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A degree condition for graphs being fractional (a, b, k)-critical covered

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Abstract. A graph G is fractional $[a, b]$-covered if for any $e \in E(G)$, G possesses a fractional $[a, b]$-factor including e. A graph G is fractional (a, b, k)-critical covered if $G - Q$ is fractional $[a, b]$-covered for any $Q \subseteq V(G)$ with $|Q| = k$. In this paper, we verify that a graph G of order n is fractional (a, b, k)-critical covered if $n \geq \frac{(a+b)((2r-3)a+b+r-2)+bk+2}{b}$, $\delta(G) \geq (r-1)(a+1)+k$ and

$$\max\{d_G(w_1), d_G(w_2), \cdots, d_G(w_r)\} \geq \frac{an+bk+2}{a+b}$$

for every independent vertex subset $\{w_1, w_2, \cdots, w_r\}$ of G. Our main result is an improvement of the previous result [S. Zhou, Y. Xu, Z. Sun, Degree conditions for fractional (a, b, k)-critical covered graphs, Information Processing Letters 152(2019)105838].

1. Introduction

A large number of real-world networks can be modelled by graphs. The nodes in the networks are represented by the vertices in the graphs, and the links between the nodes are represented by the edges in the graphs. Hence, the term *network* can be replaced by *graph*.

Consequently, we may convert the problem of network into the problem of graph theory. In data transmission network, we always require the network as a whole to meet pre-setting transmission requirements, especially when some nodes do not work properly due to failure or being under attacked for some unexpected reasons, the entire network is unobstructed. From the standpoint of graph theory, the transmission problem of data packets within a certain range in the network is equivalent to the existence of fractional $[a, b]$-factor in a graph. Besides, the smooth procedure transmission under the conditions that some nodes cannot work and some links must work is equivalent to the existence of fractional (a, b, k)-critical covered graph. The degree condition of graph is often used to measure the vulnerability and robustness of network, which is a much important parameter in network data transmission and network design, etc.

In this paper, we only discuss the simple graph. Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. For $w \in V(G)$, we use $d_G(w)$ and $N_G(w)$ to denote the degree and neighborhood of w in G, respectively. Set $\delta(G) = \min\{d_G(w) : w \in V(G)\}$ and $N_G[w] = N_G(w) \cup \{w\}$.

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For $Q \subseteq V(G)$, we write $G[Q]$ for the subgraph of G induced by Q, and set $G - Q = G[V(G) \setminus Q]$. A subset $Q \subseteq V(G)$ is independent if $G[Q]$ does not possess edges.

Let $b \geq a \geq 0$ be integers and h be a function with $0 \leq h(e) \leq 1$ for each $e \in E(G)$. If $a \leq \sum_{e \ni x} h(e) \leq b$ for every $x \in V(G)$, then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is fractional $[a, b]$-covered if for any $e \in E(G)$, G admits a fractional $[a, b]$-factor $G[F_h]$ with $h(e) = 1$. A graph G is fractional (a, b, k)-critical covered if $G - Q$ is fractional $[a, b]$-covered for any $Q \subseteq V(G)$ with $|Q| = k$, which is defined by Zhou, Xu and Sun [27]. If $h(e) \in \{0, 1\}$ for any $e \in E(G)$, then $G[F_h]$ is just an $[a, b]$-factor of G.

Many results on factors of graphs were derived by Matsuda [8], Li and Cai [4], Li [3], Cymer and Kano [2], Yashima [12], Zhou, Bian and Pan [21], Zhou, Sun and Liu [24], Zhou [18–20], Zhou, Wu and Bian [25], Zhou, Wu and Xu [26], Zhou and Liu [22], Wang and Zhang [9], Yan and Liu [11]. Many results on fractional factors of graphs were obtained by Liu and Zhang [6, 7], Cai, Wang and Yan [1], Wang and Zhang [10], Zhou [17], Zhou, Liu and Xu [23], Yuan and Hao [14, 15]. A fractional $[a, b]$-covered graph was studied by Yuan and Hao [13], and they posed a degree condition for the existence of fractional $[a, b]$-covered graphs. Zhou [16] posed a neighborhood condition for fractional (a, b, k)-critical covered graphs. Zhou, Xu and Sun [27] improved and generalised Yuan and Hao's previous result, and verified a degree condition for the existence of fractional (a, b, k)-critical covered graphs which is shown in the following.

Theorem 1 ([27]). Let a, b and k be integers with $k \geq 0$, $a \geq 1$ and $b \geq \max\{2, a\}$, and let G be a graph of order n with $n \geq \frac{(a+b)(a+b-1)+bk+3}{b}$. If $\delta(G) \geq a + k + 1$ and

$$\max\{d_G(u), d_G(v)\} \geq \frac{an + bk + 2}{a + b}$$

for every pair of nonadjacent vertices u and v of G, then G is fractional (a, b, k)-critical covered.

Next, we also investigate fractional (a, b, k)-critical covered graphs, and gain a new sufficient condition for graphs being fractional (a, b, k)-critical covered. Our main result is an improvement of Theorem 1 and it is given in Section 2.

2. Main result and proof

In this section, we first give the main theorem of this paper.

Theorem 2. Let k, r, a and b be integers with $k \geq 0$, $r \geq 2$, $a \geq 1$ and $b \geq \max\{2, a\}$, and let G be a graph of order n with $n \geq \frac{(a+b)((2r-3)a+b+r-2)+bk+2}{b}$. If $\delta(G) \geq (r-1)(a+1) + k$ and

$$\max\{d_G(w_1), d_G(w_2), \cdots, d_G(w_r)\} \geq \frac{an + bk + 2}{a + b} \tag{1}$$

for every independent vertex subset $\{w_1, w_2, \cdots, w_r\}$ of G, then G is fractional (a, b, k)-critical covered.

Let n be sufficiently large. Set $r = 2$ in Theorem 2, then we easily see that Theorem 1 is a special case of Theorem 2. Furthermore, we easily derive the following result from Theorem 2.

Corollary 1. Let r, a and b be integers with $r \geq 2$, $a \geq 1$ and $b \geq \max\{2, a\}$, and let G be a graph of order n with $n \geq \frac{(a+b)((2r-3)a+b+r-2)+2}{b}$. If $\delta(G) \geq (r-1)(a+1)$ and

$$\max\{d_G(w_1), d_G(w_2), \cdots, d_G(w_r)\} \geq \frac{an + 2}{a + b}$$

for every independent vertex subset $\{w_1, w_2, \cdots, w_r\}$ of G, then G is fractional $[a, b]$-covered.

To verify Theorem 2, we need the following theorem which was derived by Li, Yan and Zhang [5].

Theorem 3 ([5]). Let G be a graph, and let $b \geq a \geq 0$ be integers. Then G is fractional $[a,b]$-covered if and only if

$$\lambda_G(X,Y) = b|X| + d_{G-X}(Y) - a|Y| \geq \varepsilon(X)$$

for every subset X of $V(G)$, where $Y = \{w \in V(G) \setminus X, d_{G-X}(w) \leq a\}$ and $\varepsilon(X)$ is defined by

$$\varepsilon(X) = \begin{cases} 2, & if\ X\ is\ not\ independent, \\ 1, & if\ X\ is\ independent\ and\ there\ is\ an\ edge\ joining\ X\ and\ V(G) \setminus (X \cup Y),\ or \\ & there\ is\ an\ edge\ e = uv\ joining\ X\ and\ Y\ with\ d_{G-X}(v) = a\ for\ v \in Y, \\ 0, & otherwise. \end{cases}$$

Proof of Theorem 2. Let $H = G - Q$ for $Q \subseteq V(G)$ with $|Q| = k$. Note that $\delta(G) \geq (r-1)(a+1) + k$. Hence,

$$\delta(H) = \delta(G - Q) \geq \delta(G) - k \geq (r-1)(a+1). \quad (2)$$

To verify Theorem 2, it suffices to claim that H is fractional $[a,b]$-covered. Assume that H is not fractional $[a,b]$-covered. Then it follows from Theorem 3 that

$$\lambda_H(X,Y) = b|X| + d_{H-X}(Y) - a|Y| \leq \varepsilon(X) - 1 \quad (3)$$

for some $X \subseteq V(H)$, where $Y = \{w \in V(H) \setminus X : d_{H-X}(w) \leq a\}$.
Claim 1. $Y \neq \emptyset$.
Proof. If $Y = \emptyset$, then

$$\lambda_H(X,\emptyset) = b|X| \geq |X| \geq \varepsilon(X),$$

which contradicts (3). Therefore, $Y \neq \emptyset$. □

Note that $Y \neq \emptyset$ by Claim 1. Thus, we may construct a sequence $w_1, w_2, \cdots, w_t$ of vertices of Y. Define

$$\beta_1 = \min\{d_{H-X}(w) : w \in Y\}$$

and select $w_1 \in Y$ with $d_{H-X}(w_1) = \beta_1$. If $t \geq 2$ and $Y \setminus \Big(\bigcup_{i=1}^{t-1} N_Y[w_i]\Big) \neq \emptyset$, then we define

$$\beta_t = \min\Big\{d_{H-X}(w) : w \in Y \setminus \Big(\bigcup_{i=1}^{t-1} N_Y[w_i]\Big)\Big\}$$

and select $w_t \in Y \setminus \Big(\bigcup_{i=1}^{t-1} N_Y[w_i]\Big)$ with $d_{H-X}(w_t) = \beta_t$, $2 \leq t \leq r$. Obviously, $0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_t \leq a$ holds, and $\{w_1, w_2, \cdots, w_t\} \subseteq Y$ is independent.
Claim 2. $\beta_1 \leq a - 1$.
Proof. Assume that $\beta_1 = a$. Then

$$\begin{aligned} \lambda_H(X,Y) &= b|X| + d_{H-X}(Y) - a|Y| \\ &\geq b|X| + \beta_1|Y| - a|Y| = b|X| \geq |X| \geq \varepsilon(X), \end{aligned}$$

which contradicts (3). Hence, $\beta_1 \leq a - 1$. □
Claim 3. $|Y| \geq (r-1)(b+1)$.
Proof. Note that $|X| + \beta_1 = |X| + d_{H-X}(w_1) \geq d_H(w_1) \geq \delta(H) \geq (r-1)(a+1)$ by (2). Thus, we get

$$|X| \geq (r-1)(a+1) - \beta_1. \quad (4)$$

Let $|Y| \leq (r-1)(b+1) - 1$. Then it follows from (4), $a - \beta_1 \geq 0$, $r \geq 2$, $b \geq \max\{2, a\}$ and $\varepsilon(X) \leq 2$ that

$$\begin{aligned} \lambda_H(X,Y) &= b|X| + d_{H-X}(Y) - a|Y| \geq b|X| + \beta_1|Y| - a|Y| \\ &= b|X| - (a - \beta_1)|Y| \geq b((r-1)(a+1) - \beta_1) - (a - \beta_1)((r-1)(b+1) - 1) \\ &= (b-a)(r-1) + (b+1)(r-2)\beta_1 + a \geq (b-a)(r-1) + a \\ &\geq (b-a) + a = b \geq 2 \geq \varepsilon(X), \end{aligned}$$

which contradicts (3). Hence, $|Y| \geq (r-1)(b+1)$. □

Claim 4. There exists an independent subset $\{w_1, w_2, \cdots, w_r\}$ of Y.

Proof. Note that $d_{H-X}(w_1) = \beta_1 \leq a-1$ by Claim 2. Thus, $d_{H-X}(w) \leq a$ for any $w \in Y \setminus \{w_1\}$. Then using Claim 3, we can take above independent subset $\{w_1, w_2, \cdots, w_r\} \subseteq Y$ for $t = r$. □

From Claim 4 and the assumption of Theorem 2, we admit

$$\begin{aligned} \frac{an+bk+2}{a+b} &\leq \max\{d_G(w_1), d_G(w_2), \cdots, d_G(w_r)\} \\ &\leq \max\{d_{G-Q}(w_1), d_{G-Q}(w_2), \cdots, d_{G-Q}(w_r)\} + |Q| \\ &= \max\{d_H(w_1), d_H(w_2), \cdots, d_H(w_r)\} + k \\ &\leq \max\{d_{H-X}(w_1), d_{H-X}(w_2), \cdots, d_{H-X}(w_r)\} + |X| + k \\ &= \max\{\beta_1, \beta_2, \cdots, \beta_r\} + |X| + k \\ &= \beta_r + |X| + k, \end{aligned}$$

namely,

$$|X| \geq \frac{an-ak+2}{a+b} - \beta_r. \tag{5}$$

Claim 5. $|X| < \frac{an-ak+2}{a+b}$.

Proof. It follows from (3), $\varepsilon(X) \leq 2$ and $|X| + |Y| + k \leq n$ that

$$\begin{aligned} 2 &> \varepsilon(X) - 1 \geq \lambda_H(X,Y) = b|X| + d_{H-X}(Y) - a|Y| \\ &\geq b|X| - a|Y| \geq b|X| - a(n-k-|X|) \\ &= (a+b)|X| - an + ak, \end{aligned}$$

which implies $|X| < \frac{an-ak+2}{a+b}$. □

By (5), Claim 5 and the integrity of β_r, we admit

$$\beta_r \geq 1. \tag{6}$$

We easily see that

$$|N_Y[w_i]| - \Big|N_Y[w_i] \bigcap \Big(\bigcup_{j=1}^{i-1} N_Y[w_j]\Big)\Big| \geq 1 \tag{7}$$

for $2 \leq i \leq r-1$, and

$$\Big|\bigcup_{j=1}^{i} N_Y[w_j]\Big| \leq \sum_{j=1}^{i} |N_Y[w_j]| \leq \sum_{j=1}^{i} (d_{H-X}(w_j)+1) = \sum_{j=1}^{i} (\beta_j + 1) \tag{8}$$

for $1 \leq i \leq r$.

From (3), (5), (6), (7), (8), $|X| + |Y| + k \leq n$, $\varepsilon(X) \leq 2$ and $0 \leq \beta_1 \leq \beta_2 \leq \beta_r \leq a$, we admit

$$\begin{aligned} 1 &\geq \varepsilon(X) - 1 \geq \lambda_H(X,Y) = b|X| + d_{H-X}(Y) - a|Y| \\ &\geq b|X| + \beta_1|N_Y[w_1]| + \beta_2(|N_Y[w_2]| - |N_Y[w_2] \bigcap N_Y[w_1]|) + \cdots + \\ &\quad \beta_{r-1}\Big(|N_Y[w_{r-1}]| - \Big|N_Y[w_{r-1}] \bigcap \Big(\bigcup_{i=1}^{r-2} N_Y[w_i]\Big)\Big|\Big) + \beta_r(|Y| - |\bigcup_{i=1}^{r-1} N_Y[w_i]|) - a|Y| \end{aligned}$$

$$
\begin{aligned}
&\geq b|X| + (\beta_1 - \beta_r)|N_Y[w_1]| + \sum_{i=2}^{r-1} \beta_i - \beta_r \sum_{i=2}^{r-1} |N_Y[w_i]| - (a - \beta_r)|Y| \\
&\geq b|X| + (\beta_1 - \beta_r)(\beta_1 + 1) + \sum_{i=2}^{r-1} \beta_i - \beta_r \sum_{i=2}^{r-1} (\beta_i + 1) - (a - \beta_r)|Y| \\
&= b|X| + \beta_1^2 + \sum_{i=1}^{r-1} \beta_i - \beta_r \sum_{i=1}^{r-1} (\beta_i + 1) - (a - \beta_r)|Y| \\
&\geq b|X| + \beta_1^2 + \sum_{i=1}^{r-1} \beta_i - \beta_r \sum_{i=1}^{r-1} (\beta_i + 1) - (a - \beta_r)(n - k - |X|) \\
&= (a + b - \beta_r)|X| + \beta_1^2 - (\beta_r - 1) \sum_{i=1}^{r-1} \beta_i - \beta_r(r - 1) - (a - \beta_r)(n - k) \\
&\geq (a + b - \beta_r)\Big(\frac{an - ak + 2}{a + b} - \beta_r\Big) + \beta_1^2 - (r - 1)(\beta_r - 1)\beta_r \\
&\quad -(r - 1)\beta_r - (a - \beta_r)(n - k) \\
&\geq (a + b - \beta_r)\Big(\frac{an - ak + 2}{a + b} - \beta_r\Big) - (r - 1)(\beta_r - 1)\beta_r \\
&\quad -(r - 1)\beta_r - (a - \beta_r)(n - k) \\
&\geq (a + b - \beta_r)\Big(\frac{an - ak + 2}{a + b} - \beta_r\Big) - (r - 1)\beta_r^2 - (a - \beta_r)(n - k),
\end{aligned}
$$

namely,

$$
(a + b - \beta_r)\Big(\frac{an - ak + 2}{a + b} - \beta_r\Big) - (r - 1)\beta_r^2 - (a - \beta_r)(n - k) - 1 \leq 0. \tag{9}
$$

Let $\varphi(\beta_r) = (a + b - \beta_r)\big(\frac{an-ak+2}{a+b} - \beta_r\big) - (r - 1)\beta_r^2 - (a - \beta_r)(n - k) - 1$. Using $r \geq 2$, $1 \leq \beta_r \leq a$ and $n \geq \frac{(a+b)((2r-3)a+b+r-2)+bk+2}{b}$, we admit

$$
\begin{aligned}
\varphi'(\beta_r) &= -\frac{an - ak + 2}{a + b} + \beta_r - a - b + \beta_r - 2(r - 1)\beta_r + n - k \\
&= \frac{bn - bk - 2}{a + b} - 2(r - 2)\beta_r - a - b \\
&\geq \frac{bn - bk - 2}{a + b} - 2(r - 2)a - a - b \\
&= \frac{bn - bk - 2}{a + b} - (2r - 3)a - b \\
&\geq (2r - 3)a + b + r - 2 - (2r - 3)a - b \\
&= r - 2 \geq 0,
\end{aligned}
$$

which implies that $\varphi(\beta_r)$ attains its minimum value at $\beta_r = 1$ by (6). From (9), $r \geq 2$ and $n \geq \frac{(a+b)((2r-3)a+b+r-2)+bk+2}{b}$, we get

$$
\begin{aligned}
0 &\geq \varphi(\beta_r) \geq \varphi(1) = (a + b - 1)\Big(\frac{an - ak + 2}{a + b} - 1\Big) - (r - 1) - (a - 1)(n - k) - 1 \\
&= \frac{bn - bk - 2}{a + b} - a - b - r + 3 \\
&\geq (2r - 3)a + b + r - 2 - a - b - r + 3 \\
&= 2(r - 2)a + 1 \geq 1,
\end{aligned}
$$

which is a contradiction. Theorem 2 is verified. □

3. Remark

The condition (1) in Theorem 2 is best possible. It cannot be replaced by

$$\max\{d_G(w_1), d_G(w_2), \cdots, d_G(w_r)\} \geq \frac{an + bk + 2}{a + b} - 1,$$

which is claimed by constructing a graph $G = K_{amr+k} \vee (bmr + 1)K_1$, where m is a sufficiently large positive integer, and $k \geq 0$, $r \geq 2$, $a \geq 2$ and $b \geq \max\{3, a\}$ are integers. Then $n = |V(K_{amr+k})| + |V((bmr + 1)K_1)| = (a + b)mr + k + 1$ and

$$\frac{an + bk + 2}{a + b} > \max\{d_G(w_1), d_G(w_2), \cdots, d_G(w_r)\} = amr + k > \frac{an + bk + 2}{a + b} - 1$$

for any $\{w_1, w_2, \cdots, w_r\} \subseteq V((bmr + 1)K_1)$. Set $H = K_{amr} \vee (bmr + 1)K_1$. We easily see that H is not fractional $[a, b]$-covered since $b|V(K_{amr})| < a|V((bmr + 1)K_1)|$ holds. Note $H = G - Q$, where $Q \subseteq V(K_{amr+k})$ with $|Q| = k$. Hence, G is not fractional (a, b, k)-critical covered.

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