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Transmutation of conformable Sturm-Liouville operator with exactly solvable potential

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Abstract. In this paper, we proved transformation operator for fractional Sturm-Liouville operator, using conformable derivative approach, which is different from classical Sturm-Liouville operator. Especially, we obtained a Hyperbolic partial differential equation and some suitable conditions for nucleus function $K(x,t)$. Finally, we obtained a Fredholm integral equation. The proof is validated by taking $\alpha = 1$ which returns the original problem.

1. Introduction

The scattering of high energy electrons from a two center potential is of considerable interest in nuclear and molecular physics. The Schroedinger equation for the problem is separable in spheroidal coordinates and the scattering cross-section can be given in terms of spheroidal phase shifts. Schroedinger equation for a free particle is in the form

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m(V(x)-E)}{\hbar^2}\psi(x)$$

can be interpreted by saying that the left side, the rate of change of slope, is the curvature. Where, E is energy, V is potential function, m is mass of particle and $\hbar$ planck constant. Then, the curvature of the function is proportional to $(V(x)-E)\psi(x)$. It means that if $V(x) < E$ and $\psi(x)$ positive, then $\psi(x)$ negatively curving. Also, $\psi(x)$ negative, then $\psi(x)$ positively curving. The simplest case of potential is a constant potential $V(x) = V_0 < E$. In this case the wave function $\psi(x) = A.Sin(kx+\delta)$ for a δ constants and $k = \frac{2m}{\hbar^2}(E - V_0)$. Mathematically, they refers to the eigenfunctions and eigenvalus, respectively [17].
As it is known, the Schroedinger equation takes place in the literature as the Sturm-Liouville equation mathematically. In this study, we will first establish a relationship between Schroedinger problems with two different potential functions and their solutions. This relationship will lead us to a partial differential equation. Of course, it is useful to state here that we will do all these in the case of conformable derivative.

Let us consider two Sturm-Liouville operators $A = -\dfrac{d^2}{dx^2} + q(x)$ and $B = -\dfrac{d^2}{dx^2} + r(x)$, for $q(x)$ and $r(x)$ are squared integrable functions and known as potentials in the theory . The transformation operator is one of the ways to solve inverse Sturm-Liouville operator. First of all, a relation was established between two

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Sturm-Liouville problems with two different potentials by using integral equation. Then a very important formula was obtained between the potential function and the nucleus function. Many studies in details on this subject can be reach at [4],[8–12],[14], [19],[21],[23]. Using the above-mentioned relationship as a source of motivation, we tried to obtain similar results for the fractional Sturm-Liouville problem and we have seen that these results are possible to achieve.

We considered the case where E_1 and E_2 are considered as subspaces of a topological linear space E, such that h_1 and h_2 are finite, for $h_1 \in E_1$ and $h_2 \in E_2$. Also, A and B are two differential operators from E to E under the conditions that $u'(0) = h_1 u(0)$ and $u'(0) = h_2 u(0)$. These conditions based the foundation of ordinary transformation operator for the problem which is expressed as in the theorem below:

Theorem 1.1. *Defining the transformation operator between of A and B as [17],*

$$Xu(x) = u(x) + \int_0^x K(x,t)u(t)dt \tag{1}$$

Then the kernel in operator (1) *is a solution to the Hyperbolic equation;*

$$\frac{\partial^2}{\partial x^2}K(x,t) - q(x)K(x,t) = \frac{\partial^2}{\partial t^2}K(x,t) - r(x)K(x,t) \tag{2}$$

and satisfies the conditions,

$$K(x,x) = h_2 - h_1 + \frac{1}{2}\int_0^x [q(s) - r(s)]ds \tag{3}$$

$$\left[\frac{\partial K(x,t)}{\partial t} - h_1 K(x,t)\right]\Bigg|_{t=0} = 0. \tag{4}$$

Inversely, if the nucleus function $K(x,t)$ is a solution of the problem (2) - (4)*, then the operator X defined by* (1) *is a transformation operator for A and B.*

The concept of fractional differential equation is an extension of the classical differential equation. It gives the chance of taking the α-order derivative of a function, for $0 < \alpha \leq 1$. This fact has been proved to be very effective as it gives many chances of calculating and computing on some phenomenon that can't be by the classical differentiation and integration [1–3],[5–7],[13],[15], [16],[18],[20],[24–26]. The concept of operator is highly considered due to its vitality in almost all parts of mathematics, this lead us to investigate and present the possibility of providing transformation operator for the conformable fractional Sturm-lioville operator and it was achieved through the ordinary case of the problem. The proof is provided below in details with all the supporting facts and theorems to ease understanding. These definitions and Theorems were given by some authors [1], [5], [13],[22].

Definition 1.2. *Let consider the function $g : [0,\infty) \to R$. Then, the α^{th} order derivative of g is given as*

$$D_x^\alpha g(x) = \lim_{h\to 0} \frac{g(x + hx^{1-\alpha}) - g(x))}{h}, \tag{5}$$

for all $x > 0$, $\alpha \in (0,1]$.

Definition 1.3. *For a function g, the integral of g of order α given as*

$$I_\alpha g(x) = \int_0^x g(t)d_\alpha t = \int_0^x t^{\alpha-1}g(t)dt \tag{6}$$

for all $x > 0$.

Lemma 1.4. *Let us assume that $g : [a, \infty) \to \mathbb{R}$ is differentiable. Then, we have for $x > a$ (a is any real number)*

$$D_x^\alpha I_\alpha g(x) = g(x).$$

Lemma 1.5. *Assume that $g : (a, b) \to \mathbb{R}$ is differentiable. Then, for $x > a$,(a and b are any real numbers)*

$$D_x^\alpha I_\alpha g(x) = g(x) - g(a).$$

Theorem 1.6. *Let g, h are two differentiable functions. Then*

$$\int_a^b g(x) D_x^\alpha (h(x))(x) d_\alpha x = gh\Big|_a^b - \int_a^b h(x) D_x^\alpha (g(x)) d_\alpha x. \tag{7}$$

Lemma 1.7. *The $\alpha-$Leibniz integral rule is given in the form*

$$D_x^\alpha \left[\int_{a(x)}^{b(x)} g(x,t) d_\alpha t \right] = D_x^\alpha b(x) g(x, b(x)) b(x)^{\alpha-1} - D_x^\alpha a(x) g(x, a(x)) a(x)^{\alpha-1} + \int_a^b D_x^\alpha (g(x,t)) d_\alpha t. \tag{8}$$

for $a(x) \le t \le b(x)$ while $a(x)$ and $b(x)$ are both $\alpha-$differentiable for $x_0 \le x \le x_1$.

2. The proposed transformation operator

In this section, the presence of the transformation operator is given and also, a Hyperbolic type partial differential equation and some conditions is obtained from the hyperbolic partial equation including $\alpha-$derivative for the kernel $K(x,t)$ function. It is worth mentioning here that the method we used is similar to that of the classical problem. However, the results obtained will be different and more general. We considered E as a space which consisted of all real valued functions $u(x)$ for $x \ge 0$. Let the two fractional Sturm-liouville operators be expressed using the conformable derivative as follows;

$$A = -D_x^\alpha D_x^\alpha + q(x) \text{ and } B = -D_x^\alpha D_x^\alpha + r(x) \tag{9}$$

here $q(x)$ and $r(x)$, $(0 \le x < \infty)$ are continuous functions. Let E_1 and E_2 be subspaces of functions in E satisfying the boundary condition;

$$D_x^\alpha u(0) = h_1 u(0) \tag{10}$$

and

$$D_x^\alpha u(0) = h_2 u(0) \tag{11}$$

here h_1 and h_2 are any constants, then based on these, our proposed conformable fractional operator is expressed by the theorem below;

Theorem 2.1. *The transformation operator, $X = X_{A,B}$ from E_1 to E_2 is defined*

$$Xu(x) = u(x) + \int_0^x K(x,t) u(t) d_\alpha t \tag{12}$$

the Kernel in operator (12) *is a solution to the fractional differential equation;*

$$D_x^\alpha (D_x^\alpha K(x,t)) - q(x) K(x,t) = D_t^\alpha (D_t^\alpha K(x,t)) - r(x) K(x,t) \tag{13}$$

and satisfies conditions,

$$K(x,x) = h_2 - h_1 + \int_0^x \frac{[q(s) - r(s)]}{1 + s^{\alpha-1}} d_\alpha s \tag{14}$$

$$[D_t^\alpha K(x,t) - h_1 K(x,t)]\Big|_{t=0} = 0 \tag{15}$$

Inversely, the operator X which is given in (12) *is a transformation operator for A, B, on condition that $K(x,t)$ is a solution of* (13) - (15).

This is the statement of the proposed assertion and by using the facts in theorems, definitions, and lemmas above, the proof is provided as follows;

Proof. Differentiating equation (12) fractionally with respect to x gives;

$$D_x^\alpha(Xu(x)) = D_x^\alpha u(x) + D_x^\alpha[\int_0^x K(x,t)u(t)d_\alpha t]$$
$$= x^{1-\alpha}f'(x) + D_x^\alpha(x).K(x,x)u(x).x^{\alpha-1} - D_x^\alpha(0).K(x,0)u(0).(0)^{\alpha-1} + \int_0^x D_x^\alpha K(x,t)u(t)d_\alpha t$$

That is,

$$D_x^\alpha(Xu(x)) = x^{1-\alpha}u'(x) + K(x,x)u(x) + \int_0^x D_x^\alpha K(x,t)u(t)d_\alpha t \tag{16}$$

Since, $(Xu(x)) \in E_2$ and $u(x) \in E_1$, then taking as $x = 0$ in (16), we obtained,

$$\begin{aligned} D_x^\alpha(Xu(x))|_{x=0} &= h_2(Xu(x))|_{x=0} = h_2(u(0)) = D_x^\alpha u(0) + K(0,0)u(0) \\ &= h_1 u(0) + K(0,0)u(0) = [h_1 + K(0,0)]u(0) \end{aligned} \tag{17}$$

which implies that $K(0,0) = h_2 - h_1$. By differentiating equation (16) we get,

$$\begin{aligned} D_x^\alpha[D_x^\alpha(Xu(x))] &= D_x^\alpha[D_x^\alpha u(x)] + [D_x^\alpha K(x,x)]u(x) + K(x,x)[D_x^\alpha u(x)] + D_x^\alpha[D_x^\alpha K(x,x)u(x)x^{\alpha-1}](x)^{\alpha-1} \\ &\quad - D_0^\alpha[D_0^\alpha K(x,0)f(0)(0)^{\alpha-1}]x^{\alpha-1} + \int_0^x D_x^\alpha[D_x^\alpha K(x,t)u(t)]d_\alpha t \\ &= D_x^\alpha[D_x^\alpha u(x)] + [D_x^\alpha K(x,x)]u(x) + K(x,x)[D_x^\alpha u(x)] \\ &\quad + D_x^\alpha[D_x^\alpha K(x,x)u(x)x^{\alpha-1}](x)^{\alpha-1} + \int_0^x D_x^\alpha[D_x^\alpha K(x,t)u(t)]d_\alpha t \end{aligned} \tag{18}$$

therefore we have,

$$\begin{aligned} A(Xu(x)) &= -D_x^\alpha D_x^\alpha(Xu(x)) + q(x)(Xu(x)) \\ &= -D_x^\alpha D_x^\alpha u(x) - [D_x^\alpha K(x,x)]u(x) - K(x,x)[D_x^\alpha u(x)] - D_x^\alpha K(x,x)u(x)x^{\alpha-1} + q(x)(u(x)) \\ &\quad - \int_0^x [D_x^\alpha D_x^\alpha K(x,t) - q(x)K(x,t)]u(t)d_\alpha t \end{aligned} \tag{19}$$

$Bu(x)$ under the operator X is expressed as;

$$Bu(x) = [-D_x^\alpha D_x^\alpha + r(x)]u(x) = -D_x^\alpha D_x^\alpha u(x) + r(x)u(x) \tag{20}$$

then,

$$\begin{aligned} X[Bu(x)] &= (Bu(x)) + \int_0^x K(x,t)[Bu(t)]d_\alpha t \\ &= -D_x^\alpha D_x^\alpha u(x) + r(x)u(x) - \int_0^x K(x,t)[-D_t^\alpha D_t^\alpha u(t) + r(t)]u(t)]d_\alpha t \\ &= -D_x^\alpha D_x^\alpha u(x) + r(x)u(x) + [D_t^\alpha K(x,x)]u(x) - K[x,x]D_t^\alpha u(x) + K[x,0]D_t^\alpha u(0) \\ &\quad - [D_t^\alpha K(x,0)]u(0) - \int_0^x D_t^\alpha D_t^\alpha[K(x,t) - r(x)K(x,t)]u(t)d_\alpha t \end{aligned} \tag{21}$$

Since, $A(Xu) = X(Bu)$, then equating (19) and (21) gives;

$$D_x^\alpha D_x^\alpha K(x,t) - q(x)K(x,t) = D_t^\alpha D_t^\alpha K(x,t) - r(x)K(x,t) \tag{22}$$

and also, the conditions;

$$[D_t^\alpha K(x,t) - h_1 K(x,t)]|_{t=0} = 0, \tag{23}$$

and

$$D_x^\alpha K(x,x)[1 + x^{\alpha-1}] = q(x) - r(x) \tag{24}$$

which is equivalent to,

$$K(x,x) = \int_0^x \frac{[q(s) - r(s)]}{1 + s^{\alpha-1}} d_\alpha s + K(0,0), \ for\, K(0,0) = h_2 - h_1 \tag{25}$$

are obtained. □

This gives the proof of the proposed assertion and it is clear that the original problem is obtainable by taking $\alpha = 1$. Briefly, the proof is that, if the transformation operator X is in the form (12), then $K(x,t)$ is a solution of Hyperbolic problems (13) to (15).Coversely this assertion is also true. To complete the remainder of proof, the solvability of problem (13) to (15) is shown below;

First assume that the functions $q(x)$ and $r(x)$ are differentiable. Let the variables $\xi = x + t$ and $\eta = x - t$.Then,

$$Q(\xi,\eta) = K\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$$

Equations (13) to (15) in terms of ξ and η can be expressed as;

$$D_\xi^\alpha(D_\eta^\alpha Q(\xi,\eta)) = \frac{1}{4}\left[q\left(\frac{\xi+\eta}{2}\right) - r\left(\frac{\xi+\eta}{2}\right)\right] Q \tag{26}$$

and satisfies the conditions,

$$Q(\xi,0) = h_2 - h_1 + \int_0^{\frac{\xi}{2}} \frac{[q(s) - r(s)]}{1 + s^{\alpha-1}} d_\alpha s \tag{27}$$

and

$$[D_\xi^\alpha Q t^{1-\alpha} - D_\eta^\alpha Q t^{1-\alpha} - h_1 Q]\Big|_{\xi=\eta} = 0 \tag{28}$$

Integrating equation (26) according to variable η from 0 to η gives,

$$D_\xi^\alpha Q - D_\xi^\alpha Q\Big|_{\eta=0} = \frac{1}{4}\int_0^\eta \left[q\left(\frac{\xi+\eta}{2}\right) - r\left(\frac{\xi+\eta}{2}\right)\right] Q(\xi,\lambda) d_\alpha \lambda \tag{29}$$

It follows from condition (28) that,

$$D_\xi^\alpha Q\Big|_{\eta=0} = \frac{\left[q\left(\frac{\xi}{2}\right) - r\left(\frac{\xi}{2}\right)\right]}{2(1 + (\frac{\xi}{2})^{\alpha-1})}\left(\frac{\xi}{2}\right)^{\alpha-1}$$

therefore,

$$D_\xi^\alpha Q = \frac{1}{4}\int_0^\eta \left[q\left(\frac{\xi+\eta}{2}\right) - r\left(\frac{\xi+\eta}{2}\right)\right] Q d_\alpha \lambda + \frac{\left[q\left(\frac{\xi}{2}\right) - r\left(\frac{\xi}{2}\right)\right]}{2(1 + (\frac{\xi}{2})^{\alpha-1})}\left(\frac{\xi}{2}\right)^{\alpha-1} \tag{30}$$

Integrating equation (30) with respect to variable ξ from η to ξ gives,

$$Q(\xi,\eta) = \frac{1}{4}\int_\eta^\xi d_\alpha\beta \int_0^\eta \left[q\left(\frac{\beta+\eta}{2}\right) - r\left(\frac{\beta+\eta}{2}\right)\right] Q(\beta,\lambda) d_\alpha \lambda + \int_\eta^\xi \frac{\left[q\left(\frac{\xi}{2}\right) - r\left(\frac{\xi}{2}\right)\right]}{2(1 + (\frac{\xi}{2})^{\alpha-1})}\left(\frac{\xi}{2}\right)^{\alpha-1} d_\alpha \lambda + Q(\eta,\eta) \tag{31}$$

Now, to calculate $Q(\eta,\eta)$, it follows from (28) and (30) that

$$\begin{aligned} 2D^{\alpha}_{\xi}Qt^{1-\alpha}\Big|_{\xi=\eta} &= \left[D^{\alpha}_{\xi}Qt^{1-\alpha} + D^{\alpha}_{\eta}At^{1-\alpha} + h_1 Q\right]\Big|_{\xi=\eta} \\ &= \frac{1}{2}\int_0^{\eta}\left[q\left(\frac{\eta+\lambda}{2}\right) - r\left(\frac{\eta+\lambda}{2}\right)\right]Q(\eta,\lambda)d_{\alpha}\lambda + \frac{\left[q\left(\frac{\eta}{2}\right) - r\left(\frac{\eta}{2}\right)\right]}{(1+(\frac{\eta}{2})^{\alpha-1})}\left(\frac{\eta}{2}\right)^{\alpha-1} \end{aligned} \tag{32}$$

then the last equation gives;

$$\begin{aligned} D^{\alpha}_{\xi}[e^{(h_2+h_1)\eta}Q(\eta,\eta)] &= e^{(h_2+h_1)\eta}\left[\frac{\left[q\left(\frac{\eta}{2}\right) - r\left(\frac{\eta}{2}\right)\right]}{(1+(\frac{\eta}{2})^{\alpha-1})}\left(\frac{\eta}{2}\right)^{\alpha-1}\right] \\ &+ e^{(h_2+h_1)\eta}\left[\frac{1}{2}\int_0^{\eta}\left[q\left(\frac{\eta+\lambda}{2}\right) - r\left(\frac{\eta+\lambda}{2}\right)\right]Q(\eta,\lambda)d_{\alpha}\lambda\right] \end{aligned} \tag{33}$$

Integrating this equation from 0 from η and considering the equation (27) leads to,

$$\begin{aligned} Q(\eta,\eta) &= -(h_2+h_1)e^{-(h_2+h_1)\eta} \\ &+ e^{-(h_2+h_1)\eta}\int_0^{\eta} e^{(h_2+h_1)\beta}\left[\frac{1}{2}\int_0^{\eta}\left[q\left(\frac{\beta+\lambda}{2}\right) - r\left(\frac{\beta+\lambda}{2}\right)\right]Q(\beta,\lambda)d_{\alpha}\lambda\right]d_{\alpha}\beta \\ &+ e^{-(h_2+h_1)\eta}\int_0^{\eta} e^{(h_2+h_1)\beta}\left[\frac{\left[q\left(\frac{\beta}{2}\right) - r\left(\frac{\beta}{2}\right)\right]}{(1+(\frac{\beta}{2})^{\alpha-1})}\left(\frac{\beta}{2}\right)^{\alpha-1}\right]d_{\alpha}\beta \end{aligned} \tag{34}$$

Thus equation $Q(\xi,\eta)$ must satisfy the integral equation

$$\begin{aligned} Q(\xi,\eta) &= \frac{1}{4}\int_{\eta}^{\beta} d_{\alpha}\beta\int_0^{\eta}\left[q\left(\frac{\beta+\lambda}{2}\right) - r\left(\frac{\beta+\lambda}{2}\right)\right]Q(\beta,\lambda)d_{\alpha}\lambda + \int_0^{\xi}\left[\frac{\left[q\left(\frac{\lambda}{2}\right) - r\left(\frac{\lambda}{2}\right)\right]}{2(1+(\frac{\lambda}{2})^{\alpha-1})}\left(\frac{\lambda}{2}\right)^{\alpha-1}\right]d_{\alpha}\lambda \\ &+ e^{-(h_2+h_1)\eta}\Bigg(-(h_2+h_1) + \int_0^{\eta} e^{(h_2+h_1)\beta}\left[\frac{1}{2}\int_0^{\beta}\left[q\left(\frac{\beta+\lambda}{2}\right) - r\left(\frac{\beta+\lambda}{2}\right)\right]Q(\beta,\lambda)d_{\alpha}\lambda\right]d_{\alpha}\beta \\ &+ \int_0^{\eta} e^{(h_2+h_1)\beta}\left[\frac{\left[q\left(\frac{\beta}{2}\right) - r\left(\frac{\beta}{2}\right)\right]}{(1+(\frac{\beta}{2})^{\alpha-1})}\left(\frac{\beta}{2}\right)^{\alpha-1}\right]d_{\alpha}\beta\Bigg) \end{aligned} \tag{35}$$

And also, if function $Q(\xi,\eta)$ satisfies the integral equation (35) with $q(x)$ and $r(x)$ each differentiable once, it can be verified easily that (35) is a solution of problems (26) to (27).
This completed the proof and the desired result is obtained. The equivalent of equation (35) in the ordinary case is a voltera-type integral equation, also, in this case (35) behave the same. From the fact that continuous function can be approximated by smooth function, the assumption that the functions $q(x)$ and $r(x)$ are each differentiable will not affect the approximation. Then, extrapolation to the limit function $K(x,t)$ formed, is the kernel of the transformation operator.

Corollary 2.2. *If $A = -D^{\alpha}_{x}D^{\alpha}_{x} + q(x)$: and : $B = -D^{\alpha}_{x}D^{\alpha}_{x}$ we obtain two problems as;*

$$\begin{aligned} &Af = \mu f, \\ &D^{\alpha}_{x}f(0) = 0 \end{aligned} \tag{36}$$

and

$$\begin{aligned} &Bf = \mu f, \\ &D^{\alpha}_{x}f(0) = 0 \end{aligned} \tag{37}$$

and the solution of the problems (36) *and* (37) *are* $u(x)$ *and* $\cos \frac{\sqrt{\mu}}{\alpha} x^\alpha$. *In this case, we can write the transformation operator as*

$$X\left(\cos \frac{\sqrt{\mu}}{\alpha} x^\alpha\right) = \cos \frac{\sqrt{\mu}}{\alpha} x^\alpha + \int_0^x K(x,t) \cos \frac{\sqrt{\mu}}{\alpha} t^\alpha d_\alpha t. \tag{38}$$

Then we can obtain a partial differential equation for $K(x,t)$.

3. Data Availability Statement

Data usage is not applicable to this article as no data were created or analysed in this study. The study is directly on transmuting Sturm-Liouville operator into conformable fractional sense and stating the potential.

4. Conclusion

New results for the conformable fractional Sturm-Liouville problems are proposed. The method to obtain the main results is based on the transformation operator for Sturm-Liouville operator including ordinary derivative. A Hyperbolic type partial differential equation and related conditions are obtained for Nucleus function $K(x,t)$. Finally, this problem is reduced to a Volterra integral equation. When we think in terms of the solution of the inverse problem in the Sturm liouville theory, the transformation operator has an important place. Especially the relation between the potential function and the kernel in the form of $K(x,x) = \int_0^x [q(s) - r(s)]ds$ has an important place. A similar relationship is presented as $K(x,x) = \int_0^x \frac{[q(s)-r(s)]}{1+s^{\alpha-1}} d_\alpha s$ in the $\alpha-$derivative Sturm Liouville problem. It is clear that K depends on the potential function but not the spectral parameter(λ), one can give a reconstruction algorithm for the inverse conformable Sturm-Liouville case.

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