Filomat 37:10 (2023), 3291–3303
https://doi.org/10.2298/FIL2310291H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hybrid fractional differential equation with nonlocal and impulsive conditions

Khalid Hilal[a], Ahmed Kajouni[a], Samira Zerbib[a,*]

[a]*Department of Applied Mathematics, Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco*

Abstract. In this paper, we have studied the existence of solutions of the following nonlinear ψ-Hilfer hybrid fractional differential equation with non-local and impulsive conditions (non-local impulsive ψ-HHFDE)

$$\begin{cases} {}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=h(t,u(t)), \quad t\in J=[0,b]\setminus\{t_1,t_2,...,t_n\} \\ \Delta I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_k)}{f(t_k,u(t_k))}+g(t_k,u(t_k))\right]=\Gamma_k\in\mathbb{R}, \quad k\in\{1,2,...,n\} \\ I_{0_+}^{1-\zeta;\psi}\left(\frac{u(t)}{f(t,u(t))}\right)_{t=0}+\chi(u)=\mu\in\mathbb{R} \end{cases}$$

Where $0<\alpha<1$, $0<\sigma<1$, $\zeta=\alpha+\sigma(1-\alpha)$, $f\in C(J\times\mathbb{R},\mathbb{R}^*)$, $\chi\in C(\mathbb{R},\mathbb{R})$ and $g,h\in C(J\times\mathbb{R},\mathbb{R})$. The used tools in this article are the classical technique of Dhage fixed point theorem. Further, an example is provided to illustrate our results.

1. Introduction

The purpose of fractional calculus is to generalize traditional derivatives to non-integer orders. As is well known, many dynamical systems are best characterized by a fractional order dynamical model, usually based on the notion of differentiation or integration of the non-integer order. The study of fractional order systems is more delicate than for their integer order counterparts. Indeed, fractional systems are, on the one hand, considered as memory systems, especially for the consideration of initial conditions and on the other hand they present a much more complex dynamics.
Hybrid equation theory is very much useful in the study of nonlinear dynamical systems that are not easily solvable or analyzed. The non-linearity of such a dynamical system is not smooth for studying the existence or some other characterization of the solutions, however perturbing such a problem in some way allows the problem to be studied with available methods for different aspects of the solutions. The nonlinear dynamical systems perturbed in this way are called hybrid differential equations. They are

2020 *Mathematics Subject Classification*. Primary 26A33; Secondary 34G20.
Keywords. Hybrid fractional differential equation; ψ-Hilfer derivative; non-local conditions; impulsive conditions; Existence of solution; nonlinear fractional differential equations.
Received: 23 March 2022; Revised: 08 January 2023; Accepted: 11 January 2023
Communicated by Maria Alessandra Ragusa
* Corresponding author: Samira Zerbib
Email addresses: hilalkhalid2005@yahoo.fr (Khalid Hilal), kajjouni@gmail.com (Ahmed Kajouni), zerbib.sam123@gmail.com (Samira Zerbib)

results of perturbation techniques as explained in [1]. There have been many works on the theory of hybrid differential equations, and we refer the readers to the literatures(see [2–6]).
Hilal and Kajouni [5] studied boundary fractional hybrid differential equations involving Caputo differential operators of order $0 \leq \alpha \leq 1$

$$\begin{cases} {}^{c}D^{\alpha}_{0_+}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in [0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c \end{cases} \tag{1}$$

Where $f \in C^1([0,T] \times \mathbb{R}, \mathbb{R}^*)$ and $g \in Car([0,T] \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$. They proved the existence result for boundary fractional hybrid differential equations under mixed Lipschitz and Caratheodory conditions.
The main motivation for this work comes from the work [8], the authors of this work studied the following impulsive ψ-Hilfer fractional differential equation with initial condition

$$\begin{cases} {}^{H}D^{\alpha,\sigma,\psi}_{a_+}u(t) = f(t,u(t)), \quad t \in [a,T] \setminus \{t_1,t_2,...,t_n\} \\ \Delta I^{1-\zeta;\psi}_{a_+}u(t_k) = \Gamma_k \in \mathbb{R}, \quad k \in \{1,2,...,n\} \\ I^{1-\zeta;\psi}_{a_+}u(a) = \delta \in \mathbb{R} \end{cases}$$

Where $0 < \alpha < 1$, $0 < \sigma < 1$, $\zeta = \alpha + \sigma(1-\alpha)$, $a = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = T$, $\Delta I^{1-\zeta;\psi}_{0_+}u(t_k) = I^{1-\zeta;\psi}_{0_+}u(t_k^+) - I^{1-\zeta;\psi}_{0_+}u(t_k^-)$ such that $I^{1-\zeta;\psi}_{0_+}u(t_k^+) = \lim_{\varepsilon\to 0^+} I^{1-\zeta;\psi}_{0_+}u(t_k+\varepsilon)$ and $I^{1-\zeta;\psi}_{0_+}u(t_k^-) = \lim_{\varepsilon\to 0^-} I^{1-\zeta;\psi}_{0_+}u(t_k+\varepsilon)$.
In this present paper, we investigate the existence on solutions for the following impulsive ψ-Hilfer nonlinear hybrid fractional differential equation (ψ-HHFDE) with non-local initial conditions given by

$$\begin{cases} {}^{H}D^{\alpha,\sigma,\psi}_{0_+}\left(\frac{u(t)}{f(t,u(t))} + g(t,u(t))\right) = h(t,u(t)), \quad t \in J = [0,b] \setminus \{t_1,t_2,...,t_n\} \\ \Delta I^{1-\zeta;\psi}_{0_+}\left[\frac{u(t_k)}{f(t_k,u(t_k))} + g(t_k,u(t_k))\right] = \Gamma_k \in \mathbb{R}, \quad k \in \{1,2,...,n\} \\ I^{1-\zeta;\psi}_{0_+}\left(\frac{u(t)}{f(t,u(t))}\right)_{t=0} + \chi(u) = \mu \in \mathbb{R} \end{cases} \tag{2}$$

Where $0 < \alpha < 1$, $0 < \sigma < 1$, $\zeta = \alpha + \sigma(1-\alpha)$, ${}^{H}D^{\alpha,\sigma,\psi}_{0_+}(.)$ is the ψ-Hilfer fractional derivative of order α and type σ, $f \in C(J \times \mathbb{R}, \mathbb{R}^*)$, $\chi \in C(\mathbb{R},\mathbb{R})$ and $g, h \in C(J \times \mathbb{R}, \mathbb{R})$, $I^{1-\zeta;\psi}_{0_+}$ is left sided ψ-RL fractional integral operator, $0 = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = b$, $\Delta I^{1-\zeta;\psi}_{0_+}\beta(t_k,u(t_k)) = I^{1-\zeta;\psi}_{0_+}\beta(t_k^+,u(t_k^+)) - I^{1-\zeta;\psi}_{0_+}\beta(t_k^-,u(t_k^-))$, $I^{1-\zeta;\psi}_{0_+}\beta(t_k^+,u(t_k^+)) = \lim_{\varepsilon\to 0^+} I^{1-\zeta;\psi}_{0_+}\beta(t_k+\varepsilon,u(t_k+\varepsilon))$ and $I^{1-\zeta;\psi}_{0_+}\beta(t_k^-,u(t_k^-)) = \lim_{\varepsilon\to 0^-} I^{1-\zeta;\psi}_{0_+}\beta(t_k+\varepsilon,u(t_k+\varepsilon))$. Where $\beta(t_k,u(t_k)) = \left[\frac{u(t_k)}{f(t_k,u(t_k))} + g(t_k,u(t_k))\right]$
The plan of this paper is as follows : In Section 2, we present some preliminary results from fractional calculus, which will be employed throughout this paper. In section 3 we present a representation formula for the solution and we investigate the existence of the non-local impulsive ψ-HHFDE (2). As an application of our main results, illustrative example is given in the last section.

2. Preliminaries

Let $J = [0,b]$ be a finite interval of the real line $\mathbb{R}$. $C(J,\mathbb{R})$ be the Banach space of continuous real function h with the norm $\|h\| = max\{|h(t)| : t \in J\}$. $C^n(J,\mathbb{R})$ be the Banach space of n-times continuously differentiable functions on J.

Let $[a,b]$ with $(0<a<b<\infty)$ be a finite interval and $\psi \in C^1([a,b])$ be increasing function such that $\psi' \neq 0$, $\forall t \in [a,b]$, we consider the weighted space

$$C_{1-\zeta,\psi}([a,b]) = \left\{u : (a,b] \to \mathbb{R}, (\psi(t)-\psi(a))^{1-\zeta}u(t) \in C^1[a,b]\right\}.$$

Define the weighted space of piece-wise continuous functions as

$$PC_{1-\zeta,\psi}([a,b],\mathbb{R}) = \{u : (a,b] \to \mathbb{R}, u \in C_{1-\zeta,\psi}((t_k,t_{k+1}]), I_{0_+}^{1-\zeta;\psi}u(t_k^+), I_{0_+}^{1-\zeta;\psi}u(t_k^-) \quad exists \quad and$$
$$I_{0_+}^{1-\zeta;\psi}u(t_k^-) = I_{0_+}^{1-\zeta;\psi}u(t_k) \quad for \quad k=1,2,...,n\}.$$

Clearly, $PC_{1-\zeta,\psi}([a,b],\mathbb{R})$ is a Banach space with the norm

$$\|u(t)\|_{C_{1-\zeta,\psi}([a,b])} = max_{t\in[a,b]}|(\psi(t)-\psi(a))^{1-\zeta}u(t)|$$

Let us recall some definitions and properties of fractional calculus.

Definition 2.1. *[1] The left-sided ψ-Riemann-Liouville fractional integral and fractional derivative of order α, $(n-1<\alpha<n)$ for an integrable function $\Phi : [0,b] \to \mathbb{R}$ with respect to another function $\psi : [0,b] \to \mathbb{R}$, that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [0,b]$, $(b \leq +\infty)$, are respectively defined as follows*

$$I_{0_+}^{\alpha;\psi}\Phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\Phi(s)ds,$$

and

$$D_{0_+}^{\alpha;\psi}\Phi(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{0_+}^{n-\alpha;\psi}\Phi(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}\Phi(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt, z>0$$

Definition 2.2. *[7] Let $(n-1<\alpha<n)$, $n \in \mathbb{N}$, with $\psi \in C^n([0,b],\mathbb{R})$ a function such that $\psi(t)$ is increasing and $\psi'(t) \neq 0$ for all $t \in [0,b]$.*
The ψ-Hilfer fractional derivative (left-sided) of function $\Phi \in C^n([0,b],\mathbb{R})$ of order α and type $\sigma \in [0,1]$ is determined as

$${}^{H}D_{0_+}^{\alpha,\sigma,\psi}\Phi(t) = I_{0_+}^{\sigma(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{0_+}^{(1-\sigma)(n-\alpha);\psi}\Phi(t).$$

In other way

$${}^{H}D_{0_+}^{\alpha,\sigma,\psi}\Phi(t) = I_{0_+}^{\sigma(n-\alpha);\psi}D_{0_+}^{\zeta,\psi}\Phi(t),$$

where

$$D_{0_+}^{\zeta,\psi}\Phi(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{(n)} I_{0_+}^{(1-\sigma)(n-\alpha);\psi}\Phi(t),$$

with $\zeta = \alpha + \sigma(n-\alpha)$

In particular, the ψ-Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\sigma \in [0,1]$, can be written in the following form

$${}^{H}D_{0_+}^{\alpha;\sigma;\psi}\Phi(t) = \frac{1}{\Gamma(\zeta-\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\zeta-\alpha-1}D_{0_+}^{\zeta,\psi}\Phi(s)ds,$$

where
$\zeta = \alpha + \sigma(1-\alpha)$, and $D_{0_+}^{\zeta,\psi}\Phi(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{(1)} I_{0_+}^{1-\zeta;\psi}\Phi(t)$.

Lemma 2.3. *[1, 9]. Let $\alpha, \beta > 0$. Then we have*
i) $I_{0_+}^{\alpha;\psi}(\psi(t)-\psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(0))^{\alpha+\beta-1}$
ii) ${}^{H}D_{0_+}^{\alpha;\sigma;\psi}(\psi(t)-\psi(0))^{\zeta-1} = 0$

Lemma 2.4. *[11]. Let $\alpha > 0$ and $\beta > 0$. Then the relation*

$$I_{0_+}^{\alpha;\psi} I_{0_+}^{\beta;\psi} h(t) = I_{0_+}^{\alpha+\beta;\psi} h(t)$$

holds almost every where for $t \in J$, for $h \in L^p(J, \mathbb{R})$ and $p \geq 1$. If $\alpha + \beta > 1$, then the relation holds at any point of J.

Lemma 2.5. *[1, 10] Let $\alpha > 0$ and $h \in C([a, b])$, then $I_{a_+}^{\alpha;\psi} h(t) \in C([a, b])$ and*
i) $I_{a_+}^{\alpha;\psi}(.)$ *maps $C([a, b], \mathbb{R})$ into $C([a, b], \mathbb{R})$*
ii) $\lim_{t\to a^+} I_{a_+}^{\alpha;\psi} h(t) = I_{a_+}^{\alpha;\psi} h(a) = 0$

Lemma 2.6. *[7] Let $h \in C^n[a, b]$, $n - 1 < \alpha < n$, $0 \leq \sigma \leq 1$, and $\zeta = \alpha + \sigma - \sigma\alpha$. Then for all $t \in (a, b]$*

$${}^{H}D_{a_+}^{\alpha;\sigma;\psi} I_{a_+}^{\alpha;\psi} h(t) = h(t),$$

and

$$I_{a_+}^{\alpha;\psi H} D_{a_+}^{\alpha;\sigma;\psi} h(t) = h(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{(\zeta-k)}}{\Gamma(\varsigma - k + 1)} h_{\psi}^{[n-k]} I_{a_+}^{(n-\sigma)(n-\alpha);\psi} h(a), \quad \text{where} \quad h_{\psi}^{[n-k]} h(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^{n-k} h(t).$$

In particular, if $0 < \alpha < 1$, we have

$$I_{a_+}^{\alpha;\psi H} D_{a_+}^{\alpha;\sigma;\psi} h(t) = h(t) - \Omega_{\psi}^{\zeta}(t, a) I_{a_+}^{(1-\sigma)(1-\alpha);\psi} h(a),$$

where

$$\Omega_{\psi}^{\zeta}(t, a) = \frac{(\psi(t)-\psi(a))^{\zeta-1}}{\Gamma(\varsigma)}.$$

Theorem 2.7. *[2] Let S be a closed, convex and bounded subset of the Banach algebra X. Suppose that $A : X \to X$ and $B : S \to X$ are two operators such that*

a) A is Lipschitzian with a Lipschitz constant δ.

b) B is completely continuous.

c) $u = AuBv \Rightarrow u \in S$, for all $v \in S$.

d) $\delta M < 1$, Where $M = \|B(S)\|$
Then, the operator equation $u = AuBu$ has a solution in S.

3. Existence Results

In this partition, we prove the existence of the solution of the given problem (2). We first present the following important result through which we can prove our major results.

Representation formula for the solution:

We will give a lemma which plays an important role for the construction of an equivalent fractional integral equation of the non-local impulsive ψ-HHFDE (2).
We consider the following problem

$$\begin{cases} {}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\varphi(t), \quad t\in J=[0,b]\setminus\{t_1,t_2,...,t_n\} \\ \Delta I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_k)}{f(t_k,u(t_k))}+g(t_k,u(t_k))\right]=\Gamma_k\in\mathbb{R}, \quad k\in\{1,2,...,n\} \\ I_{0_+}^{1-\zeta;\psi}\left(\frac{u(t)}{f(t,u(t))}\right)_{t=0}+\chi(u)=\mu\in\mathbb{R} \end{cases} \tag{3}$$

Lemma 3.1. *Let $f\in C(J\times\mathbb{R},\mathbb{R}^*)$, $g\in C(J\times\mathbb{R},\mathbb{R})$, $u\in C_{1-\zeta,\psi}(J,\mathbb{R})$ and $\varphi: J\longrightarrow\mathbb{R}$ be continuous. Then for any $\tau\in J$ a function $u: J\longrightarrow\mathbb{R}$ defined by*

$$u(t)=f(t,u(t))\left\{-g(t,u(t))+\Omega_{\psi}^{\zeta}(t,0)\{I_{0_+}^{1-\zeta;\psi}(\frac{u(\tau)}{f(\tau,u(\tau))}+g(\tau,u(\tau)))-I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=\tau}\}+I_{0_+}^{\alpha;\psi}\varphi(t)\right\}, \tag{4}$$

is the solution of the problem

$${}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\varphi(t), \quad t\in J=[0,b]$$

Proof. From (4) we have

$$\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\Omega_{\psi}^{\zeta}(t,0)\{I_{0_+}^{1-\zeta;\psi}(\frac{u(\tau)}{f(\tau,u(\tau))}+g(\tau,u(\tau)))-I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=\tau}\}+I_{0_+}^{\alpha;\psi}\varphi(t).$$

Applying the ψ-Hilfer fractional derivative operator ${}^{H}D_{0_+}^{\alpha,\sigma,\psi}$ on both sides of the above equation, we get

$${}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\{I_{0_+}^{1-\zeta;\psi}\beta(\tau)-I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=\tau}\}{}^{H}D_{0_+}^{\alpha,\sigma,\psi}\Omega_{\psi}^{\zeta}(t,0)+{}^{H}D_{0_+}^{\alpha,\sigma,\psi}I_{0_+}^{\alpha;\psi}\varphi(t), \quad t\in J,$$

using the Lemma (2.3), (ii) and Lemma (2.6), we get

$${}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\varphi(t), \quad t\in J.$$

This completes the proof of the lemma. □

Lemma 3.2. *Let $f\in C(J\times\mathbb{R},\mathbb{R}^*)$, $g\in C(J\times\mathbb{R},\mathbb{R})$, $u\in PC_{1-\zeta,\psi}(J,\mathbb{R})$, and $\varphi\in C(J,\mathbb{R})$ with $J=[0,b]$. Then the problem* (3) *has a solution given by*

$$u(t)=\begin{cases} f(t,u(t))\left\{-g(t,u(t))+\Omega_{\psi}^{\zeta}(t,0)(\mu-\chi(u))+I_{0_+}^{\alpha;\psi}\varphi(t)\right\} \quad t\in[0,t_1] \\ f(t,u(t))\left\{-g(t,u(t))+\Omega_{\psi}^{\zeta}(t,0)(\mu-\chi(u)+\sum_{i=1}^{k}\Gamma_i)+I_{0_+}^{\alpha;\psi}\varphi(t)\right\} \quad t\in(t_k,t_{k+1}], k=1,2,...,n \end{cases} \tag{5}$$

Proof. Assume that $u\in PC_{1-\zeta,\psi}(J,\mathbb{R})$ satisfies the nonlocal impulsive ψ-HHFDE (3).
If $t\in[0,t_1]$ we have

$$\begin{cases} {}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\varphi(t) \\ I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t)}{f(t,u(t))}\right]_{t=0}+\chi(u)=\mu\in\mathbb{R} \end{cases} \tag{6}$$

Applying the ψ-RL fractional integral operator $I_{0_+}^{\alpha,\psi}$ on both sides of the problem (6) and using Lemma (2.5) we get

$$\frac{u(t)}{f(t,u(t))}+g(t,u(t))-\Omega_\psi^\zeta(t,0)I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right]_{t=0}=I_{0_+}^{\alpha;\psi}\varphi(t),$$

this implies that

$$u(t)=f(t,u(t))\left\{-g(t,u(t))+\Omega_\psi^\zeta(t,0)(\mu-\chi(u))+I_{0_+}^{\alpha;\psi}\varphi(t)\right\}. \tag{7}$$

Now, if $t\in(t_1,t_2]$ we have

$$\begin{cases} {}^HD_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\varphi(t), \quad t\in(t_1,t_2] \\ I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_1^+)}{f(t_1^+,u(t_1^+))}+g(t_1^+,u(t_1^+))\right]-I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_1^-)}{f(t_1^-,u(t_1^-))}+g(t_1^-,u(t_1^-))\right]=\Gamma_1\in\mathbb{R}\end{cases} \tag{8}$$

By Lemma (3.1), we have

$$u(t)=f(t,u(t))\left\{-g(t,u(t))+\Omega_\psi^\zeta(t,0)\left\{I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_1^+)}{f(t_1^+,u(t_1^+))}+g(t_1^+,u(t_1^+))\right]-I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=t_1}\right\}+I_{0_+}^{\alpha;\psi}\varphi(t)\right\}.$$

Then

$$u(t)=f(t,u(t))\left\{-g(t,u(t))+\Omega_\psi^\zeta(t,0)\left\{I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_1^-)}{f(t_1^-,u(t_1^-))}+g(t_1^-,u(t_1^-))\right]+\Gamma_1-I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=t_1}\right\}+I_{0_+}^{\alpha;\psi}\varphi(t)\right\} \tag{9}$$

Now from (7) we obtain

$$\frac{u(t)}{f(t,u(t))}+g(t,u(t))=\Omega_\psi^\zeta(t,0)(\mu-\chi(u))+I_{0_+}^{\alpha;\psi}\varphi(t),$$

this implies

$$I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right]=\mu-\chi(u)+I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t),$$

this gives

$$I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_1^-)}{f(t_1^-,u(t_1^-))}+g(t_1^-,u(t_1^-))\right]-I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=t_1}=\mu-\chi(u). \tag{10}$$

By (9) and (10), we get

$$u(t)=f(t,u(t))\left\{-g(t,u(t))+\Omega_\psi^\zeta(t,0)\left(\mu-\chi(u)+\Gamma_1\right)+I_{0_+}^{\alpha;\psi}\varphi(t)\right\}, \quad t\in(t_1,t_2]. \tag{11}$$

Next, if $t\in(t_2,t_3]$ then

$$\begin{cases} {}^HD_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\varphi(t), (t_2,t_3] \\ I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_2^+)}{f(t_2^+,u(t_2^+))}+g(t_2^+,u(t_2^+))\right]-I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_2^-)}{f(t_2^-,u(t_2^-))}+g(t_2^-,u(t_2^-))\right]=\Gamma_2\in\mathbb{R}\end{cases} \tag{12}$$

Again by using Lemma (3.1), we have

$$u(t)=f(t,u(t))\left\{-g(t,u(t))+\Omega_\psi^\zeta(t,0)\left\{I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_2^+)}{f(t_2^+,u(t_2^+))}+g(t_2^+,u(t_2^+))\right]-I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=t_2}\right\}+I_{0_+}^{\alpha;\psi}\varphi(t)\right\}.$$

Then

$$u(t) = f(t,u(t))\left\{-g(t,u(t)) + \Omega_{\psi}^{\zeta}(t,0)\left\{I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_2^-)}{f(t_2^-,u(t_2^-))} + g(t_2^-,u(t_2^-))\right] + \Gamma_2 - I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=t_2}\right\} + I_{0_+}^{\alpha;\psi}\varphi(t)\right\} \quad (13)$$

From (11) we have

$$I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t)}{f(t,u(t))} + g(t,u(t))\right] = \mu - \chi(u) + \Gamma_1 + I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t),$$

this gives

$$I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_2^-)}{f(t_2^-,u(t_2^-))} + g(t_2^-,u(t_2^-))\right] - I_{0_+}^{1-\zeta+\alpha;\psi}\varphi(t)|_{t=t_2} = \mu - \chi(u) + \Gamma_1. \quad (14)$$

Using (14) in (13), we get

$$u(t) = f(t,u(t))\left\{-g(t,u(t)) + \Omega_{\psi}^{\zeta}(t,0)\left(\mu - \chi(u) + \Gamma_1 + \Gamma_2\right) + I_{0_+}^{\alpha;\psi}\varphi(t)\right\}, \quad t \in (t_2, t_3].$$

Continuing the above process, we obtain

$$u(t) = f(t,u(t))\left\{-g(t,u(t)) + \Omega_{\psi}^{\zeta}(t,0)\left(\mu - \chi(u) + \sum_{i=1}^{k}\Gamma_i\right) + I_{0_+}^{\alpha;\psi}\varphi(t)\right\} \quad t \in (t_k, t_{k+1}], k = 1,2,...,n.$$

Conversely, suppose that $u \in PC_{1-\zeta,\psi}(J,\mathbb{R})$ satisfies the fractional integral equation (5), Then for $t \in [0,t_1]$ we have

$$\frac{u(t)}{f(t,u(t))} + g(t,u(t)) = \Omega_{\psi}^{\zeta}(t,0)(\mu - \chi(u)) + I_{0_+}^{\alpha;\psi}\varphi(t).$$

Applying the ψ-H fractional derivative operator ${}^{H}D_{0_+}^{\alpha,\sigma,\psi}$ on both sides, we get

$${}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))} + g(t,u(t))\right) = {}^{H}D_{0_+}^{\alpha,\sigma,\psi}\Omega_{\psi}^{\zeta}(t,0)(\mu - \chi(u)) + {}^{H}D_{0_+}^{\alpha,\sigma,\psi}I_{0_+}^{\alpha;\psi}\varphi(t),$$

using the Lemma (2.3), (ii) and Lemma (2.6), we get

$${}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))} + g(t,u(t))\right) = \varphi(t), \quad t \in [0,t_1].$$

Now, for $t \in (t_k, t_{k+1}], k = 1,2,...,n$, we have

$$\left(\frac{u(t)}{f(t,u(t))} + g(t,u(t))\right) = \Omega_{\psi}^{\zeta}(t,0)\left(\mu - \chi(u) + \Gamma_1 + \Gamma_2\right) + I_{0_+}^{\alpha;\psi}\varphi(t).$$

Applying the ψ-H fractional derivative operator ${}^{H}D_{0_+}^{\alpha,\sigma,\psi}$ on both sides and using Lemma (2.3), (ii) and Lemma (2.6), we get

$${}^{H}D_{0_+}^{\alpha,\sigma,\psi}\left(\frac{u(t)}{f(t,u(t))} + g(t,u(t))\right) = \varphi(t), \quad t \in (t_k, t_{k+1}], k = 1,2,...,n.$$

It remains to verify the initial conditions. For $t \in [0,t_1]$, we have

$$\frac{u(t)}{f(t,u(t))} + g(t,u(t)) = \Omega_{\psi}^{\zeta}(t,0)(\mu - \chi(u)) + I_{0_+}^{\alpha;\psi}\varphi(t).$$

Applying the operator $I_{0_+}^{1-\zeta;\psi}(.)$ on both sides of the above equation, we get

$$I_{0_+}^{1-\zeta;\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=I_{0_+}^{1-\zeta;\psi}\Omega_{\psi}^{\zeta}(t,0)(\mu-\chi(u))+I_{0_+}^{1-\zeta;\psi}I_{0_+}^{\alpha;\psi}\varphi(t).$$

Note that for a fixed function $u(.)$, $\chi(u(.))$ is a fixed element in $\mathbb{R}$.
By using the Lemma (2.5, *ii*), Lemma (2.3, *i*) and for $t=0$ we obtain

$$I_{0_+}^{1-\zeta;\psi}\left(\frac{u(t)}{f(t,u(t))}\right)_{t=0}+\chi(u)=\mu\in\mathbb{R}.$$

Next, for $t\in(t_k,t_{k+1}]$ we have

$$\frac{u(t)}{f(t,u(t))}+g(t,u(t))=\Omega_{\psi}^{\zeta}(t,0)\left(\mu-\chi(u)+\sum_{i=1}^{k}\Gamma_i\right)+I_{0_+}^{\alpha;\psi}\varphi(t).$$

Applying the operator $I_{0_+}^{1-\zeta;\psi}(.)$ on both sides of the above equation and using the Lemma (2.4) and Lemma (2.3, *i*) we get

$$I_{0_+}^{1-\zeta;\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\mu-\chi(u)+\sum_{i=1}^{k}\Gamma_i+I_{0_+}^{1-\zeta\alpha;\psi}\varphi(t). \tag{15}$$

And for $t\in(t_{k-1},t_k]$ we have

$$I_{0_+}^{1-\zeta;\psi}\left(\frac{u(t)}{f(t,u(t))}+g(t,u(t))\right)=\mu-\chi(u)+\sum_{i=1}^{k-1}\Gamma_i+I_{0_+}^{1-\zeta\alpha;\psi}\varphi(t). \tag{16}$$

Then from (15) and (16), we have

$$\Delta I_{0_+}^{1-\zeta;\psi}\left[\frac{u(t_k)}{f(t_k,u(t_k))}+g(t_k,u(t_k))\right]=\sum_{i=1}^{k}\Gamma_i-\sum_{i=1}^{k-1}\Gamma_i=\Gamma_k.$$

We have proved that u satisfies the non-local impulsive ψ-HHFDE (5). This completes the proof. □

Lemma 3.3. *Let $f\in C(J\times\mathbb{R},\mathbb{R}^*)$, $g\in C(J\times\mathbb{R},\mathbb{R})$, $u\in PC_{1-\zeta,\psi}(J,\mathbb{R})$, and $h\in C_{1-\zeta,\psi}(J\times\mathbb{R},\mathbb{R})$ with $J=[0,b]$. Then the problem* (2) *has a solution given by*

$$u(t)=\begin{cases} f(t,u(t))\left\{-g(t,u(t))+\Omega_{\psi}^{\zeta}(t,0)(\mu-\chi(u))+I_{0_+}^{\alpha;\psi}h(t,u(t))\right\} & t\in[0,t_1]\\ f(t,u(t))\left\{-g(t,u(t))+\Omega_{\psi}^{\zeta}(t,0)(\mu-\chi(u)+\sum_{i=1}^{k}\Gamma_i)+I_{0_+}^{\alpha;\psi}h(t,u(t))\right\} & t\in(t_k,t_{k+1}],k=1,2,...,n\end{cases}$$

Next, we introduce the following hypotheses:

(H_1) The function $f\in C(J\times\mathbb{R},\mathbb{R}^*)$ and $g\in C(J\times\mathbb{R},\mathbb{R})$ are bounded and there exists constant $\delta>0$ such that for all $p,q\in\mathbb{R}$, and $t\in J$ we have:

$$|f(t,p)-f(t,q)|\le\delta|p-q|.$$

(H_2) The function $h\in C(J\times\mathbb{R},\mathbb{R})$ and there exists a function $K\in PC_{1-\zeta,\psi}(J,\mathbb{R})$ such that

$$|h(t,p)|\le(\psi(t)-\psi(0))^{1-\zeta}K(t)\quad t\in J,p\in\mathbb{R}.$$

(H_3) assume that $\chi : PC_{1-\zeta,\psi}(J,\mathbb{R}) \to \mathbb{R}$ is a continuous function that satisfy

$$|\chi(u)| \leq L_\chi$$

Let $X := (PC_{1-\zeta,\psi}(J,\mathbb{R}), \|.\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})})$. Then X is a Banach algebra with the product of vectors defined by $(uv)(t) = u(t)v(t)$, $t \in (0,b]$. Define,

$$S = \left\{ v \in X, I_{0_+}^{1-\zeta;\psi}\left[\frac{v(t)}{f(t,v(t))}\right]_{t=0} + \chi(v) = \mu \quad and \quad \|v\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} \leq R \right\},$$

where

$$R = L_1 \left\{ (\psi(b)-\psi(0))^{1-\zeta} L_2 + \frac{1}{\Gamma(\zeta)}\left(|\mu| + \sum_{i=1}^{k} |\Gamma_i| + L_\chi \right) + \frac{(\psi(b)-\psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)} \|K\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} \right\},$$

and $L_1, L_2 > 0$ are the constants such that $|f(t,p)| \leq L_1$, $\quad |g(t,p)| \leq L_2$ for all $t \in J$.
From the definition of the set S it is clear that it is closed and bounded.
Now we check for convexity. Let $\lambda \in [0,1]$, $u, v \in S$, and $w = \lambda u + (1-\lambda)v$ such that $I_{0_+}^{1-\zeta;\psi}\left[\frac{w(t)}{f(t,w(t))}\right]_{t=0} + \chi(w) = \mu$, then we have

$$\begin{aligned}
\|w\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} &= \|\lambda u + (1-\lambda)v\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} \\
&\leq \lambda\|u\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} + (1-\lambda)\|v\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} \\
&\leq \lambda R + (1-\lambda)R \\
&\leq R
\end{aligned}$$

This implies that the set S is convex.
Therefore, S is closed, convex and bounded subset of X. Define the operators $A : X \to X$ and $B : S \to X$ by:

$$Au(t) = f(t,u(t))$$

$$Bu(t) = \left\{ -g(t,u(t)) + \Omega_\psi^\zeta(t,0)(\mu - \chi(u) + \sum_{i=1}^{k} \Gamma_i) + I_{0_+}^{\alpha;\psi} h(t,u(t)) \right\}$$

We consider the mapping $T : S \to X$ defined by:

$$Tu(t) = Au(t)Bu(t)$$

Theorem 3.4. *Assume that the hypotheses $(H_1), (H_2) and (H_3)$ hold. Then, the non-local impulsive ψ-HHFDE* 2 *has a solution $u \in PC_{1-\zeta,\psi}(J,\mathbb{R})$ provided:*

$$\delta \left\{ (\psi(b)-\psi(0))^{1-\zeta} L_2 + \frac{1}{\Gamma(\zeta)}\left(|\mu + \sum_{i=1}^{k} \Gamma_i| + L_\chi \right) + \frac{(\psi(b)-\psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)} \|K\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} \right\} < 1 \tag{17}$$

Proof. To prove that $u \in C_{1-\zeta,\psi}(J,\mathbb{R})$ is a solution of the problem (2) is equivalent to prove that the mapping T has a fixed point, we show that the operators A and B satisfies the conditions of the Theorem 2.7.
The proof is given in the several steps:
Step 1: $A : X \to X$ is Lipschitz operator:
Using the hypothesis (H_1), we obtain

$$\begin{aligned}
|(\psi(t)-\psi(0))^{1-\zeta}(Au(t) - Av(t))| &= |(\psi(t)-\psi(0))^{1-\zeta}(f(t,u(t)) - f(t,v(t)))| \\
&\leq \delta|(\psi(t)-\psi(0))^{1-\zeta}(u(t) - v(t))| \\
&\leq \delta\|u - v\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})}.
\end{aligned}$$

Therefore, A is Lipschitz operator with Lipschitz constant δ.
Step 2: $B : S \to X$ is completely continuous:

i) $B : S \to X$ is continuous:
Let $(u_n)_{n\in\mathbb{N}}$ be any sequence in S such that $u_n \to u$ as $n \to \infty$ in S. We prove that $Bu_n \to Bu$ as $n \to \infty$ in S.
We have

$$\begin{aligned}
\|Bu_n - Bu\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} &= max_{t\in J}|(\psi(t)-\psi(0))^{1-\zeta}(-g(t,u_n(t)) - \Omega_{\psi}^{\zeta}(t,0)\chi(u_n) + I_{0_+}^{\alpha;\psi}h(t,u_n(t)) \\
&+ g(t,u(t)) + \Omega_{\psi}^{\zeta}(t,0)\chi(u) - I_{0_+}^{\alpha;\psi}h(t,u(t)))| \\
&\leq max_{t\in J}(\psi(t)-\psi(0))^{1-\zeta}\{|g(t,u_n(t)) - g(t,u(t))| + \Omega_{\psi}^{\zeta}(t,0)|\chi(u_n) - \chi(u)| \\
&+ |I_{0_+}^{\alpha;\psi}h(t,u_n(t)) - I_{0_+}^{\alpha;\psi}h(t,u(t))|\}.
\end{aligned}$$

By continuity of g, χ and Lebesgue dominated convergence theorem, from the above inequality, we obtain:
$\|Bu_n - Bu\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} \to 0$ as $n \to \infty$.
This proves that $B : S \to X$ is continuous.

ii) $B(S) = \{Bu : u \in S\}$ is uniformly bounded.
Using hypotheses (H_1), (H_2) and (H_3) for any $u \in S$ and $t \in J$, we have:

$$\begin{aligned}
|(\psi(t)-\psi(0))^{1-\zeta}Bu(t)| &= |(\psi(t)-\psi(0))^{1-\zeta}(-g(t,u(t)) + \Omega_{\psi}^{\zeta}(t,0)(\mu - \chi(u) + \sum_{i=1}^{k}\Gamma_i) + I_{0_+}^{\alpha;\psi}h(t,u(t)))| \\
&\leq (\psi(t)-\psi(0))^{1-\zeta}|g(t,u(t))| + \frac{1}{\Gamma(\zeta)}|\mu + \sum_{i=1}^{k}\Gamma_i| + \frac{1}{\Gamma(\zeta)}|\chi(u)| \\
&+ (\psi(t)-\psi(0))^{1-\zeta}\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|h(s,u(s))|ds \\
&\leq (\psi(b)-\psi(0))^{1-\zeta}L_2 + \frac{1}{\Gamma(\zeta)}|\mu + \sum_{i=1}^{k}\Gamma_i| + \frac{L_\chi}{\Gamma(\zeta)} \\
&+ (\psi(b)-\psi(0))^{1-\zeta}\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|(\psi(s)-\psi(0))^{1-\zeta}K(s)|ds \\
&\leq (\psi(b)-\psi(0))^{1-\zeta}L_2 + \frac{1}{\Gamma(\zeta)}|\mu + \sum_{i=1}^{k}\Gamma_i| + \frac{L_\chi}{\Gamma(\zeta)} \\
&+ (\psi(b)-\psi(0))^{1-\zeta}\|K\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})}\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}ds \\
&\leq (\psi(b)-\psi(0))^{1-\zeta}L_2 + \frac{1}{\Gamma(\zeta)}\left(|\mu + \sum_{i=1}^{k}\Gamma_i| + L_\chi\right) \\
&+ \frac{(\psi(b)-\psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})}.
\end{aligned}$$

Therefore

$$\|Bu\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})} \leq (\psi(b)-\psi(0))^{1-\zeta}L_2 + \frac{1}{\Gamma(\zeta)}\left(|\mu + \sum_{i=1}^{k}\Gamma_i| + L_\chi\right) + \frac{(\psi(b)-\psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})}. \tag{18}$$

iii) $B(S)$ is equicontinuous: Let any $u \in S$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Then using hypothesis (H_2), we have

$$\begin{aligned}
&|(\psi(t_2)-\psi(0))^{1-\zeta}Bu(t_2)-(\psi(t_1)-\psi(0))^{1-\zeta}Bu(t_1)|\\
&=|-(\psi(t_2)-\psi(0))^{1-\zeta}g(t_2,u(t_2))+(\psi(t_2)-\psi(0))^{1-\zeta}I_{0_+}^{\alpha;\psi}h(t_2,u(t_2))\\
&+(\psi(t_1)-\psi(0))^{1-\zeta}g(t_1,u(t_1))-(\psi(t_1)-\psi(0))^{1-\zeta}I_{0_+}^{\alpha;\psi}h(t_1,u(t_1))|\\
&\le |(\psi(t_2)-\psi(0))^{1-\zeta}g(t_2,u(t_2))-(\psi(t_1)-\psi(0))^{1-\zeta}g(t_1,u(t_1))|\\
&+|(\psi(t_2)-\psi(0))^{1-\zeta}\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi^{'}(s)(\psi(t_2)-\psi(s))^{\alpha-1}|h(s,u(s))|ds\\
&-(\psi(t_1)-\psi(0))^{1-\zeta}\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}|h(s,u(s))|ds|\\
&\le |(\psi(t_2)-\psi(0))^{1-\zeta}g(t_2,u(t_2))-(\psi(t_1)-\psi(0))^{1-\zeta}g(t_1,u(t_1))|\\
&+|\frac{(\psi(t_2)-\psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}||K||_{PC_{1-\zeta,\psi}(J,\mathbb{R})}-\frac{(\psi(t_1)-\psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}||K||_{PC_{1-\zeta,\psi}(J,\mathbb{R})}|\\
&\le |(\psi(t_2)-\psi(0))^{1-\zeta}g(t_2,u(t_2))-(\psi(t_1)-\psi(0))^{1-\zeta}g(t_1,u(t_1))|\\
&+\frac{||K||_{PC_{1-\zeta,\psi}(J,\mathbb{R})}}{\Gamma(\alpha+1)}\{(\psi(t_2)-\psi(0))^{\alpha+1-\zeta}-(\psi(t_1)-\psi(0))^{\alpha+1-\zeta}\}.
\end{aligned}$$

By the continuity of ψ and g, from the above inequality it follows that:
if $|t_1-t_2|\to 0$ then $|(\psi(t_2)-\psi(0))^{1-\zeta}Bu(t_2)-(\psi(t_1)-\psi(0))^{1-\zeta}Bu(t_1)|\to 0$ From the parts (ii) and (iii), it follows that B(S) is uniformly bounded and equicontinous set in X. Then by Arzela-Ascoli theorem, B(S) is relatively compact. Since $B: S\to X$ is the continuous and compact operator, it is completely continuous.

Step 3:

Let any $u\in PC_{1-\zeta,\psi}(J,\mathbb{R})$ and $v\in S$ such that $u(t)=Au(t)Bv(t)$ then, for all $t\in[a,b]$ we have

$$\begin{aligned}
|(\psi(t)-\psi(0))^{1-\zeta}u(t)| &= |(\psi(t)-\psi(0))^{1-\zeta}Au(t)Bv(t)|\\
&=|f(t,u(t))|\{(\psi(t)-\psi(0))^{1-\zeta}|-g(t,v(t))+\Omega_{\psi}^{\zeta}(t,0)(\mu-\chi(v)+\sum_{i=1}^{k}\Gamma_i)+I_{0_+}^{\alpha;\psi}h(t,v(t))|\}\\
&\le |f(t,u(t))|\{(\psi(t)-\psi(0))^{1-\zeta}|g(t,v(t))|+\frac{1}{\Gamma(\zeta)}(|\mu+\sum_{i=1}^{k}\Gamma_i|+|\chi(v)|)\\
&+|(\psi(t)-\psi(0))^{1-\zeta}I_{0_+}^{\alpha;\psi}h(t,v(t))|\}\\
&\le L_1\{(\psi(b)-\psi(0))^{1-\zeta}L_2+\frac{1}{\Gamma(\zeta)}(|\mu+\sum_{i=1}^{k}\Gamma_i|+L_\chi)+\frac{(\psi(b)-\psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}||K||_{PC_{1-\zeta,\psi}(J,\mathbb{R})}\}.
\end{aligned}$$

This gives that $||u||_{PC_{1-\zeta,\psi}(J,\mathbb{R})}\le R$. Then, $u\in S$.

Step 4:

Let $M=||B(S)||_{PC_{1-\zeta,\psi}(J,\mathbb{R})}=sup\{||Bu||_{PC_{1-\zeta,\psi}(J,\mathbb{R})}: u\in S\}$.
From inequality (17) and (18), we have

$$\begin{aligned}
\delta M &\le \delta\left\{(\psi(b)-\psi(0))^{1-\zeta}L_2+\frac{1}{\Gamma(\zeta)}\left(|\mu+\sum_{i=1}^{k}\Gamma_i|+L_\chi\right)+\frac{(\psi(b)-\psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}||K||_{PC_{1-\zeta,\psi}(J,\mathbb{R})}\right\}\\
&<1
\end{aligned}$$

From steps 1 to 4, it follows that all the conditions of the theorem (2.7) are fulfilled. Hence the operator T has a solution in S.
This implies that the non-local impulsive ψ-HHFDE (2) has a solution in $PC_{1-\zeta,\psi}(J,\mathbb{R})$. □

4. Example

This section is consisted on appropriate example which is relevant to demonstrate our results. We consider the particular case when $\psi(t) = t$ and $\sigma = 0$.
Consider the non-local impulsive ψ-HHFDE involving Riemann Liouville fractional derivative.
Where $f(t, u(t)) = \frac{1}{4}tu(t) + 1$, $g(t, u(t)) = \frac{\cos(t)|u(t)|}{1+|u(t)|}$, $h(t, u(t)) = \frac{|u(t)|}{2+u(t)^2}$, and $\chi(u) = \frac{1}{3(1+|u(\frac{1}{2})|)}$

$$\begin{cases} {}^{RL}D^{\frac{1}{2}}_{0_+}\left(\frac{u(t)}{\frac{1}{4}tu(t)+1} + \frac{\cos(t)|u(t)|}{1+|u(t)|}\right) = \frac{|u(t)|}{2+u(t)^2}, \quad t \in [0,1] \setminus \{\frac{1}{2}\} \\ \Delta I^{1-\zeta;\psi}_{0_+}\left[\frac{u(t)}{\frac{1}{4}tu(t)+1} + \frac{\cos(t)|u(t)|}{1+|u(t)|}\right]_{t=\frac{1}{2}} = \Gamma_{\frac{1}{2}} = \frac{2}{3} \\ I^{1-\zeta;\psi}_{0_+}\left[\frac{u(t)}{\frac{1}{4}tu(t)+1}\right]_{t=0} + \frac{1}{3(1+|u(\frac{1}{2})|)} = \frac{1}{2} \end{cases} \tag{19}$$

Comparing the problem (19) with system of non-local impulsiveψ-HHFDE (2). Then
$\alpha = \frac{1}{2}$, $\sigma = 0$, $\zeta = \frac{1}{2}$, $\psi(t) = t$ and $J = [0, 1]$.
It is clear that
$|f(t, u(t)) - f(t, v(t))| \leq \frac{1}{4}|u - v|$, $|g(t, u(t))| \leq 1$ and $|h(t, u(t))| \leq 1$, here $k(t) = 1$, $L_2 = 1$, $|\chi(u)| \leq \frac{1}{3} = L_\chi$, and $\delta = \frac{1}{4}$.
Now we check for condition (17). Further, consider

$$\begin{aligned} &\delta\left\{(\psi(b) - \psi(0))^{1-\zeta}L_2 + \frac{1}{\Gamma(\zeta)}\left(|\mu + \sum_{i=1}^{k}\Gamma_i| + L_\chi\right) + \frac{(\psi(b) - \psi(0))^{\alpha+1-\zeta}}{\Gamma(\alpha+1)}\|K\|_{PC_{1-\zeta,\psi}(J,\mathbb{R})}\right\} \\ &= \frac{1}{4}\left\{1 + \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{1}{2} + \frac{2}{3} + \frac{1}{3}\right) + \frac{1}{\Gamma(\frac{3}{2})}\right\} \\ &\approx 0,787 \\ &< 1 \end{aligned}$$

We observe that all the conditions of Theorem (3.4) are satisfied. Therefore, the system of non-local impulsive ψ-HHFDEs involving Riemann Liouville fractional derivative (19) has a solution in $PC_{1-\zeta,\psi}(J, \mathbb{R})$.

5. Conclusion

Impulsive differential equations play a very important role in modeling real world physical phenomena involving in the study of population dynamics, biotechnology and chemical technology. In this work we have established existence theory on solutions of the system of hybrid fractional differential equation introduced by the ψ-Hilfer fractional derivative with non-local impulsive conditions The technique used is based on Dhage fixed point theorem. Also we presented an example to illustrate our main results.

References

[1] Kilbas, Anatoliĭ Aleksandrovich, Hari M. Srivastava, and Juan J. Trujillo. *Theory and applications of fractional differential equations.* Vol. **204**. elsevier, 2006.
[2] Dhage, Bapurao C. *"On a fixed point theorem in Banach algebras with applications."* Applied Mathematics Letters **18**.3 (2005): 273-280.
[3] Abbas M.I., Ragusa M.A.,*Nonlinear fractional differential inclusions with nonsingular Mittag-Leffler kernel*, AIMS Mathematics, **7** (11), 20328–20340, (2022);
[4] Akdemir A.O., Karaoglan A., Ragusa M.A., E. Set,*Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions*, Journal of Function Spaces, vol.**2021**, art.n.1055434, (2021);
[5] Hilal, Khalid, and Ahmed Kajouni. *"Boundary value problems for hybrid differential equations with fractional order."*Advances in Difference Equations **2015**.1 (2015): 1-19.

[6] Ucar D., *New conformable fractional operator and some related inequalities*, Filomat, **35** (11), 3597-3606, (2021);

[7] J. Vanterler da C. Sousa, and E. Capelas de Oliveira, *"On the $\psi-$Hilfer fractional derivative"*, Commun. Nonlinear Sci. Numer. Simulat., vol. **60**, pp. 72–91, 2018.

[8] Kucche, Kishor D., Jyoti P. Kharade, and J. V. D. C. Sousa. *"On the Nonlinear Impulsive Ψ-Hilfer Fractional Differential Equations."* arXiv preprint arXiv:1901.01814 (2019).

[9] Almeida, Ricardo. *"A Caputo fractional derivative of a function with respect to another function."* Communications in Nonlinear Science and Numerical Simulation **44** (2017): 460-481.

[10] Abdo, Mohammed S., Satish K. Panchal, and Abdulkafi M. Saeed. *"Fractional boundary value problem with ψ-Caputo fractional derivative."* Proceedings-Mathematical Sciences **129**.5 (2019): 1-14.

[11] ABDO, MOHAMMED S., AHMED G. IBRAHIM, and SATISH K. PANCHAL. *"Nonlinear implicit fractional differential equation involving-Caputo fractional derivative."* Proceedings of the Jangjeon Mathematical Society. Vol. **22**. No. 3. 2019.