$

Hence, we find

$$\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \|QR_\lambda^0 \Psi_{mk}\| < \infty \quad (\lambda \in \sigma(L_0)).$$

Since $\{\Psi\}_{m=0,k=1}^{\infty\ \ \infty}$ is an orthonormal basis of H_1 space,

$$\|QR_\lambda^0\|_{\sigma_1(H_1)} \le \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \|QR_\lambda^0 \Psi_{mk}\| \tag{26}$$

can be written [1].From (25) and (26)

$$\|QR_\lambda^0\|_{\sigma_1(H_1)} \le \sum_{k=1}^{\infty} \|Q(x)\varphi_k\| \sum_{m=0}^{\infty} |(m+\frac{1}{2})^{2n} - \lambda|^{-1} \tag{27}$$

is obtained. On $|\lambda| = b_p$ circle □

$$\sum_{m=0}^{\infty} |(m+\frac{1}{2})^{2n} - \lambda|^{-1} = \sum_{m=0}^{p} |(m+\frac{1}{2})^{2n} - \lambda|^{-1} + \sum_{m=p+1}^{\infty} |(m+\frac{1}{2})^{2n} - \lambda|^{-1}$$

$$< \sum_{m=0}^{p} (|\lambda| - (m+\frac{1}{2})^{2n})^{-1} + \sum_{m=p+1}^{\infty} ((m+\frac{1}{2})^{2n} - |\lambda|)^{-1} = \frac{1}{2}\sum_{m=0}^{p} \left((p+\frac{3}{2})^{2n} - (p+\frac{1}{2})^{2n}\right)^{-1} + \sum_{m=p+1}^{\infty} ((m+\frac{1}{2})^{2n} - |\lambda|)^{-1}$$

$$< const\sum_{m=0}^{p} p^{-2n+1} + \sum_{m=p+1}^{\infty}((m+\frac{1}{2})^{2n}-|\lambda|)^{-1} < constp^{-2n+2} + \sum_{m=p+1}^{\infty}((m+\frac{1}{2})^{2n}-|\lambda|)^{-1}, \tag{28}$$

and

$$\sum_{m=p+1}^{\infty}((m+\frac{1}{2})^{2n}-\lambda)^{-1} = ((p+\frac{3}{2})^{2n}-(p+\frac{1}{2})^{2n})^{-1} + \sum_{m=p+2}^{\infty}((m+\frac{1}{2})^{2n}-|\lambda|)^{-1}$$

$$< constp^{1-2n} + \sum_{m=p+2}^{\infty}((m+\frac{1}{2})^{2n}-(p+\frac{3}{2})^{2n})^{-1}. \tag{29}$$

Moreover,

$$\sum_{m=p+2}^{\infty}((m+\frac{1}{2})^{2n}-(p+\frac{3}{2})^{2n})^{-1} \leq ((p+\frac{5}{2})^{2n}-(p+\frac{3}{2})^{2n})^{-1} + \int_{p+\frac{5}{2}}^{\infty}(x^{2n}-(p+\frac{3}{2})^{2n})^{-1}dx. \tag{30}$$

Let $x^{2n}-(p+\frac{3}{2})^{2n}=t$, then

$$\int_{p+\frac{5}{2}}^{\infty}(x^{2n}-(p+\frac{3}{2})^{2n})^{-1}dx = \frac{1}{2n}\int_{(p+\frac{5}{2})^{2n}-(p+\frac{3}{2})^{2n}}^{\infty}[t^{-1}(t+(p+\frac{3}{2})^{2n})^{\frac{1}{2n}-1}]dt < \int_{(p+\frac{5}{2})^{2n}-(p+\frac{3}{2})^{2n}}^{\infty} t^{\frac{1}{2n}-2}dt$$

$$< \left.\frac{t^{\frac{1}{2n}-1}}{\frac{1}{2n}-1}\right|_{(p+\frac{5}{2})^{2n}-(p+\frac{3}{2})^{2n}}^{\infty} < const(p^{2n-1})^{\frac{1-2n}{2n}} = const(p^{-\frac{(2n-1)^2}{2n}}). \tag{31}$$

From (30) and (31), we obtain

$$\sum_{m=p+2}^{\infty}((m+\frac{1}{2})^{2n}-(p+\frac{3}{2})^{2n})^{-1} < const(p^{1-2n}+p^{-\frac{(2n-1)^2}{2n}}) \tag{32}$$

Moreover, from (28), (29) and (32), we get

$$\sum_{m=0}^{\infty}|(m+\frac{1}{2})^{2n}-\lambda|^{-1} < const(p^{2-2n}+p^{-\frac{(2n-1)^2}{2n}}) \tag{33}$$

and from (27) and (33), we get

$$||QR_{\lambda}^{0}||_{\sigma_1(H_1)} < const(p^{2-2n})$$

Now we will show that $\lim_{p\to\infty} M_{p2}=0$. According to Theorem 2.1,

$$M_{p2} = \frac{1}{4\pi i}\int_{|\lambda|=b_p} tr[(QR_{\lambda}^{0})^2]d\lambda.$$

Since $tr(QR_{\lambda}^{0})^2 = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}((QR_{\lambda}^{0})^2\Psi_{mk},\Psi_{mk})$, then

$$M_{p2} = \frac{1}{4\pi i}\int_{|\lambda|=b_p}[\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}((QR_{\lambda}^{0})^2\Psi_{mk},\Psi_{mk})]d\lambda. \tag{34}$$

Also we have

$$QR_\lambda^0\Psi_{mk} = \frac{Q\Psi_{mk}}{(m+\frac{1}{2})^{2n}-\lambda}$$

and

$$(QR_\lambda^0)^2\Psi_{mk} = \frac{1}{(m+\frac{1}{2})^{2n}-\lambda}QR_\lambda^0 Q\Psi_{mk} = ((m+\frac{1}{2})^{2n}-\lambda)^{-1}QR_\lambda^0\sum_{m_1=0}^{\infty}\sum_{k_1=1}^{\infty}(Q\Psi_{mk},\Psi_{m_1k_1})\Psi_{m_1k_1}$$

$$= ((m+\frac{1}{2})^{2n}-\lambda)^{-1}\sum_{m_1=0}^{\infty}\sum_{k_1=1}^{\infty}\frac{(Q\Psi_{mk},\Psi_{m_1k_1})}{(m_1+\frac{1}{2})^{2n}-\lambda}Q\Psi_{m_1k_1}. \tag{35}$$

By using (34) and (35), we write

$$M_{p2} = \frac{1}{4\pi i}\int_{|\lambda|=b_p}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{k_1=1}^{\infty}\frac{(Q\Psi_{mk},\Psi_{m_1k_1})(Q\Psi_{m_1k_1},\Psi_{mk})}{(\lambda-(m+\frac{1}{2})^{2n})(\lambda-(m_1+\frac{1}{2})^{2n})}d\lambda. \tag{36}$$

By taking the following equality into consideration for $(m+\frac{1}{2})^{2n} < b_p$ and $(m_1+\frac{1}{2})^{2n} < b_p$ ($(m+\frac{1}{2})^{2n} > b_p$ and $(m_1+\frac{1}{2})^{2n} > b_p$),

$$\int_{|\lambda|=b_p}\frac{1}{(\lambda-(m+\frac{1}{2})^{2n})(\lambda-(m_1+\frac{1}{2})^{2n})}d\lambda = \int_{|\lambda|=b_p}\left(\frac{1}{(\lambda-(m+\frac{1}{2})^{2n})}-\frac{1}{(\lambda-(m_1+\frac{1}{2})^{2n})}\right)\frac{1}{(m+\frac{1}{2})^{2n}-(m_1+\frac{1}{2})^{2n}}d\lambda = 0$$

we can write the equation (36) as

$$M_{p2} = \frac{1}{2\pi i}\sum_{m=0}^{p}\sum_{k=1}^{\infty}\sum_{m_1=p+1}^{\infty}\sum_{k_1=1}^{\infty}(Q\Psi_{mk},\Psi_{m_1k_1})(Q\Psi_{m_1k_1},\Psi_{mk})\times\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-(m+\frac{1}{2})^{2n})(\lambda-(m_1+\frac{1}{2})^{2n})}.$$

For $m \le p$ and $m_1 \ge p+1$,

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-(m+\frac{1}{2})^{2n})(\lambda-(m_1+\frac{1}{2})^{2n})} = \frac{1}{(m+\frac{1}{2})^{2n}-(m_1+\frac{1}{2})^{2n}}.$$

Therefore

$$M_{p2} = \frac{1}{2\pi i}\sum_{m=0}^{p}\sum_{k=1}^{\infty}\sum_{m_1=p+1}^{\infty}\sum_{k_1=1}^{\infty}\frac{1}{(m+\frac{1}{2})^{2n}-(m_1+\frac{1}{2})^{2n}}|(Q\Psi_{mk},\Psi_{m_1k_1})|^2.$$

From here, we get

$$|M_{p2}| \le \sum_{m_1=p+1}^{\infty}\sum_{k_1=1}^{\infty}\frac{1}{(m_1+\frac{1}{2})^{2n}-(p+\frac{1}{2})^{2n}}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}|(Q\Psi_{mk},\Psi_{m_1k_1})|^2 = \sum_{m_1=p+1}^{\infty}\frac{1}{(m_1+\frac{1}{2})^{2n}-(p+\frac{1}{2})^{2n}}\sum_{k=1}^{\infty}||Q\varphi_{m_1k_1}||^2. \tag{37}$$

If the operator function $Q(x)$ satisfies its 3rd condition, then

$$\sum_{k=1}^{\infty}||Q\Psi_{mk}||^2 = \sum_{k=1}^{\infty}\int_0^{\pi}||Q(x)\sqrt{\frac{2}{\pi}}\cos(m+\frac{1}{2})x\varphi_k||_H^2dx \le \sum_{k=1}^{\infty}\int_0^{\pi}||Q(x)\varphi_k||_H^2dx = \sum_{k=1}^{\infty}||Q(x)\varphi_k||^2 < const \tag{38}$$

is obtained. From (32), (37) and (38), we have

$$|M_{p2}| < const(p^{1-2n} + p^{-\frac{(2n-1)^2}{2n}}),$$

thus,

$$\lim_{p\to\infty} M_{p2} = 0 \tag{39}$$

is obtained. The equality

$$\lim_{p\to\infty} M_{p3} = 0 \tag{40}$$

can be proved similarly.

Theorem 3.3. *If the operator function satisfies the conditions 1-3, then*

$$\sum_{m=0}^{\infty}\left\{\sum_{k=1}^{\infty}[\lambda_{mk} - (m+\frac{1}{2})^{2n}] - \frac{1}{\pi}\int_0^{\pi} trQ(x)dx\right\} = \frac{1}{4}[trQ(0) - trQ(\pi)].$$

Proof. By using Theorem 2.1, Lemma 3.1 and Lemma 3.2, we have

$$|M_{p4}| \le |\int\limits_{|\lambda|=b_p} tr(QR_\lambda^0)^4 d\lambda| \le \int\limits_{|\lambda|=b_p} |tr(QR_\lambda^0)^4||d\lambda| \le \int\limits_{|\lambda|=b_p} ||(QR_\lambda^0)^4||_{\sigma_1(H_1)}|d\lambda| \le \int\limits_{|\lambda|=b_p} ||(QR_\lambda^0)^3||||QR_\lambda^0||_{\sigma_1(H_1)}|d\lambda|$$

$$\le \int_{|\lambda|=b_p} p^{3(1-2n)}p^{(2-2n)}|d\lambda| < const(p^{5-8n}p^{2n}) = const(p^{5-6n}) \tag{41}$$

From (9) and again by using Lemma 3.1 and Lemma 3.2, we get

$$|M_p| = |\int\limits_{|\lambda|=b_p} \lambda tr[R_\lambda(QR_\lambda^0)^5]d\lambda| \le \int\limits_{|\lambda|=b_p} |\lambda tr[R_\lambda(QR_\lambda^0)^5]||d\lambda| \le b_p \int\limits_{|\lambda|=b_p} ||R_\lambda(QR_\lambda^0)^5]||_{\sigma_1(H_1)}|d\lambda|$$

$$\le b_p \int\limits_{|\lambda|=b_p} ||R_\lambda||||(QR_\lambda^0)^4]||||(QR_\lambda^0)||_{\sigma_1(H_1)}|d\lambda| < const b_p \int\limits_{|\lambda|=b_p} p^{5(1-2n)}p^{2-2n}|d\lambda| \le const p^{4n+5(1-2n)}$$

$$= const(p^{5-6n}) \tag{42}$$

From (41) and (42) we obtain

$$\lim_{p\to\infty} M_{p4} = \lim_{p\to\infty} M_p = 0. \tag{43}$$

From (10) and (21),we get

$$\sum_{m=0}^{p}\sum_{k=1}^{\infty}[\lambda_{mk} - (m+\frac{1}{2})^{2n}] = \frac{p+1}{\pi}\int_0^{\pi} trQ(x)dx + \frac{1}{\pi}\sum_{m=0}^{p}\int_0^{\pi} trQ(x)\cos(2m+1)xdx + \sum_{j=2}^{4} M_{pj} + M_p. \tag{44}$$

Also from (39), (40), (43) and (44) we get

$$\sum_{m=0}^{\infty}\left\{\sum_{k=1}^{\infty}[\lambda_{mk} - (m+\frac{1}{2})^{2n}] - \frac{1}{\pi}\int_0^{\pi} trQ(x)dx\right\} = \frac{1}{\pi}\sum_{m=0}^{\infty}\int_0^{\pi} trQ(x)\cos(2m+1)xdx \tag{45}$$

Using the fact that the operator function $Q(x)$ satisfies the first condition,for the expression on the right side of equation (45), we get

$$\frac{1}{\pi}\sum_{m=0}^{\infty}\int_0^{\pi} trQ(x)\cos(2m+1)xdx = \frac{1}{2\pi}\sum_{m=1}^{\infty}\int_0^{\pi} trQ(x)\cos mxdx - \frac{1}{2\pi}\sum_{m=1}^{\infty}(-1)^m\int_0^{\pi} trQ(x)\cos mxdx$$

$$= \frac{1}{4}\sum_{m=1}^{\infty}[\frac{2}{\pi}\int_0^{\pi} trQ(x)\cos mxdx]\cos m0 + \frac{1}{4}[\frac{1}{\pi}\int_0^{\pi} trQ(x)dx]\cos 0 - \frac{1}{4}\sum_{m=1}^{\infty}[\frac{2}{\pi}\int_0^{\pi} trQ(x)\cos mxdx]\cos m\pi$$

$$+\frac{1}{4}[\frac{1}{\pi}\int_0^{\pi} trQ(x)dx]\cos 0\pi = \frac{1}{4}[trQ(0) - trQ(\pi)] \tag{46}$$

Finally, from (45) and (46), we obtain.

$$\sum_{m=0}^{\infty}\left\{\sum_{k=1}^{\infty}[\lambda_{mk} - (m+\frac{1}{2})^{2n}] - \frac{1}{\pi}\int_0^{\pi} trQ(x)dx\right\} = \frac{1}{4}[trQ(0) - trQ(\pi)]. \tag{47}$$

□

References

[1] Adıguzelov EE. About the trace of the difference of two Sturm-Liouville operator with operator coefficient.iz AN AZ SSR, seriya fiz-tekn i Mat. Nauk. 1976; 5: 20-24

[2] Adıguzelov EE, Avcı H, Gul E.The trace formula for Sturm-Liouville operator with operator coefficient. Journal of Math. Physics. 2001; 42(6): 2611-2624

[3] Adıguzelov EE, Baksi O, Baykal O. On a regularized trace of a differential operator with bounded operator coefficient. Int. Math. Journal.2004; 5(3): 273-286

[4] Adıguzelov EE, Kanar P. The regularized trace of a second order differential operator with unbounded operator coefficient. Int. Journal of Pure and Appl. Math. 2005; 22(3): 349-365

[5] Aslanova NM, Bayramoglu M, Aslanov KM. Eigenvalue problem associated with the fourth order differential-operator equation. Rocky Mountain Journal of Mathematics. 2018; 48(6): 1763-1779

[6] Chalilova RZ. On regularization of the trace of the Sturm-Liouville operator equation. Funks. Analiz, Teoriya Funksiy i ik pril.-Mahackala. 1976; 3(1): 154-161

[7] Cohberg IC, Krein MG. Introduction to the theory of linear non-self adjoint operators. Translation of Mathematical Monographs (AMS, Providence, RI). 1969; 18

[8] Dikiy LA. About a formula of Gelfand-Levitan. Usp. Mat. Nauk. 1953; 8(2): 119-123

[9] Gelfand IM, Levitan BM. On a formula for eigenvalues of a differential operator of second order.Dokl. Akad. Nauk SSSR. 1953; 88(4): 593-596

[10] Gul E, Ceyhan A. A second regularized trace formula for a fourth order differential operator. Symmetry. 2021; 13(4): 629.

[11] Halberg CJ, Kramer VA. A generalization of the trace concept. Duke Math. J. 1960; 27(4): 607-618

[12] Levitan BM. The computation of the regularized trace of Sturm-Liouville operator. Usp. Math. Nauk.1964; 19: 161-164

[13] Levitan BM, Sargsyan IS. Sturm-Liouville and Dirac operators. Kluwer, Dordrecht, 1991

[14] Fulton TC,, Pruess SA. Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems.J. Math. Anal. Appl.1994; 188 :297-340

[15] Maksudov FG, Bayramoglu M, Adıguzelov EE. On regularized trace of Sturm-Liouville operator on a finite interval with unbounded operator coefficient. Dokl. Akad. Nauk SSR. 1984; 30(1): 169-173

[16] Sezer Y, Baksi Ö, Karayel S. On Higher Regularized Traces of a Differential Operator with Bounded Operator Coefficient Given in a Finite Interval. Mediterranean Journal of Mathematics. 2021; 18(3): 1-10