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Quantile-based entropy function in past lifetime for order statistics and its properties

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Abstract. In this paper, we introduce a quantile version of past entropy for order statistics and study some of its properties. It is shown that this measure uniquely determine the quantile function. Two nonparametric classes of distributions are also defined based on the proposed measure. The closure of these classes under increasing convex (concave) transformations and weighted variables are discussed. Moreover, a new stochastic order based on this measure is defined and some features of it are investigated. We give desirable conditions for a function of a random variable to have more quantile past entropy for order statistics than original random variable.

1. Introduction

In recent years, the measurement of uncertainty of probability distribution has gained much interest. An important uncertainty measure is called entropy and is introduced by Shannon (1948). Let X be an absolutely continuous non-negative random variable representing the lifetime of a component or a system with distribution function $F(x)$, survival function $\bar{F}(x)$ and density function $f(x)$. The Shannon entropy of X is defined by

$$\eta(X) = -E(\log f(X)) = -\int_0^\infty (\log f(x))f(x)dx. \tag{1}$$

Shannon entropy has been used by many authors in the context of information theory, we refer to Ebrahimi *et al.* (2004), Baratpour *et al.* (2007), Madadi and Tata (2011, 2014), Zamani and Madadi (2018) and the references therein. Sometimes, the uncertainty about the remaining lifetime of a system if it is working at time t is important. Based on this idea, Ebrahimi (1996) introduced the residual entropy of random variable X at time t as follows

$$\eta(X;t) = -\int_t^\infty \left(\log \frac{f(x)}{\bar{F}(t)}\right)\frac{f(x)}{\bar{F}(t)}dx = \log \bar{F}(t) - \frac{1}{\bar{F}(t)}\int_t^\infty (\log f(x))f(x)dx. \tag{2}$$

In many realistic situations, the random lifetime variable is not necessarily related to the future but can also refer to the past. If at time t the system is observed failed for the first time, then the uncertainty is related

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to the past. The random variable $X^{(t)} = (t - X|X \leq t)$ is called the past lifetime and its entropy is named past entropy. The past entropy shows the uncertainty of a system which is observed failed at time t and has been introduced by Di Crescenzo and Longobardi (2002) as

$$\bar{\eta}(X;t) = -\int_0^t \left(\log \frac{f(x)}{F(t)}\right) \frac{f(x)}{F(t)} dx = \log F(t) - \frac{1}{F(t)} \int_0^t (\log f(x)) f(x) dx, \tag{3}$$

where $\bar{\eta}(X;t) = \eta(X^{(t)})$. For further studies on entropy measures and its applications in past lifetime, we refer to Di Crescenzo and Longobardi (2002, 2006), Nanda and Paul (2006), Kundu *et al.* (2010), Sachlas and Papaioannou (2014), Di Crescenzo and Toomaj (2015), Krishnan *et al.* (2020) and Calì *et al.* (2020) and the references therein.

All of the investigations of uncertainty measures are based on distribution function, however might not be suitable for the models that have no tractable distribution. An offered way in this condition is using the quantile function which definition is given in the following. Suppose that X is an absolutely continuous non-negative random variable with distribution function $F(x)$ and probability density function $f(x)$ as described before. The quantile function $Q(u)$ of X is defined by

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1. \tag{4}$$

The equation $F(Q(u)) = u$ is obtained using (4). Now, differentiating of this relation with respect to u, yields

$$q(u) f(Q(u)) = 1,$$

where $f(Q(u))$ and $q(u) = Q'(u)$ are called the density quantile function and the quantile density function of X, respectively and the prime denotes the differentiation. One important measure in reliability analysis is the reversed hazard rate function $\lambda(x) = \frac{f(x)}{F(x)}$ which its quantile version is defined by $\Lambda(u) = \lambda(Q(u)) = (uq(u))^{-1}$.

A variety of literature deal with information measures using the quantile function. For example, Sunoj and Sankaran (2012) have considered the quantile-based Shannon entropy and its residual form as follows

$$\eta = \int_0^1 \log q(p) dp, \tag{5}$$

and

$$\eta(u) = \log(1-u) + (1-u)^{-1} \int_u^1 \log q(p) dp, \tag{6}$$

respectively. The past entropy in relation (3) in terms of the quantile function is introduced by Sunoj *et al.* (2013) as

$$\bar{\eta}(u) = \log u + u^{-1} \int_0^u \log q(p) dp. \tag{7}$$

Different approaches of employing the quantile version of various entropies were also introduced to providing alternative methodology, new results and different methods of stochastic comparisons. We refer to Sunoj and Sankaran (2012), Yu and Wang (2013), Nanda *et al.* (2014) and Qiu (2019) for more details.

Suppose that $X_1, X_2, \cdots, X_n$ are n independent and identically distributed (iid) random variables with an absolutely continuous distribution function $F(x)$ and probability density function $f(x)$. The order statistics of this sample is defined by the arrangement of $X_1, X_2, \cdots, X_n$ from the smallest to the largest and is denoted as $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$. The probability density function of ith order statistics $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} (F(x))^{i-1} (\bar{F}(x))^{n-i} f(x), \tag{8}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx; \quad a, b > 0,$$

is the beta function.
The density function of ith order statistics in terms of the quantile function is obtained as

$$f_{i:n}(u) = f_{i:n}(Q(u)) = \frac{1}{B(i, n-i+1)} u^{i-1}(1-u)^{n-i} \frac{1}{q(u)} = \frac{g_i(u)}{q(u)}, \tag{9}$$

where $g_i(u) = \frac{1}{B(i,n-i+1)} u^{i-1}(1-u)^{n-i}$ is the probability density function of Beta distribution with parameters $(i, n-i+1)$. Because of the applications of order statistics in various applied practical fields, many authors have considered some of entropy measures for order statistics based on the quantile function. For example, Sunoj *et al.* (2017) obtained the quantile versions of entropy and residual entropy of order statistics and studied their properties. Kumar and Singh (2018) obtained the quantile version of generalized entropy of order (α, β) for order statistics and obtained this measure for residual and past lifetime variables. Nisa and Biag (2019) presented some characterization results for the proposed measure by Kumar and Singh (2018).
Various stochastic orders are defined to compare two random variables. Suppose X and Y be two non-negative random variables with absolutely continuous distribution functions $F(x)$ and $G(x)$, densities $f(x)$ and $g(x)$ and quantile functions $Q_X(u)$ and $Q_Y(u)$, respectively. Furthermore, suppose $\Lambda_X(u)$ and $\Lambda_Y(u)$ indicate their respective reversed hazard quantile functions. Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are independent random samples of size n from populations X and Y, respectively. In the following, we recall some stochastic orders which will be used later.

Definition 1.1. X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$), if $Q_X(u) \leq Q_Y(u)$ for all $u \in (0, 1)$.

Definition 1.2. (Nair *et al.*, 2013). X is said to be smaller than Y in the reversed hazard quantile function order ($X \leq_{rhq} Y$), if and only if, $\Lambda_X(u) \leq \Lambda_Y(u)$ for all $u \in (0, 1)$.

Definition 1.3. $X_{i:n}$ is smaller than $Y_{i:n}$ in the dispersive order, (denoted by $X_{i:n} \leq_{disp} Y_{i:n}$) if $G^{-1}_{i:n}F_{i:n}(x) - x$ is increasing in x.

Remark 1.4. *Barlow and Proschan (1975) showed that $G^{-1}_{i:n}F_{i:n}(x) = G^{-1}F(x)$, therefore we can say that $X_{i:n} \leq_{disp} Y_{i:n}$ if $Q_Y(u) - Q_X(u)$ is increasing in u.*

There are different distributions that do not have closed-form distribution functions, even though they have tractable quantile functions. Thus, the study of entropy measures is difficult for these distributions. Thus, the presentation of entropy measures in terms of the quantile function is useful in this direction and has several advantages. Firstly, here we can derive the quantile-based past entropy of order statistics for certain quantile functions which do not have an explicit form for distribution functions and their computations are simple. Secondly, our approach gives an alternative methodology in the study of past entropy of order statistics. Further there are certain properties of quantile functions that are not shared by the distribution function approach. Because of this, we study the quantile form of entropy function in past lifetime for order statistics in this work. The rest of the paper is organized as follows: In Section 2, we define the quantile version of past entropy for order statistics and obtain some of its features. We study also this measure for some lifetime distributions. In Section 3, we define two nonparametric classes of distribution based on the proposed measure and study the closure of these classes under increasing convex (concave) transformations and weighting. A new stochastic order based on this measure is also introduced in this section. Finally, the paper is concluded in Section 4.
Throughout this paper, the terms "increasing" and "decreasing" mean "non-decreasing" and "non-increasing", respectively. All integrals and expectations are implicitly assumed to exist whenever they are written.

2. Quantile-based past entropy for order statistics

In some situations, such as reliability or neurobiology a shift-dependent measure of uncertainty is needed. In this section, we derive past entropy for order statistics from the quantile point of view and study its properties.

Theorem 2.1. *The quantile form of entropy function in past lifetime for ith order statistics is defined by*

$$\bar{\eta}_{X_{i:n}}(u) = -\frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\int_0^u (\log g_i(p))g_i(p)dp + \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\int_0^u (\log q(p))g_i(p)dp + \log\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)}, \tag{10}$$

where $\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)}$ *is the quantile version of* $F_{i:n}(u)$ *with* $\beta_u(i,n-i+1) = \int_0^u p^{i-1}(1-p)^{n-i}dp$, *the incomplete beta function.*

Proof. We know that the past entropy for ith order statistics is defined as

$$\bar{\eta}(X_{i:n}, t) = -\int_0^t \left(\log\frac{f_{i:n}(x)}{F_{i:n}(t)}\right)\frac{f_{i:n}(x)}{F_{i:n}(t)}dx. \tag{11}$$

It is easy to show that $F_{i:n}(t) = \frac{\beta_{F(t)}(i,n-i+1)}{\beta(i,n-i+1)}$. Now, by substituting $x = Q(p)$, $t = Q(u)$ in (11) and using relations (4) and (9), we reach to the desired result. □

Remark 2.2. *The proposed measure in relation (10) is a special case of the quantile-based generalized entropy of order* (α, β), *for order statistics when* $\beta = 1$ *and* $\alpha \to 1$. *Considering this fact, the present study concentrates only on those results which are not presented in Kumar and Singh (2018) and Nisa and Baig (2019).*

Corollary 2.3. *An equivalent representation of (10) is*

$$\begin{aligned}\bar{\eta}_{X_{i:n}}(u) &= -\frac{1}{\beta_u(i,n-i+1)}\int_0^u (\log g_i(p))p^{i-1}(1-p)^{n-i}dp + \frac{1}{\beta_u(i,n-i+1)}\int_0^u (\log q(p))p^{i-1}(1-p)^{n-i}dp\\ &+ \log\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)}.\end{aligned} \tag{12}$$

Order statistics have many applications in the characterization of probability distributions, goodness-of-fit tests and also in reliability theory. In reliability theory, order statistics are used for statistical modeling. The $(n-k+1)$th order statistics in a sample of size n represents the life length of a k-out-of-n system. Particularly, $X_{1:n}$ and $X_{n:n}$ give the lifetimes of series and parallel systems, respectively (see Arnold *et al.* (1992) and David and Nagaraja (2003)).

Corollary 2.4. *The past entropy for the sample minimum* $X_{1:n}$ *and the sample maximum* $X_{n:n}$ *are respectively given by*

$$\bar{\eta}(X_{1:n}, t) = -\int_0^t \left(\log\frac{nf(x)(1-F(x))^{n-1}}{1-(1-F(t))^n}\right)\left(\frac{nf(x)(1-F(x))^{n-1}}{1-(1-F(t))^n}\right)dx, \tag{13}$$

and

$$\bar{\eta}(X_{n:n}, t) = -\int_0^t \left(\log\frac{nf(x)(F(x))^{n-1}}{F^n(t)}\right)\left(\frac{nf(x)(F(x))^{n-1}}{F^n(t)}\right)dx. \tag{14}$$

Corollary 2.5. *The corresponding quantile-based past entropy of the first and the nth order statistics are obtained as*

$$\begin{aligned}\bar{\eta}_{X_{1:n}}(u) &= -\log n + \frac{n-1}{n} + (n-1)\frac{(1-u)^n}{1-(1-u)^n}\log(1-u) + \log(1-(1-u)^n)\\ &+ \frac{n}{1-(1-u)^n}\int_0^u (\log q(p))(1-p)^{n-1}dp,\end{aligned} \tag{15}$$

and

$$\bar{\eta}_{X_{n:n}}(u) = -\log n + \frac{n-1}{n} + \log u + \frac{n}{u^n}\int_0^u (\log q(p))p^{n-1}dp, \tag{16}$$

respectively.

$\bar{\eta}_{X_{1:n}}(u)$ and $\bar{\eta}_{X_{n:n}}(u)$ measure the quantile-based past entropy of series and parallel systems with i.i.d components, respectively. We obtain various properties using $\bar{\eta}_{X_{1:n}}(u)$, results of $\bar{\eta}_{X_{n:n}}(u)$ can be derived similarly. Now, we consider one example which deals with the Cox proportional reversed hazard (PRH) model.

Example 2.6. *The Cox proportional reversed hazards model, defined by $\Lambda_Y(x) = \theta\Lambda_X(x)$. Further, this model can be expressed in terms of the quantile function by,*

$$Q_Y(u) = Q_X(u^{\frac{1}{\theta}}), \;\; \theta > 0.$$

In addition, in terms of the quantile density function, we have $q_Y(u) = \frac{1}{\theta}u^{\frac{1}{\theta}-1}q_X(u^{\frac{1}{\theta}})$. The quantile-based past entropy of the first order statistics is obtained as

$$\begin{aligned}\bar{\eta}_{Y_{1:n}}(u) &= -\log n + (n-1)\frac{(1-u)^n}{1-(1-u)^n}\log(1-u) + \frac{n-1}{n} + \log(1-(1-u)^n) - \log\theta \\ &+ (\frac{1}{\theta}-1)\frac{n}{1-(1-u)^n}\int_0^u (\log p)(1-p)^{n-1}dp + \frac{n}{1-(1-u)^n}\int_0^u (\log q_X(p^{\frac{1}{\theta}}))(1-p)^{n-1}dp.\end{aligned}$$

There are some models that do not have closed form for distribution functions and probability distribution functions, but have simple quantile functions. For instance, the following characterization example shows this fact.

Example 2.7. *Suppose X is distributed with the quantile density function as*

$$q(u) = Ku^{\alpha}(1-u)^{-(A+\alpha)}, \tag{17}$$

where K, α and A are real constants. The past quantile entropy of the sample minimum $X_{1:n}$ is equal to

$$\begin{aligned}\bar{\eta}_{X_{1:n}}(u) &= -\log n + \log K + (n-1+A+\alpha)\frac{(1-u)^n}{1-(1-u)^n}\log(1-u) + \log(1-(1-u)^n) + \frac{A+\alpha+n-1}{n} \\ &+ \frac{n\alpha}{1-(1-u)^n}\int_0^u (\log p)(1-p)^{n-1}dp.\end{aligned}$$

Remark 2.8. *The family of distributions (17) contains some important probability distributions such as, Exponential ($\alpha = 0$, $A = 1$), Pareto ($\alpha = 0$, $A < 1$), Rescaled beta ($\alpha = 0$, $A > 1$), Log-logistic ($\alpha = \lambda - 1$, $A = 2$) and the Life distribution proposed by Govindarajulu (1977) ($\alpha = \beta - 1$, $A = -\beta$).*

The following example computes the quantile-based past entropy of the first order statistics, $\bar{\eta}_{X_{1:n}}(u)$, where the distribution function has no closed form.

Example 2.9. *Let X be a random variable with the quantile function $Q(u) = 2u - u^2$, which has no closed form distribution function. The distribution of X is a special case of Govindarajulu distribution (1977). Then*

$$\bar{\eta}_{X_{1:n}}(u) = -\log n + \log 2 + 1 - \frac{2}{n} + \log(1-(1-u)^n) + (n-2)\frac{(1-u)^n}{1-(1-u)^n}\log(1-u).$$

Table 1 provides the quantile functions and the corresponding quantile-based past entropy of the first order statistics $\bar{\eta}_{X_{1:n}}(u)$ of some important lifetime models.
In the case of Generalized Pareto distribution, we want to consider the behavior of the quantile-based past entropy of the first order statistics. We have

$$\frac{\partial}{\partial b}\bar{\eta}_{X_{1:n}}(u) = \frac{1}{b} > 0,$$

therefore the entropy is an increasing function of b. Furthermore,

$$\begin{aligned}\frac{\partial}{\partial a}\bar{\eta}_{X_{1:n}}(u) &= -\frac{1}{a+1} + \frac{1}{n(a+1)^2} + \frac{1}{(a+1)^2}\frac{(1-u)^n}{1-(1-u)^n}\log(1-u) \\ &= \frac{-na-(n-1)}{n(a+1)^2} + \frac{1}{(a+1)^2}\frac{(1-u)^n}{1-(1-u)^n}\log(1-u) < 0,\end{aligned}$$

Table 1: Quantile function and quantile-based past entropy of the first order statistics

Distribution	Quantile function	$\bar{\eta}_{X_{1:n}}(u)$
Exponential	$-\frac{\log(1-u)}{\lambda}$	$-\log n - \log\lambda + 1 + \log(1-(1-u)^n)$ $+n\frac{(1-u)^n}{1-(1-u)^n}\log(1-u)$
Uniform	$a+(b-a)u$	$-\log n + (n-1)\frac{(1-u)^n}{1-(1-u)^n}\log(1-u) + \frac{n-1}{n}$ $+\log(1-(1-u)^n) + \log(b-a)$
Pareto II	$\alpha[(1-u)^{-\frac{1}{c}}-1]$	$-\log n + \log(\frac{\alpha}{c}) + (1+\frac{1}{nc}) + \log(1-(1-u)^n)$ $+(n+\frac{1}{c})\frac{(1-u)^n}{1-(1-u)^n}\log(1-u)$
Rescaled beta	$R[1-(1-u)^{\frac{1}{c}}]$	$-\log n + 1 - \frac{1}{nc} + \log(1-(1-u)^n)$ $+\log(\frac{R}{c}) + (n-\frac{1}{c})\frac{(1-u)^n}{1-(1-u)^n}\log(1-u)$
Pareto I	$\sigma(1-u)^{-\frac{1}{\alpha}}$	$-\log n + \log(\frac{\sigma}{\alpha}) + 1 + \frac{1}{n\alpha} + \log(1-(1-u)^n)$ $+(n+\frac{1}{\alpha})\frac{(1-u)^n}{1-(1-u)^n}\log(1-u)$
Generalized Pareto	$\frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}}-1\right]$	$-\log n + \log(1-(1-u)^n) + \log(\frac{b}{a+1})$ $+(1+\frac{a}{n(a+1)}) + (n+\frac{a}{a+1})\frac{(1-u)^n}{1-(1-u)^n}\log(1-u)$

so, $\bar{\eta}_{X_{1:n}}(u)$ is a decreasing function of a.
Figure 1 gives plots of $\bar{\eta}_{X_{1:n}}(u)$ of Generalized Pareto distribution with different parameters a and b for the sample sizes $n = 5$ and $n = 20$.
Next, we provide an association between the quantile density function and $\bar{\eta}_{X_{i:n}}(u)$.

Corollary 2.10. *The quantile-based past entropy of order statistics, $\bar{\eta}_{X_{i:n}}(u)$, uniquely determines the quantile density function by*

$$q(u) = \frac{u^{i-1}(1-u)^{n-i}}{\beta_u(i,n-i+1)}\exp\left\{\bar{\eta}_{X_{i:n}}(u) - 1 - \frac{\bar{\eta}'_{X_{i:n}}(u)\beta_u(i,n-i+1)}{u^{i-1}(1-u)^{n-i}}\right\}. \tag{18}$$

Proof. Differentiating (12) with respect to u, we have

$$\bar{\eta}'_{X_{i:n}}(u) = \frac{u^{i-1}(1-u)^{n-i}}{\beta_u(i,n-i+1)}\left\{-\bar{\eta}_{X_{i:n}}(u) + \log q(u) + 1 - \log\frac{u^{i-1}(1-u)^{n-i}}{\beta_u(i,n-i+1)}\right\}. \tag{19}$$

Now, by simplifying the relation (19), we get the desired result. □

Example 2.11. *If X is distributed as Pareto II with quantile function*

$$Q(u) = \alpha\left[(1-u)^{-\frac{1}{c}} - 1\right], \quad \alpha, c > 0.$$

Then the quantile-based past entropy of $X_{1:n}$ is given by

$$\bar{\eta}_{X_{1:n}}(u) = -\log n + \log(\frac{\alpha}{c}) + (1+\frac{1}{nc}) + \log(1-(1-u)^n) + (n+\frac{1}{c})\frac{(1-u)^n}{1-(1-u)^n}\log(1-u). \tag{20}$$

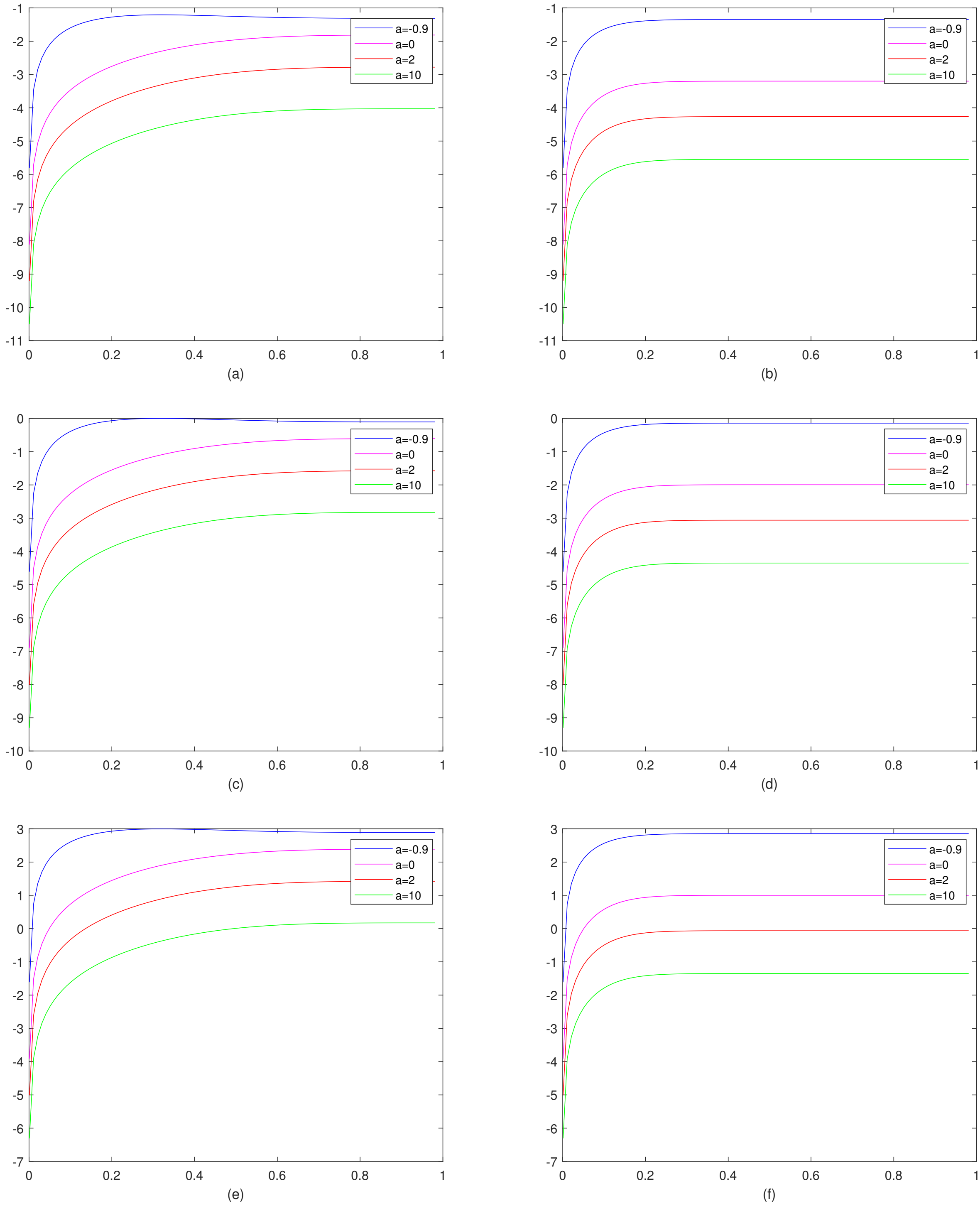

Figure 1: Plots of $\bar{\eta}_{X_{1:n}}(u)$ of Generalized Pareto distribution ($GP(a,b)$) against u. Figures: (a) $b = 0.3$, $n = 5$, (b) $b = 0.3$, $n = 20$, (c) $b = 1$, $n = 5$, (d) $b = 1$, $n = 20$, (e) $b = 20$, $n = 5$, (f) $b = 20$, $n = 20$.

Assume that (20) holds. Using (18), we have

$$q(u) = \exp\left\{\bar{\eta}_{X_{1:n}}(u) - 1 + \log\frac{(1-u)^{n-1}}{\beta(1,n)} - \frac{\bar{\eta}'_{X_{1:n}}(u)\beta_u(1,n)}{(1-u)^{n-1}} - \log\frac{\beta_u(1,n)}{\beta(1,n)}\right\},$$

or equivalently

$$\begin{aligned} q(u) &= \exp\left\{-\log n + \log(\frac{\alpha}{c}) + (1+\frac{1}{nc}) + \log(1-(1-u)^n) + (n+\frac{1}{c})\frac{(1-u)^n}{1-(1-u)^n}\log(1-u) - 1\right. \\ &+ \left.\log(n(1-u)^{n-1}) - \log(1-(1-u)^n) - \frac{1}{nc} - (n+\frac{1}{c})\log(1-u) - (n+\frac{1}{c})\frac{(1-u)^n}{1-(1-u)^n}\log(1-u)\right\}. \end{aligned}$$

Now, with simplifying we can write

$$q(u) = \frac{\alpha}{c}(1-u)^{-\frac{c+1}{c}},$$

which is the quantile density function of Pareto II distribution.

Theorem 2.12. *Let X and Y be two non-negative random variables having common support χ with absolutely continuous distribution functions, F and G and quantile functions Q_X and Q_Y, respectively. Then $F(x) = G(x)$, $\forall\, x \in \chi$, if and only if*

$$\bar{\eta}_{X_{i:n}}(u) = \bar{\eta}_{Y_{i:n}}(u),\ \forall\ 0 < u < 1. \tag{21}$$

Proof. The necessity part is clear. Now, for proving the sufficiency part, let $\bar{\eta}_{X_{i:n}}(u) = \bar{\eta}_{Y_{i:n}}(u)$ for all $u \in (0,1)$. Using equation (10) and on simplification, we have

$$\int_0^u \log q_X(p) g_i(p)dp = \int_0^u \log q_Y(p) g_i(p)dp.$$

Therefore the desired result is obtained. □

Remark 2.13. *The quantile-based past entropy of order statistics $\bar{\eta}_{X_{i:n}}(u)$ can be expressed in terms of quantile-based Shannon entropy of order statistics, $\eta_{X_{i:n}}$, as*

$$\begin{aligned} \bar{\eta}_{X_{i:n}}(u) &= \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\left(\int_0^u (\log q(p))g_i(p)dp - \int_0^u (\log g_i(p))g_i(p)dp\right) + \log\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)} \\ &= \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\eta_{X_{i:n}} + \int_u^1 [\log(g_i(p)q^{-1}(p))]\frac{p^{i-1}(1-p)^{n-i}}{\beta_u(i,n-i+1)}dp + \log\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)}, \end{aligned}$$

where $\eta_{X_{i:n}} = \eta_{g_i} + E_{g_i}(\log q(U))$. Note that η_{g_i} is given by Ebrahimi et al. *(2004) as*

$$\eta_{g_i} = \log B(i,n-i+1) - (i-1)[\Psi(i) - \Psi(n+1)] - (n-i)[\Psi(n-i+1) - \Psi(n+1)], \tag{22}$$

in which $\Psi(z) = \frac{d}{dz}(\log\Gamma(z))$ is digamma function.

The weighted distributions are widely used in many fields such as medicine, ecology, reliability, analysis of family data, for the improvement of proper statistical models. The concept of weighted distributions was initiated by Fisher (1934). Then Rao (1965) and Rao (1985) introduced weighted distributions in a unified way. Let X be a univariate random variable with distribution function F and probability distribution function f. Then the weighted version of X with univariate non-negative weight function w with $0 < E[w(X)] < \infty$, is denoted by X^w. The probability density function of X^w is given by $f_w(x) = \frac{w(x)f(x)}{E[w(X)]}$. When $w(t) = t$, X^w is called the length (or size) biased random variable. The corresponding density quantile function using f_w is obtained as

$$f_w(Q(u)) = w(Q(u))f(Q(u))/\mu,$$

where $\mu = \int_0^1 w(Q(p))f(Q(p))d(Q(p)) = \int_0^1 w(p)dp$. So, the quantile density function of X^w is given by $\frac{1}{q_w(u)} = \frac{w(Q(u))}{\mu q(u)}$.

Theorem 2.14. *The quantile-based past entropy for ith order statistics corresponding to X^w, $\bar{\eta}_{X^w_{i:n}}(u)$, is obtained as*

$$\begin{aligned}\bar{\eta}_{X^w_{i:n}}(u) &= \bar{\eta}_{X_{i:n}}(u) + \log\mu - \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\int_0^u (\log w(Q(p)))g_i(p)dp \\ &= \bar{\eta}_{X_{i:n}}(u) + \log\mu - \frac{\int_0^u (\log w(Q(p)))p^{i-1}(1-p)^{n-i}dp}{\int_0^u p^{i-1}(1-p)^{n-i}dp}. \end{aligned} \tag{23}$$

3. Ageing classes and stochastic comparisons

It is often useful to identify different classes of probability distributions based on the uncertainty measures. In the following, we define two nonparametric classes of distribution based on $\bar{\eta}_{X_{i:n}}(u)$.

Definition 3.1. X is said to have increasing (decreasing) past lifetime quantile entropy of order statistics (IPQEO (DPQEO)), if $\bar{\eta}_{X_{i:n}}(u)$ is increasing (decreasing) in $u \geq 0$.

If X is IPQEO (DPQEO), then from (19) we can show

$$\bar{\eta}_{X_{i:n}}(u) \leq (\geq) 1 + log\frac{q(u)\beta_u(i,n-i+1)}{u^{i-1}(1-u)^{n-i}},$$

which gives the upper (lower) bound for $\bar{\eta}_{X_{i:n}}(u)$. According to Table 1 and for Uniform distribution, we have

$$\bar{\eta}'_{X_{1:n}}(u) = \frac{(1-u)^{n-1}}{1-(1-u)^n} - n(n-1)\frac{(1-u)^{n-1}}{[1-(1-u)^n]^2}\log(1-u) > 0.$$

So, the uniform random variable belongs to increasing past lifetime quantile entropy of first order statistics. For Power function distribution with quantile function $Q(u) = \alpha u^{\frac{1}{\beta}}$, we obtain $\bar{\eta}_{X_{n:n}}(u) = -\log n + \log(\frac{\alpha}{\beta}) + 1 + \frac{1}{\beta}\log u - \frac{1}{n\beta}$ and $\bar{\eta}'_{X_{n:n}}(u) = \frac{1}{\beta u} > 0$, therefore the Power function random variable has increasing past lifetime quantile entropy of nth order statistics. For Exponential distribution

$$\bar{\eta}'_{X_{1:n}}(u) = -n^2\frac{(1-u)^{n-1}}{[1-(1-u)^n]^2}\log(1-u) > 0,$$

so, this distribution belongs to increasing past lifetime quantile entropy of first order statistics class.

Remark 3.2. *The reversed hazard rate ordering do not imply quantile past lifetime entropy of the first order statistics.*

Example 3.3. *In Exponential distribution, $\Lambda(u) = \lambda(\frac{1}{u} - 1)$ and $\Lambda'(u) < 0$, therefore this distribution is DRHR. While we showed that this distribution is IPQEO in first order statistics.*

The closure of the IPQEO (DPQEO) class under increasing convex (concave) transformations is studied in the following.

Theorem 3.4. *If X is IPQEO (DPQEO) and $\phi(.)$ is a non-negative, increasing and convex (concave) function, then $\phi(X)$ is also IPQEO (DPQEO).*

Proof. Let $Y = \phi(X)$ be a non-negative, increasing and convex (concave) function. In addition, let $g(y)$ be density function of Y, then

$$g(y) = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))} = \frac{1}{\phi'(Q_X(u))q_X(u)},$$

and $q_Y(u) = q_X(u)\phi'(Q_X(u))$. Now using (10), the past quantile entropy of $Y_{i:n}$ is obtained as

$$\bar{\eta}_{Y_{i:n}}(u) = -\frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\int_0^u (\log g_i(p))g_i(p)dp + \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\int_0^u (\log q_Y(p))g_i(p)dp + \log\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)},$$

which is equivalent to

$$\begin{aligned}\bar{\eta}_{Y_{i:n}}(u) &= -\frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\int_0^u (\log g_i(p))g_i(p)dp + \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\int_0^u (\log q_X(p))g_i(p)dp \\ &+ \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)}\int_0^u (\log \phi'(Q_X(p)))g_i(p)dp + \log\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)} \\ &= \bar{\eta}_{X_{i:n}}(u) + \frac{\int_0^u (\log\phi'(Q_X(p)))p^{i-1}(1-p)^{n-i}dp}{\beta_u(i,n-i+1)}. \qquad (24)\end{aligned}$$

Differentiating the relation (24) with respect to u yields

$$\bar{\eta}'_{Y_{i:n}}(u) = \bar{\eta}'_{X_{i:n}}(u) + \frac{u^{i-1}(1-u)^{n-i}\int_0^u [\log\phi'(Q_X(u)) - \log\phi'(Q_X(p))]p^{i-1}(1-p)^{n-i}dp}{(\int_0^u p^{i-1}(1-p)^{n-i}dp)^2}.$$

Since X is IPQEO (DPQEO), we obtain $\bar{\eta}'_{X_{i:n}}(u) > (<)0$. By regarding convexity (convcavity) of $\phi(u)$, we have $\phi'(u)$ is an increasing (a decreasing) function in u. Hence for any $p \in (0,u)$, we obtain $[\log\phi'(Q_X(u)) - \log\phi'(Q_X(p))] \geq (\leq)0$ and so $\bar{\eta}'_{Y_{i:n}}(u) > (<)0$. Therefore, Y is IPQEO (DPQEO) which completes the proof. □

The usefulness of Theorem 3.4 is illustrated by the following example.

Example 3.5. *Let X has Exponential distribution with the quantile function* $Q_X(u) = -\frac{1}{\lambda}\log(1-u)$ *and let* $Y = X^{\frac{1}{\alpha}}$, $\alpha > 0$. *The distribution of Y is Weibull distribution with the quantile function* $Q_Y(u) = \lambda^{-\frac{1}{\alpha}}(-\log(1-u))^{\frac{1}{\alpha}}$. *We know that X is IPQEO and the non-negative increasing function* $\phi(x) = x^{\frac{1}{\alpha}}$, $x > 0$, $\alpha > 0$, *is convex (concave) if* $0 < \alpha < 1$ $(\alpha > 1)$. *Hence due to Theorem 3.4, the Weibull distribution is IPQEO for* $0 < \alpha < 1$.

Now, we give appropriate conditions under which the IPQEO (DPQEO) class is preserved by weighted variables.

Theorem 3.6. *(a) If X is IPQEO and if* $w(x)$ *is a non-negative decreasing function of x, then* X^w *is also IPQEO.*
(b) If X is DPQEO and if $w(x)$ *is a non-negative increasing function of x, then* X^w *is also DPQEO.*

Proof. (a). Differentiating both sides of (23) with respect to u, we get

$$\bar{\eta}'_{X^w_{i:n}}(u) = \bar{\eta}'_{X_{i:n}}(u) + \frac{u^{i-1}(1-u)^{n-i}\int_0^u [\log w(Q(p)) - \log w(Q(u))]p^{i-1}(1-p)^{n-i}dp}{(\int_0^u p^{i-1}(1-p)^{n-i}dp)^2}. \qquad (25)$$

Since the quantile function is a nondecreasing function and $w(x)$ is a non-negative decreasing function of x, we have $\log w(Q(p)) > \log w(Q(u))$ for each $0 < p < u$ .Thus, we can deduce the second term in the expression (25) is positive. In addition, $\bar{\eta}'_{X_{i:n}}(u) > 0$, because X is IPQEO. Therefore $\bar{\eta}'_{X^w_{i:n}}(u)$ is positive i.e. X^w also belongs to IPQEO class. Part (b) can be proved in an analogous manner. □

Here, we define a new stochastic order based on the comparison of $\bar{\eta}_{X_{i:n}}(u)$ and $\bar{\eta}_{Y_{i:n}}(u)$ corresponding to two non-negative random variables X and Y, respectively.

Definition 3.7. X is said to be smaller than Y in PQEO order (written as $X \leq_{PQEO} Y$) if $\bar{\eta}_{X_{i:n}}(u) \leq \bar{\eta}_{Y_{i:n}}(u)$ for all $u \in (0,1)$.

Example 3.8. *Let X and Y be two exponential random variables with quantile functions*

$$Q_X(u) = -\frac{1}{\lambda_1}\log(1-u), \quad Q_Y(u) = -\frac{1}{\lambda_2}\log(1-u), \quad \textit{for all } u \in (0,1),$$

respectively. By means of Table 1 and Definition 3.7, we get the following results for any $u \in (0,1)$.
(a). If $\lambda_1 < \lambda_2$, *then* $\bar{\eta}_{X_{1:n}}(u) \geq \bar{\eta}_{Y_{1:n}}(u)$ *or equivalently* $X \geq_{PQEO} Y$.
(b). If $\lambda_1 > \lambda_2$, *then* $\bar{\eta}_{X_{1:n}}(u) \leq \bar{\eta}_{Y_{1:n}}(u)$ *or equivalently* $X \leq_{PQEO} Y$.

In the following, we give sufficient conditions for a function of a random variable to have more (less) past quantile entropy of order statistics than itself.

Theorem 3.9. *Let ϕ is a non-negative increasing function of x. (a). If $\phi'(x) \geq 1$ for all $x > 0$, then $X \leq_{PQEO} \phi(X)$. (b). If $\phi'(x) \leq 1$ for all $x > 0$, then $\phi(X) \leq_{PQEO} X$.*

Proof. For Part (a), Let $Z = \phi(X)$. From (24), we have

$$\bar{\eta}_{Z_{i:n}}(u) = \bar{\eta}_{X_{i:n}}(u) + \frac{\int_0^u (\log \phi'(Q_X(p))) p^{i-1}(1-p)^{n-i} dp}{\beta_u(i, n-i+1)}. \tag{26}$$

First notice that the function $\log(\phi'(Q_X(p)))$ can be defined because $\phi(x)$ is an increasing function of x i.e. $\phi'(x) > 0$. The condition $\phi'(x) \geq 1$ implies $\log(\phi'(Q_X(p))) \geq 0$. Therefore from (26), we obtain $\bar{\eta}_{Z_{i:n}}(u) \geq \bar{\eta}_{X_{i:n}}(u)$ which is equivalent to $X \leq_{PQEO} \phi(X)$ according to Definition 3.7. This completes the proof.
For Part (b), The proof can be done in a similar manner and hence is omitted. □

In the following example, we illustrate the usefulness of Theorem 3.9.

Example 3.10. *If X is a random variable following the Exponential distribution with quantile function $Q_X(u) = -\frac{1}{\lambda}\log(1-u)$, $\lambda > 0$ and let*

$$\phi(x) = \frac{1}{\gamma}\log(1 + \frac{\lambda\gamma}{\beta}x), \quad \gamma > 0, \ \lambda \leq \beta.$$

Then $\phi(X)$ has the Gompertz–Makeham distribution with quantile function

$$Q_{\phi(X)}(u) = \frac{1}{\gamma}\log(1 - \frac{\gamma}{\beta}\log(1-u)).$$

From Theorem 3.9 Part (b), it follows that $\phi(X) \leq_{PQEO} X$. Note that using this issue and Definition 3.7, we obtain an upper bound for quantile-based past entropy of order statistics for the Gompertz–Makeham distribution.

Cox and Oakes (1984) proposed the following accelerated life model (ALM) to study the relationship between F and G,

$$F(x) = G(W(x)),$$

where $W(x)$ is strictly increasing function with $W(0) = 0$, and $W(x) \to \infty$ as $x \to \infty$ and it is a time-dependent scale transformation function.
In general, assume that $W(x)$ is continuous and differentiable in $[0, \infty)$ and

$$W(x) = \int_0^x w(t)dt, \quad w(t) \geq 0, \ \ x \in [0, \infty).$$

Furthermore, suppose that F and G are distribution functions for two lifetime variables X and Y, respectively. For ALM, it is interesting to investigate the role of $W(x)$ or $w(x)$. For example, the role of $W(x)$ or $w(x)$ in establishing the aging properties of X via the aging properties of Y, is important. Here, we give sufficient condition under which X has less quantile past entropy of order statistics than Y.

Theorem 3.11. *If $W(x) - x$ is increasing in x, then $X \leq_{PQEO} Y$.*

Proof. We have $X = \phi(Y)$, where $\phi = W^{-1}$, from above discussion. The conditions of Theorem 3.9 Part(b) is satisfied, thus the proof is complete. □

Theorem 3.12. *If $q_X(u) \leq q_Y(u)$ for all $u \in (0, 1)$, then $X \leq_{PQEO} Y$.*

Proof. The proof directly follows from relation (10) and Definition 3.7. □

Example 3.13. *Let X_i, $i = 1, 2$ be distributed as Rescaled beta with quantile functions $Q_i(u) = R_i[1 - (1 - u)^{\frac{1}{c}}]$ and quantile density function $q_i(u) = \frac{R_i}{c}(1 - u)^{\frac{1}{c}-1}$, $i = 1, 2$. If we have $R_1 < R_2$, then $q_1(u) < q_2(u)$ for all $u \in (0, 1)$ and therefore from Theorem 3.12, we conclude that $X_1 \leq_{PQEO} X_2$.*

The PQEO order is closed under non-negative increasing convex transformations as the following theorem shows.

Theorem 3.14. *Let X and Y be two non-negative random variables such that $X \leq_{PQEO} Y$ and $X \leq_{st} Y$. Then for any non-negative increasing convex function $\phi(.)$, we have $\phi(X) \leq_{PQEO} \phi(Y)$.*

Proof. Let $Z = \phi(X)$ and $T = \phi(Y)$. From (24), we have

$$\bar{\eta}_{Z_{i:n}}(u) = \bar{\eta}_{X_{i:n}}(u) + \frac{\int_0^u (\log \phi^{'}(Q_X(p)))p^{i-1}(1-p)^{n-i}g_i(p)dp}{\beta_u(i, n-i+1)},$$

and

$$\bar{\eta}_{T_{i:n}}(u) = \bar{\eta}_{Y_{i:n}}(u) + \frac{\int_0^u (\log \phi^{'}(Q_Y(p)))p^{i-1}(1-p)^{n-i}g_i(p)dp}{\beta_u(i, n-i+1)}.$$

According to Definition 1.1, $X \leq_{st} Y$ implies that $Q_X(u) \leq Q_Y(u)$ for all $u \in (0, 1)$. Since $\phi(u)$ is convex, $\phi^{'}(u)$ is an increasing function of u, hence $\phi^{'}(Q_X(u)) \leq \phi^{'}(Q_Y(u))$, $\forall u \in (0, 1)$. The stochastic order $X \leq_{PQEO} Y$ also asserts that $\bar{\eta}_{X_{i:n}}(u) \leq \bar{\eta}_{Y_{i:n}}(u)$ for all $u \in (0, 1)$. So, we can easily show that $\bar{\eta}_{Z_{i:n}}(u) \leq \bar{\eta}_{T_{i:n}}(u)$ for all $u \in (0, 1)$ and so using Definition 3.7, we reach the desired result. □

The following example deals with use of Theorem 3.14.

Example 3.15. *It is known that if X follows Pareto II distribution with quantile function $Q_X(u) = \alpha[(1 - u)^{-\frac{1}{c}} - 1]$, $\alpha, c > 0$, then $Z = X^{\beta}$ follows Burr type XII distribution with parameters (α, c, β) and the quantile function $Q_Z(u) = \alpha^{\beta}[(1 - u)^{-\frac{1}{c}} - 1]^{\beta}$. Now, assume that X and Y follow two independent Pareto II distributions with quantile functions $Q_X(u) = \alpha_1[(1 - u)^{-\frac{1}{c}} - 1]$ and $Q_Y(u) = \alpha_2[(1 - u)^{-\frac{1}{c}} - 1]$, respectively. For $\alpha_1 < \alpha_2$, we obtain $Q_X(u) \leq Q_Y(u)$ for all $u \in (0, 1)$ which means $X \leq_{st} Y$. From Table 1,*

$$\bar{\eta}_{X_{i:n}}(u) = -\log n + \log(\frac{\alpha_1}{c}) + (1 + \frac{1}{nc}) + \log(1 - (1 - u)^n) + (n + \frac{1}{c})\frac{(1 - u)^n}{1 - (1 - u)^n} \log(1 - u).$$

The condition $\alpha_1 < \alpha_2$, implies that $\bar{\eta}_{X_{i:n}}(u) \leq \bar{\eta}_{Y_{i:n}}(u)$ for all $u \in (0, 1)$ which is equivalent to $X \leq_{PQEO} Y$. Furthermore, we know that $\phi(u) = u^{\beta}$ is a non-negative increasing function of u and for $\beta > 1$ is convex. Therefore the conditions of Theorem 3.14 is satisfied and we conclude that $Z_1 \leq_{PQEO} Z_2$, where $Z_1 = X^{\beta}$ and $Z_2 = Y^{\beta}$. We remark that the past quantile entropy of order statistics for two Burr type XII densities can be obtained from Theorem 3.4 by taking $\phi_1(X) = X^{\beta}$ and $\phi_2(Y) = Y^{\beta}$.

Theorem 3.16. *Let X and Y be two random variables such that $X_{i:n} \leq_{disp} Y_{i:n}$, then $X \leq_{PQEO} Y$.*

Proof. According to Remark 1.4, the condition $X_{i:n} \leq_{disp} Y_{i:n}$ implies that $Q_Y(u) - Q_X(u)$ is increasing in u and therefore $q_X(u) \leq q_Y(u)$ for all $u \in (0, 1)$. Now, using Theorem 3.12, we conclude that $X \leq_{PQEO} Y$ which completes the proof. □

We consider the following example, to illustrate Theorem 3.16.

Example 3.17. *Let X and Y be two random variables with quantile functions $Q_X(u) = u$ and $Q_Y(u) = -4\log(1-u)$, respectively. Let $g(u) = Q_Y(u) - Q_X(u) = -4\log(1 - u) - u$, then $g'(u) = \frac{3+u}{1-u} \geq 0$. Therefore $Q_Y(u) - Q_X(u)$ is increasing function in u, and $X_{i:n} \leq_{disp} Y_{i:n}$ which implies $X \leq_{PQEO} Y$.*

Theorem 3.18. *If $X \leq_{rhq} Y$, then $X \geq_{PQEO} Y$.*

Proof. According to Definition 1.2, $X \le_{rhq} Y$ asserts that

$$-\log \Lambda_X(u) \ge -\log \Lambda_Y(u), \quad for\ all\ u \in (0,1).$$

Using definition of the quantile reversed hazard rate function, we obtain $q(u) = \frac{1}{u\Lambda(u)}$ and $\log q(u) = -\log u - \log \Lambda(u)$. Next from relation (10), we have

$$\begin{aligned}
\bar{\eta}_{X_{i:n}}(u) &= -\frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)} \int_0^u (\log g_i(p)) g_i(p) dp + \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)} \int_0^u (-\log p) g_i(p) dp \\
&+ \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)} \int_0^u (-\log \Lambda_X(p)) g_i(p) dp + \log \frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)} \\
&\ge -\frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)} \int_0^u (\log g_i(p)) g_i(p) dp + \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)} \int_0^u (-\log p) g_i(p) dp \\
&+ \frac{\beta(i,n-i+1)}{\beta_u(i,n-i+1)} \int_0^u (-\log \Lambda_Y(p)) g_i(p) dp + \log \frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)} \\
&= \bar{\eta}_{Y_{i:n}}(u)
\end{aligned}$$

So, $X \ge_{PQEO} Y$. $\square$

To show the usefulness of Theorem 3.18, we consider the following examples.

Example 3.19. *If X and Y follow Exponential distributions with parameters λ_1 and λ_2 and the quantile reversed hazard rate functions $\Lambda_X(u) = \lambda_1 \frac{1-u}{u}$ and $\Lambda_Y(u) = \lambda_2 \frac{1-u}{u}$, respectively. The condition $\lambda_1 < \lambda_2$ implies $\Lambda_X(u) < \Lambda_Y(u)$ for all $u \in (0,1)$ which is equivalent to $X \le_{rhq} Y$. So using Theorem 3.18, we reach to the desired issue.*

Example 3.20. *Let X be a random variable with quantile function $Q_X(u) = \frac{u}{(1-u)^2}$ which is a special case of Power-Pareto distribution due to Hankin and Lee (2006). In addition, let Y follows Exponential distribution with the quantile function $Q_Y(u) = -\log(1-u)$. The reversed hazard quantile functions of X and Y are given by $\Lambda_X(u) = \frac{(1-u)^3}{u(1+u)}$ and $\Lambda_Y(u) = \frac{1-u}{u}$, respectively. Hence $\frac{\Lambda_X(u)}{\Lambda_Y(u)} = \frac{(1-u)^2}{1+u}$ and*

$$\frac{d}{du}\left(\frac{\Lambda_X(u)}{\Lambda_Y(u)}\right) = \frac{-(1-u)(3+u)}{(1+u)^2} < 0,$$

or equivalently $\frac{\Lambda_X(u)}{\Lambda_Y(u)}$ is a decreasing function of u. For $u = 0$, we obtain $\frac{\Lambda_X(u)}{\Lambda_Y(u)} = 1$, therefore we can conclude that $\frac{\Lambda_X(u)}{\Lambda_Y(u)} \le 1$ for all $u \in (0,1)$ which means $X \le_{rhq} Y$. Now applying Theorem 3.18, we get to the relation $X \ge_{PQEO} Y$.

4. Conclusions

In the present work, we have introduced past entropy of order statistics based on the quantile function. Specially, we obtained the quantile version of entropy function in past lifetime for parallel and series systems. Furthermore, this measure is computed for some lifetime models. The proposed measure is useful for variables which have not closed form for distribution functions. It is defined two nonparametric classes of distribution based on the proposed measure. The closure property of these classes under increasing convex (concave) function and weighting are also discussed. One stochastic order based on this measure is defined and several properties for it are studied. In addition, the effect of non-negative increasing convex function on this order is investigated. Finally, some results based on the proposed order are also provided.

References

[1] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, A First Course in Order Statistics, New York, John Wiley and Sons, 1992.

[2] S. Baratpour, J. Ahmadi and N. R. Arghami, Some characterizations based on entropy of order statistics and record values, Communications in Statistics—Theory and Methods 36 (2007) 47–57.

[3] R. Barlow and F. Proschan, Statistical theory of reliability and life testing: probability models, International series in decision processes, Holt, Rinehart and Winston, 1975.

[4] C. Calì, M. Longobardi and J. Navarro, Properties for generalized cumulative past measures of information, Probability in the Engineering and Informational Sciences 34 (2020) 92–111.

[5] D. R. Cox and D. Oakes, Analysis of survival data, New York: Chapman and Hall, 1984.

[6] H. A. David and H. N. Nagaraja, Order Statistics, (Third edition), John Wiley and Sons, Hoboken, New Jersey, 2003.

[7] A. Di Crescenzo and M. Longobardi, Entropy-based measure of uncertainty in past lifetime distributions, Journal of Applied Probability 39 (2002) 434–440.

[8] A. Di Crescenzo and M. Longobardi, On weighted residual and past entropies, Scientiae Math. Japon 64 (2006) 255–266.

[9] A. Di Crescenzo and A. Toomaj, Extension of the past lifetime and its connection to the cumulative entropy, Journal of Applied Probability 52 (2015) 1156–1174.

[10] N. Ebrahimi, How to measure uncertainty in the residual life time distribution, Sankhyā: The Indian Journal of Statistics, Series A (1996) 48–56.

[11] N. Ebrahimi, E. S. Soofi. and H. Zahedi, Information properties of order statistics and spacings, IEEE Transactions on Information Theory 50 (2004) 177–183.

[12] R. A. Fisher, The effect of methods of ascertainment upon the estimation of frequencies, Annals of eugenics 6 (1934) 13–25.

[13] Z. Govindarajulu, A class of distributions useful in life testing and reliability with applications to nonparametric testing, In The Theory and Applications of Reliability with Emphasis on Bayesian and Nonparametric Methods (pp. 109-129), Academic Press, 1977.

[14] R. K. Hankin. and A. Lee, A new family of non-negative distributions, Australian and New Zealand Journal of Statistics 48 (2006) 67–78.

[15] A. S. Krishnan, S. M. Sunoj and N. U. Nair, Some reliability properties of extropy for residual and past lifetime random variables, Journal of the Korean Statistical Society 49 (2020) 457–474.

[16] V. Kumar and N. Singh, Quantile-based generalized entropy of order (α, β) for order statistics, Statistica 78 (2018) 299–318.

[17] C. Kundu, A. K. Nanda and S. S. Maiti, Some distributional results through past entropy, Journal of statistical planning and inference 140 (2010) 1280–1291.

[18] M. Madadi and M. Tata, Shannon information in record data, Metrika 74 (2011) 11–31.

[19] M. Madadi and M. Tata, Shannon information in k-records, Communications in Statistics-Theory and Methods 43 (2014) 3286–3301.

[20] A. K. Nanda and P. Paul, Some properties of past entropy and their applications, Metrika 64 (2006) 47–61.

[21] A. K. Nanda, P. G. Sankaran and S. M. Sunoj, Renyi's residual entropy: A quantile approach, Statistics and Probability Letters 85 (2014) 114–121.

[22] N. U. Nair, P. G. Sankaran and N. Balakrishnan, Quantile-based reliability analysis, Springer, New York, 2013.

[23] R. Nisa and M. A. K. Baig, Characterization Results based on Quantile version of Two Parametric Generalized Entropy of Order Statistics, Journal of Probability and Mathematical Statistics 1 (2019) 1–17.

[24] G. Qiu, Further results on quantile entropy in the past lifetime, Probability in the Engineering and Informational Sciences 33 (2019) 146–159.

[25] C.R. Rao, On discrete distributions arising out of methods of ascertainment, Sankhya: The Indian Journal of Statistics, Series A (1965) 311–324.

[26] C. R. Rao, Weighted distributions arising out of methods of ascertainment: What population does a sample represent?, A celebration of statistics (1985) 543–569.

[27] A. Sachlas and T. Papaioannou, Residual and past entropy in actuarial science and survival models, Methodology and Computing in Applied Probability 16 (2014) 79–99.

[28] C. E. Shannon, A mathematical theory of communication, The Bell system technical journal 27 (1948) 379–423.

[29] S. M. Sunoj and P. G. Sankaran, Quantile based entropy function, Statistics and Probability Letters 82 (2012) 1049–1053.

[30] S. M. Sunoj, P. G. Sankaran and A. K. Nanda, Quantile based entropy function in past lifetime, Statistics and Probability Letters 83 (2013) 366–372.

[31] S. M. Sunoj, A. S. Krishnan and P. G. Sankaran, Quantile-based entropy of order statistics, Journal of the Indian Society for Probability and Statistics 18 (2017) 1–17.

[32] H. L. Yu and C. H. Wang, Quantile-based bayesian maximum entropy approach for spatiotemporal modeling of ambient air quality levels, Environmental Science and Technology 47 (2013) 1416–1424.

[33] Z. Zamani and M. Madadi, Shannon information in k-records for Pareto-type distributions, Journal of Statistical Theory and Applications 17 (2018) 419–438.