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Four double series involving $\zeta(3)$

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Abstract. Four double series involving $\zeta(3)$ are evaluated in closed form by calculating definite integrals. Three examples are also illustrated by the hypergeometric series approach.

1. Introduction and Outline

Let $\zeta(z)$ be the usual Riemann zeta function defined by

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \text{where} \quad \Re(z) > 1.$$

In a letter to Euler, Goldbach posed the problem to evaluate the double series

$$\zeta(\lambda,\mu) := \sum_{n=1}^{\infty} \frac{1}{n^\lambda} \sum_{k=1}^{n} \frac{1}{k^\mu}, \quad \text{where} \quad \lambda,\mu \in \mathbb{N} \quad \text{with} \quad \lambda > 1.$$

This led Euler to examine the nowadays so-called "multiple zeta functions" extensively. One of his beautiful formulae is recorded below

$$2\zeta(\lambda,1) = \lambda\zeta(\lambda+1) - \sum_{j=1}^{\lambda-2} \zeta(j+1)\zeta(\lambda-j), \quad \text{where} \quad \lambda > 1.$$

In particular for $\lambda = 2$, we get immediately

$$\zeta(2,1) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k} = \sum_{n\geq k} \frac{1}{n^2 k} = \zeta(3).$$

Recently, there have been growing interests (cf. [1, 2, 4, 5, 7–10] and [12–15, 17–20]) in finding closed form expressions and interrelations for the multiple Euler sums. Observe that the above series can be interpreted

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as the subseries of the divergent one "$\sum_{n,k}\frac{1}{n^2k}$" consisting of only the terms with indices $n > k$ (under the main diagonal). This suggests the author to recall the following well–known series

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} \quad \text{and} \quad \ln 2 = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}$$

and consider their tensor products

$$\Omega(1,-1) = \sum_{n,k=1}^{\infty}\frac{(-1)^k}{n^2k} = -\frac{\pi^2}{6}\ln 2,$$
$$\Omega(-1,-1) = \sum_{n,k=1}^{\infty}\frac{(-1)^{n+k}}{n^2k} = \frac{\pi^2}{12}\ln 2.$$

Define, in general, the bivariate series by

$$\Omega(x,y) = \sum_{n,k=1}^{\infty}\frac{x^n y^k}{n^2k}. \tag{1}$$

Then there are four subseries divided by the main diagonal "$n = k$":

$$\Omega_>(x,y) = \sum_{n>k}\frac{x^n y^k}{n^2k}, \quad \Omega_\geq(x,y) = \sum_{n\geq k}\frac{x^n y^k}{n^2k};$$
$$\Omega_<(x,y) = \sum_{n<k}\frac{x^n y^k}{n^2k}, \quad \Omega_\leq(x,y) = \sum_{n\leq k}\frac{x^n y^k}{n^2k}.$$

The aim of this short article is to focus entirely on the subseries of $\Omega(\pm1,\pm1)$ and to evaluate them in closed form. Four remarkable formulae are highlighted, in anticipation, as follows:

$$\Omega_>(1,1) = \sum_{n>k}\frac{1}{n^2k} = \zeta(2,1) = \zeta(3),$$
$$\Omega_>(1,-1) = \sum_{n>k}\frac{(-1)^k}{n^2k} = \zeta(3) - \frac{\pi^2}{4}\ln 2,$$
$$\Omega_>(-1,1) = \sum_{n>k}\frac{(-1)^n}{n^2k} = \frac{\zeta(3)}{8},$$
$$\Omega_>(-1,-1) = \sum_{n>k}\frac{(-1)^{n+k}}{n^2k} = \frac{\pi^2}{4}\ln 2 - \frac{13}{8}\zeta(3);$$

where the first one is well–known, while the other three values also involve $\zeta(3)$.

These values will be determined in the next section by calculating definite integrals in conjunction with power series expansions. Some of them will alternatively be illustrated in Section 3 by the hypergeometric series approach.

2. Integration Method

For $|x| \le 1$, write the sum in terms of a definite integral

$$\begin{aligned}\sum_{n=k+1}^{\infty} \frac{x^n}{n^2} &= \sum_{n=k+1}^{\infty} \int_0^x \frac{d\tau}{\tau} \int_0^{\tau} T^{n-1} dT \\ &= \sum_{n=k+1}^{\infty} \int_0^x T^{n-1} dT \int_T^x \frac{d\tau}{\tau} \\ &= \int_0^x \Big\{ \sum_{n=k+1}^{\infty} T^{n-1} \Big\} \ln(x/T) dT \\ &= \int_0^x \frac{T^k \ln(x/T)}{1-T} dT.\end{aligned}$$

By substitution, we can further reformulate the double series

$$\begin{aligned}\Omega_{>}(x,y) = \sum_{n>k} \frac{x^n y^k}{n^2 k} &= \sum_{k=1}^{\infty} \frac{y^k}{k} \int_0^x \frac{T^k \ln(x/T)}{1-T} dT \\ &= \int_0^x \frac{\ln(x/T)}{1-T} \Big\{ \sum_{k=1}^{\infty} \frac{(Ty)^k}{k} \Big\} dT\end{aligned}$$

and

$$\begin{aligned}\Omega_{\geq}(x,y) = \sum_{n \geq k} \frac{x^n y^k}{n^2 k} &= \sum_{k=1}^{\infty} \frac{y^k}{k} \int_0^x \frac{T^{k-1} \ln(x/T)}{1-T} dT \\ &= \int_0^x \frac{\ln(x/T)}{T(1-T)} \Big\{ \sum_{k=1}^{\infty} \frac{(Ty)^k}{k} \Big\} dT,\end{aligned}$$

which yield the following definite integral expressions

$$\Omega_{>}(x,y) = \int_0^x \frac{\ln(T/x)\ln(1-Ty)}{1-T} dT, \tag{2}$$

$$\Omega_{\geq}(x,y) = \int_0^x \frac{\ln(T/x)\ln(1-Ty)}{T(1-T)} dT. \tag{3}$$

Now we are in position to determine the values of the double series by computing the corresponding integrals for specific "$x,\ y = \pm 1$".

2.1. $\mathbf{\Omega_{>}(1,1)}$

There are different proofs (see [5, 6] for example) for the value of $\Omega_{>}(1,1)$. For completeness, we show it by making use of two integrals:

$$\int_0^1 T^{n-1} \ln(T) dT = \frac{-1}{n^2} \quad \text{and} \quad \int_0^1 T^{n-1} \ln^2(T) dT = \frac{2}{n^3}.$$

In fact, by integration by parts, it is almost routine check that

$$\begin{aligned}\Omega_>(1,1) &= \int_0^1 \frac{\ln(T)\ln(1-T)}{T}dT\\ &= -\int_0^1 \ln(T)\Big\{\sum_{n=1}^\infty \frac{T^{n-1}}{n}\Big\}dT\\ &= -\sum_{n=1}^\infty \frac{1}{n}\int_0^1 T^{n-1}\ln(T)dT\\ &= \sum_{n=1}^\infty \frac{1}{n^3} = \zeta(3).\end{aligned}$$

The same value can alternatively be obtained as follows:

$$\begin{aligned}\Omega_>(1,1) &= \int_0^1 \frac{\ln(T)\ln(1-T)}{T}dT\\ &= \frac{\ln^2(T)\ln(1-T)}{2}\Big|_0^1 + \int_0^1 \frac{\ln^2(T)}{2(1-T)}dT\\ &= \int_0^1 \frac{\ln^2(T)}{2}\Big\{\sum_{n=1}^\infty T^{n-1}\Big\}dT\\ &= \frac{1}{2}\sum_{n=1}^\infty \int_0^1 T^{n-1}\ln^2(T)dT\\ &= \sum_{n=1}^\infty \frac{1}{n^3} = \zeta(3).\end{aligned}$$

□

2.2. $\Omega_>(1,-1)$

Expanding $(1-T)^{-1}$ into the geometric series

$$\frac{1}{1-T} = \sum_{n=1}^\infty T^{n-1} \quad \text{and} \quad \int_0^T T^{n-1}\ln(T) = \frac{T^n}{n}\ln T - \frac{T^n}{n^2}$$

we can proceed by making use of integration by parts

$$\begin{aligned}\int_0^1 T^{n-1}\ln(T)\ln(1+T)dT &= \Big\{\frac{T^n}{n}\ln(T) - \frac{T^n}{n^2}\Big\}\ln(1+T)\Big|_0^1\\ &\quad - \int_0^1 \Big\{\frac{T^n}{n(1+T)}\ln(T) - \frac{T^n}{n^2(1+T)}\Big\}dT\\ &= \frac{-\ln 2}{n^2} - \int_0^1 \Big\{\frac{T^n\ln(T)}{n(1+T)} - \frac{T^n}{n^2(1+T)}\Big\}dT.\end{aligned}$$

This leads us to the expression

$$\begin{aligned}\Omega_>(1,-1) &= \int_0^1 \frac{\ln(T)\ln(1+T)}{1-T}dT\\ &= \sum_{n=1}^\infty \int_0^1 \frac{T^n}{n^2(1+T)}dT - \sum_{n=1}^\infty \int_0^1 \frac{T^n\ln(T)}{n(1+T)}dT - \frac{\pi^2}{6}\ln 2.\end{aligned}$$

Evaluating further the integrals

$$\int_0^1 \frac{T^n}{1+T}dT = \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 T^{n+k-1}dT = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k} \tag{4}$$

and

$$\int_0^1 \frac{T^n\ln(T)}{1+T}dT = \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 T^{n+k-1}\ln(T)dT = \sum_{k=1}^{\infty}\frac{(-1)^k}{(n+k)^2}; \tag{5}$$

then making substitution, we can simplify the expression

$$\begin{aligned}\Omega_>(1,-1) &= \int_0^1 \frac{\ln(T)\ln(1+T)}{1-T}dT\\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k} - \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\frac{(-1)^k}{(n+k)^2} - \frac{\pi^2}{6}\ln 2\\ &= -\sum_{n<m}\frac{(-1)^{m+n}}{n^2 m} - \sum_{m>n}\frac{(-1)^{m+n}}{m^2 n} - \frac{\pi^2}{6}\ln 2 \qquad \boxed{m=n+k}\\ &= -\Omega_<(-1,-1) - \Omega_>(-1,-1) - \frac{\pi^2}{6}\ln 2\\ &= \zeta(3) - \frac{\pi^2}{4}\ln 2,\end{aligned}$$

where the last passage is justified by

$$\Omega(-1,-1) = \frac{\pi^2}{12}\ln 2 = \zeta(3) + \Omega_<(-1,-1) + \Omega_>(-1,-1).$$

2.3. $\Omega_>(-1,1)$

Analogously, from the integral expression

$$\begin{aligned}\Omega_>(-1,1) &= \int_0^{-1} \frac{\ln(-T)\ln(1-T)}{1-T}dT\\ &= -\int_0^1 \frac{\ln(T)\ln(1+T)}{1+T}dT \qquad \boxed{T\to -T}\\ &= \int_0^1 \frac{\ln^2(1+T)}{2T}dT,\end{aligned}$$

we can manipulate further the series

$$\begin{aligned}\Omega_>(-1,1) &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n}\int_0^1 T^{n-1}\ln(1+T)dT\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n}\left\{\frac{T^n}{n}\ln(1+T)\Big|_0^1 - \int_0^1 \frac{T^n}{n(1+T)}dT\right\}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2}\int_0^1 \frac{T^n}{1+T}dT - \sum_{n=1}^{\infty}\frac{(-1)^n\ln 2}{2n^2}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2}\int_0^1 \frac{T^n}{1+T}dT + \frac{\pi^2}{24}\ln 2.\end{aligned}$$

By invoking (4), the above sum can further be reduced to

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2}\int_0^1\frac{T^n}{1+T}dT &= \sum_{n,k=1}^{\infty}\frac{(-1)^{n+k-1}}{2n^2(n+k)} = \frac{-1}{2}\sum_{n<m}\frac{(-1)^m}{mn^2} \qquad \boxed{m=n+k}\\ &= \frac{-1}{2}\Omega_<(1,-1) = \frac{-1}{2}\Big\{\Omega(1,-1)+\frac{3}{4}\zeta(3)-\Omega_>(1,-1)\Big\}\\ &= \frac{-1}{2}\Big\{-\frac{\pi^2}{6}\ln 2+\frac{3}{4}\zeta(3)-\zeta(3)+\frac{\pi^2}{4}\ln 2\Big\} = \frac{\zeta(3)}{8}-\frac{\pi^2}{24}\ln 2.\end{aligned}$$

Consequently we arrive at the closed formula

$$\Omega_>(-1,1) = \int_0^1\frac{\ln^2(1+T)}{2T}dT = \frac{\zeta(3)}{8}.$$

Alternatively, for $0 < T < 1$, if making use of the geometric series

$$\frac{1}{T} = \frac{1}{(1+T)-1} = \sum_{k=0}^{\infty}\frac{1}{(1+T)^{k+1}},$$

and then evaluating the integrals

$$\begin{aligned}&\int_0^1\frac{\ln^2(1+T)}{1+T}dT = \frac{\ln^3(2)}{3},\\ &\int_0^1\frac{\ln^2(1+T)}{(1+T)^{k+1}}dT = \frac{2}{k^3}-\frac{2}{k^3\cdot 2^k}-\frac{2\ln 2}{k^2\cdot 2^k}-\frac{\ln^2(2)}{k\cdot 2^k},\end{aligned}$$

we can derive the following expression

$$\begin{aligned}\Omega_>(-1,1) &= \int_0^1\frac{\ln^2(1+T)}{2T}dT\\ &= \frac{\ln^3(2)}{6}+\frac{1}{2}\sum_{k=1}^{\infty}\Big\{\frac{2}{k^3}-\frac{2}{k^3\cdot 2^k}-\frac{2\ln 2}{k^2\cdot 2^k}-\frac{\ln^2(2)}{k\cdot 2^k}\Big\}\\ &= \zeta(3)-\mathrm{Li}_3(\tfrac{1}{2})-\ln 2\,\mathrm{Li}_2(\tfrac{1}{2})-\frac{\ln^3(2)}{3},\end{aligned}$$

where the polylogarithm function is defined by the power series

$$\mathrm{Li}_n(y) = \sum_{k=1}^{\infty}\frac{y^k}{k^n}.$$

Thanks to the two known equations

$$\begin{aligned}\mathrm{Li}_2(\tfrac{1}{2}) &= \frac{\pi^2}{12}-\frac{\ln^2(2)}{2},\\ \mathrm{Li}_3(\tfrac{1}{2}) &= \frac{\ln^3(2)}{6}-\frac{\pi^2}{12}\ln 2+\frac{21}{24}\zeta(3);\end{aligned}$$

we confirm again

$$\Omega_>(-1,1) = \int_0^1\frac{\ln^2(1+T)}{2T}dT = \frac{\zeta(3)}{8}$$

2.4. $\Omega_>(-1,-1)$

Finally, we turn to evaluate the integral

$$\Omega_>(-1,-1) = \int_0^{-1} \frac{\ln(-T)\ln(1+T)}{1-T} dT$$
$$= -\int_0^1 \frac{\ln(T)\ln(1-T)}{1+T} dT \qquad \boxed{T \to -T}.$$

By means of integration by parts, we have

$$\Omega_>(-1,-1) = -\int_0^1 \frac{\ln(T)\ln(1-T)}{1+T} dT = \int_0^1 \frac{\ln(1+T)\ln(1-T)}{T} dT$$
$$-\ln(T)\ln(1+T)\ln(1-T)\Big|_0^1 - \int_0^1 \frac{\ln(T)\ln(1+T)}{1-T} dT$$
$$= \int_0^1 \frac{\ln(1+T)\ln(1-T)}{T} dT - \Omega_>(1,-1).$$

Denote by H_n the harmonic number

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for} \quad n \in \mathbb{N}.$$

According to the equality

$$\int_0^1 T^{n-1} \ln(1-T) dT = -\frac{H_n}{n},$$

we can evaluate the integral

$$\int_0^1 \frac{\ln(1+T)\ln(1-T)}{T} dT = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 T^{n-1} \ln(1-T) dT$$
$$= \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^2} = \Omega_\geq(-1,1)$$
$$= \Omega_>(-1,1) - \frac{3}{4}\zeta(3).$$

Therefore, we find that

$$\Omega_>(-1,-1) = \Omega_>(-1,1) - \frac{3}{4}\zeta(3) - \Omega_>(1,-1)$$
$$= \frac{\zeta(3)}{8} - \frac{3}{4}\zeta(3) - \zeta(3) + \frac{\pi^2}{4}\ln 2$$
$$= \frac{\pi^2}{4}\ln 2 - \frac{13}{8}\zeta(3).$$

From the four summation formulae established for $\Omega_>(\pm1,\pm1)$ in this section, we can deduce other double series $\Omega(\pm1,\pm1)$ labeled by "$<$, $\leq$, $\geq$". For example, among the four series $\Omega_<(\pm1,\pm1)$, two series "$\Omega_<(1,1)$ and $\Omega_<(-1,1)$" are divergent, while two convergent ones are evaluated by

$$\Omega_<(1,-1) = \frac{\pi^2}{12}\ln 2 - \frac{\zeta(3)}{4},$$
$$\Omega_<(-1,-1) = \frac{5}{8}\zeta(3) - \frac{\pi^2}{6}\ln 2.$$

3. Hypergeometric Series Approach

For an indeterminate α and a nonnegative integer n, define the shifted factorial by $(\alpha)_0 \equiv 1$ and

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

Then the classical hypergeometric series (cf. Bailey [3]) reads as

$${}_pH_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}.$$

There exist numerous hypergeometric series identities in the literature. Some of them have been shown powerful to prove summation formulae involving harmonic numbers (see [6, 11]). The strategy consists of two steps. The first one is to extract the initial coefficient of x from hypergeometric terms. Let $[x^m]\phi(x)$ stand for the coefficient of x^m in the formal power series $\phi(x)$. Then it is trivial to check the following relations:

$$[x]\frac{(1+x)_n}{n!} = H_n \quad \text{and} \quad [x]\frac{n!}{(1-x)_n} = H_n.$$

Another step is to do the same from the Γ-function quotient. Recalling, for the Γ-function (cf. [16, §11]), the Weierstrass product

$$\Gamma(z) = z^{-1}\prod_{n=1}^{\infty}\{(1+1/n)^z/(1+z/n)\}$$

and the logarithm–differentiation

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty}\frac{z-1}{(n+1)(n+z)}$$

with the Euler constant

$$\gamma = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{1}{k} - \ln n\right\},$$

we can derive the following expansions (cf. [6])

$$\Gamma(1-z) = \exp\left\{\sum_{k=1}^{\infty}\frac{\sigma_k}{k}z^k\right\},$$
$$\Gamma\Big(\frac{1}{2}-z\Big) = \sqrt{\pi}\exp\left\{\sum_{k=1}^{\infty}\frac{\tau_k}{k}z^k\right\},$$

where the Riemann Zeta sequences $\{\sigma_k, \tau_k\}$ are defined by

$$\sigma_1 = \gamma, \qquad \sigma_m = \zeta(m), \; m = 2, 3, \cdots$$
$$\tau_1 = \gamma + 2\ln 2, \qquad \tau_m = (2^m - 1)\zeta(m), \; m = 2, 3, \cdots.$$

Now we are going to illustrate the hypergeometric approach through three examples.

3.1. $\mathbf{\Omega_{>}(1,1)}$

Recall the Gauss summation theorem (cf. Bailey [3, §1.3])

$$ {}_2F_1\left[\begin{matrix} x, & x \\ & 1 \end{matrix}\middle| 1\right] = \frac{\Gamma(1-2x)}{\Gamma^2(1-x)}. $$

Then we can express $\Omega_{>}(1,1)$ in terms of the coefficient

$$ \begin{aligned} \Omega_{>}(1,1) &= \sum_{n>k} \frac{1}{n^2 k} = \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} \\ &= \frac{1}{2}[x^3]\, {}_2F_1\left[\begin{matrix} x, & x \\ & 1 \end{matrix}\middle| 1\right] \\ &= \frac{1}{2}[x^3]\, \frac{\Gamma(1-2x)}{\Gamma^2(1-x)} = \zeta(3). \end{aligned} $$

3.2. $\mathbf{\Omega_{>}(-1,1)}$

In view of the Kummer summation theorem (cf. Bailey [3, §2.3])

$$ {}_2F_1\left[\begin{matrix} x, & x \\ & 1 \end{matrix}\middle| -1\right] = \frac{\Gamma(1+\frac{x}{2})}{\Gamma(1+x)\Gamma(1-\frac{x}{2})}, $$

we can express $\Omega_{>}(-1,1)$ in terms of the coefficient

$$ \begin{aligned} \Omega_{>}(-1,1) &= \sum_{n>k} \frac{(-1)^n}{n^2 k} = \sum_{n=1}^{\infty} (-1)^n \frac{H_{n-1}}{n^2} \\ &= \frac{1}{2}[x^3]\, {}_2F_1\left[\begin{matrix} x, & x \\ & 1 \end{matrix}\middle| -1\right] \\ &= \frac{1}{2}[x^3]\, \frac{\Gamma(1+\frac{x}{2})}{\Gamma(1+x)\Gamma(1-\frac{x}{2})} = \frac{\zeta(3)}{8}. \end{aligned} $$

However, we fail to rederive the formulae for both $\Omega_{>}(1,-1)$ and $\Omega_{>}(-1,-1)$. Instead, we succeed in proving an extra identity in the next subsection.

3.3. $\mathbf{\Omega_{\geq}(\frac{1}{2},1)}$

Recall Bailey's summation theorem (cf. Bailey [3, §2.4])

$$ \mathcal{B}(x,y) = {}_2F_1\left[\begin{matrix} x, & 1-x \\ & 1+y \end{matrix}\middle| \frac{1}{2}\right] = \frac{\Gamma(\frac{1+y}{2})\Gamma(\frac{2+y}{2})}{\Gamma(\frac{1+x+y}{2})\Gamma(\frac{2-x+y}{2})}. $$

According to the linear relation "$2x = (k+x)-(k-x)$", the contiguous series can be written in terms of $\mathcal{B}$-series

$$ {}_2F_1\left[\begin{matrix} x, -x \\ 1+y \end{matrix}\middle| \frac{1}{2}\right] = \frac{1}{2}\mathcal{B}(x,y) + \frac{1}{2}\mathcal{B}(-x,y). $$

Then we can evaluate the series

$$ \begin{aligned} \sum_{n=1}^{\infty} \frac{H_n}{n^2 \cdot 2^n} &= [x^2 y]\, {}_2F_1\left[\begin{matrix} x, -x \\ 1+y \end{matrix}\middle| \frac{1}{2}\right] \\ &= [x^2 y]\left\{\frac{\mathcal{B}(x,y)+\mathcal{B}(-x,y)}{2}\right\} \\ &= [x^2 y]\mathcal{B}(x,y) = \zeta(3) - \frac{\pi^2}{12}\ln 2. \end{aligned} $$

This is equivalent to the following interesting identity

$$\Omega_{\geq}(\tfrac{1}{2},1)=\zeta(3)-\frac{\pi^2}{12}\ln 2.$$

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