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Fixed points of monotone asymptotic pointwise ρ-nonexpansive semigroups in modular spaces

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Abstract. In this paper, we prove common fixed point results for monotone asymptotic pointwise ρ-nonexpansive semigroups in modular function spaces. Thus, utilizing the monotonicity of semigroups, our findings generalize and extend some prominent recent results of the literature in the setting of modular function spaces.

1. Introduction

The theory of semigroups has recently become a very interesting subject area for future research and applications. As in the case of theory of dynamical systems the modular function space X_ρ on which the semigroup S is defined represents the states space, and the mapping $R^+ \times C \to C$, $(t,s) \to T_t(f)$ defines the evolution function of the dynamical system (see for instance, [14, 18]). In 1992 the existence of semigroups of nonexpansive mappings in modular function spaces was investigated by Khamsi [8] in the context of Musielak-Orlicz spaces and discussed applications to differential equations. The problem of finding common fixed points, and the structure of the set of common fixed points, for semigroups of mappings is in its beginning stage. There are a lot of interesting features of the theory that are to be explored.

In recent years, an analogous variant of the Banach contraction principle [9] for monotone mappings in ordered metric spaces was investigated by Turinici [23, 24], and modified by Ran and Reurings [22] which was later extended and generalized by several authors [2, 17, 19–21]. The respective authors also presented applications of these newly obtained fixed point results to linear and nonlinear matrix equations and integro-differential equations. For recent development of the theory of fixed points via monotone mappings, we must quote the survey article by Bachar and Khamsi [3].

On another point of note, Goebel and Kirk [7] established the existence of fixed points for asymptotic nonexpansive mappings. Kirk and Xu [12] extended these results for asymptotic pointwise contractions

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and asymptotic pointwise nonexpansive mappings in the Banach spaces. Afterwards, Khamsi and Kozlowski [10, 11] and Kozlowski [13] extended these results to modular function spaces. In this continuation, Kozlowski [15] proved that the set of common fixed points of any semigroup of ρ-nonexpansive mappings, for a ρ-closed convex and ρ-bounded subset of a uniformly convex modular space, is nonempty ρ-closed and convex.

Most recently, Bin Dehaish et al. [5] presented the existence of common fixed points for asymptotic pointwise nonexpansive semigroups in modular spaces, Kozlowski [14] for monotone Lipschitzian semigroups in Banach spaces and El Harmouchi et al. [6] for the monotone ρ-nonexpansive semigroups in modular function spaces. Motivated by the above cited work, and with a view to utilize the monotonicity of the semigroups, it is our aim in this paper, to establish the existence of common fixed points for monotone asymptotic pointwise ρ-nonexpansive semigroups in modular function spaces.

2. Preliminaries

In this section, we provide some basic definitions which will work as a relevant necessary background for further presentations. Throughout this paper, X denotes a real vector space.

Definition 2.1. *[1] A function $\rho : X \to [0, +\infty]$ is called a modular if the following holds:*
(i) $\rho(f) = 0$ if and only if $f = 0$;
(ii) $\rho(-f) = \rho(f)$;
(iii) $\rho(\alpha f + (1-\alpha)g) \leq \rho(f) + \rho(g)$ for all $\alpha \in [0,1]$ and $f, g \in X$.
If (iii) is replaced by $\rho(\alpha f + (1-\alpha)g) \leq \alpha\rho(f) + (1-\alpha)\rho(g)$ for all $\alpha \in [0,1]$ and $f, g \in X$, then ρ is called a convex modular. A modular ρ defines the corresponding modular function space, that is, the vector space

$$X_\rho = \{f \in X : \lim_{\lambda \to 0} \rho(\lambda f) = 0\}.$$

Let ρ be a convex modular. Then the modular space X_ρ is equipped with a norm called the Luxemburg norm, defined by

$$\|f\|_\rho = \inf\{\lambda > 0 : \rho(\frac{f}{\lambda}) \leq 1\}.$$

Definition 2.2. *[1, 9] Let $\{f_n\}$ be a sequence in a modular function space X_ρ and let C be a non-empty subset of X_ρ. Then*
(i) $\{f_n\}$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to +\infty$.
(ii) $\{f_n\}$ is called ρ-convergent to $f \in X_\rho$ if and only if $\rho(f_n - f) \to 0$, as $n \to +\infty$, $f \in X_\rho$.
(iii) X_ρ is called ρ-complete if every ρ-Cauchy sequence ρ-converges to a point in X_ρ.
(iv) C is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.
(v) C is called ρ-bounded if $\delta_\rho(C) = \sup\{\rho(f-g) : f, g \in C\} < \infty$.
(vi) C is called ρ-compact if for any $\{f_n\}$ in C there exists a subsequence that ρ-converges to f in C.
(vii) ρ satisfies the Fatou property if $\rho(f-g) \leq \lim_{n\to+\infty} \rho(f-g_n)$ for any f whenever $\{g_n\}$ ρ-converges to g in X_ρ.

Note that the ρ-convergence does not imply the ρ-Cauchy condition. Also, $f_n \xrightarrow{\rho} f$ does not imply in general that $\lambda f_n \xrightarrow{\rho} \lambda f$ for every $\lambda > 1$.

The property connected with a function modular, which plays a crucial role in the modular function spaces, is the Δ_2-condition.

Definition 2.3. *[1, 16] Let ρ be a modular defined on a vector space X. Then ρ satisfies Δ_2-condition if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$ as $n \to +\infty$, and Δ_2-type condition, if there exists $K > 0$ such that $\rho(2f) \leq K\rho(f)$.*

Definition 2.4. *[2, 24] (a) Let X be a non-empty set endowed with a partial order relation (anti-symmetric, reflexive and transitive) denoted by '$\preceq$'. Then the pair $(X, \preceq)$ is called a partially ordered set or an ordered set.*
(b) The element f is called comparable to the element g, if either $f \preceq g$ or $f \succeq g$ and we denote this as '<>'.
(c) X is linearly ordered or totally ordered if any two elements of X are comparable.

Definition 2.5. *[6] Let ρ be a modular, and let C be a nonempty subset of the modular space X_ρ. A mapping $T : C \to C$ is called*
(i) monotone if $T(f) \leq T(g)$ for all $f, g \in C$ such that $f \leq g$,
(ii) monotone ρ-nonexpansive if T is monotone such that $\rho(T(f) - T(g)) \leq \rho(f - g)$ for all $f, g \in X_\rho$ such that $f \leq g$,
(iii) ρ-continuous if $\{T(f_n)\}$ ρ-converges to $T(f)$ whenever $\{f_n\}$ ρ-converges to f.

It is not true that a monotone ρ-nonexpansive mapping is ρ-continuous since this result is not true in general when ρ is a norm. Further we assume that ρ is a convex modular.

Definition 2.6. *[1] Let ρ be a modular and let $r > 0$ and $\epsilon > 0$. Define a relation $D_i(r, \epsilon)$ on X_ρ for $i \in \{1, 2\}$ such that*

$$D_i(r, \epsilon) = \{(f, g) \in X_\rho \times X_\rho : \rho(f) \leq r,\ \rho(g) \leq r,\ \rho\Big(\frac{f - g}{i}\Big) \geq r\}.$$

If $D_i(r, \epsilon) \neq \phi$, then

$$\delta_i(r, \epsilon) = \{1 - \frac{1}{r}\rho\Big(\frac{f - g}{2}\Big) : (f, g) \in D_i(r, \epsilon)\}.$$

If $D_i(r, \epsilon) = \phi$, then we set $\delta_i(r, \epsilon) = 1$.

We say that ρ satisfies
(i) uniform convexity (UCi) if for all $r > 0$ and $\epsilon > 0$, we have $\delta_i(r, \epsilon) > 0$,
(ii) unique uniform convexity (UUCi) if for all $s \geq 0$ and $\epsilon > 0$, there exists $\eta(s, \epsilon) > 0$ such that $\delta_i(r, \epsilon) > \eta(s, \epsilon)$ for $r > s$,
(iii) strictly convex (SC) if for all $f, g \in X_\rho$ such that $\rho(f) = \rho(g)$ and $\rho(\frac{f+g}{2}) = \frac{\rho(f)+\rho(g)}{2}$, we have $f = g$.
The following proposition shows the connection between the above notions.

Proposition 2.7. *[1] (a) (UUCi) implies (UCi) for $i = 1, 2$,*
(b) $\delta_1(r, \epsilon) \leq \delta_2(r, \epsilon)$ for $r > 0$ and $\epsilon > 0$,
(c) (UC1) implies (UC2),
(d) (UC2) implies (SC),
(e) (UUC1) implies (UUC2).

Motivated by the above notion, El Harmouchi et al. [6] introduced the notion of uniform convexity in every direction (UCED) for modular spaces as follows.

Definition 2.8. *[6] Let ρ be a modular. Then ρ is*
(i) uniformly convex in every direction (UCED) if for any $r > 0$ and nonzero $h \in X_\rho$, we have

$$\delta(r, h) = \inf\{1 - \frac{1}{r}\rho(f + \frac{h}{2}) : \rho(f) \leq r,\ \rho(f + h) \leq r\} > 0;$$

(ii) unique uniform convexity in every direction (UUCED) if there exists $\eta(s, h) > 0$ for $s \geq 0$ and nonzero $h \in X_\rho$ such that $\delta(r, h) > \eta(s, h)$ for $r > s$.

The following proposition shows the connection between the above notions.

Proposition 2.9. *[6] (a) (UCi) (resp., (UUCi)) implies (UCED) (resp., (UUCED)) for i = 1,2,*
(b) (UUCED) implies (UCED),
(c) (UCED) implies (SC).

The following property plays a similar role as the reflexivity in Banach spaces for modular spaces.

Definition 2.10. *[16] Let ρ be a modular. Then the modular space X_ρ is said to satisfy property (R) if for every decreasing sequence $\{C_n\}$ of nonempty ρ-closed convex and ρ-bounded subsets of X_ρ, we have*

$$\bigcap_{n \in \mathbb{N}} C_n \neq \phi.$$

Lemma 2.11. *[1] Let ρ be a convex modular satisfying the Fatou property such that X_ρ is ρ-complete and ρ is (UUC2). Then X_ρ satisfies the property (R).*

Proposition 2.12. *[1] Let ρ be a convex modular such that X_ρ is ρ-complete and ρ is (UUC2). Let C be a ρ-closed convex and ρ-bounded nonempty subset of X_ρ . Let (C_i) $i \in I$ be a family of ρ-closed convex nonempty subsets of C such that $\cap_{i\in I} C_i$ is nonempty for any finite subset F of I. Then*

$$\bigcap_{i\in I} C_i \neq \phi.$$

The ρ-type function is a powerful technical tool to prove the existence of a fixed point.

Definition 2.13. *[9] Let $\{f_n\}$ be a sequence in X_ρ, and let K be a nonempty subset of X_ρ. The function $\tau : K \to [0, +\infty]$ defined by*

$$\tau(f) = \limsup_{n\to+\infty} \rho(f_n - f)$$

is called a ρ-type function.

The next definition is an alternation of the definition of ρ-type functions to a one parameter family of mappings.

Definition 2.14. *[9] Let $C \subset X_\rho$ be convex ρ-bounded. A function $\tau : C \to R^+$ is a ρ-type function if there exists a one-parameter family $\{T_t : t \geq 0\}$ of elements of a nonempty subset K of X_ρ such that for all $f \in K$,*

$$\tau(f) = \limsup_{n\to+\infty} \rho(T_t(f) - g),$$

for all $g \in K$.
A sequence $\{c_n\} \subset K$ is a minimizing sequence of τ if

$$\lim_{n\to+\infty} \tau(c_n) = \inf_{f\in K} \tau(f).$$

Note that the ρ-type function τ is convex since ρ is convex.
Recall the definition of the uniform continuity of a modular.

Definition 2.15. *[9] A modular ρ is said to be uniformly continuous if for any $\epsilon > 0$ and $R > 0$, there exists $\eta > 0$ such that*

$$|\rho(g) - \rho(f + g)| \leq \epsilon,$$

whenever $\rho(f) \leq \eta$ and $\rho(g) \leq R$.

Lemma 2.16. *[6] Let ρ be a convex modular uniformly continuous and (UUCED). Assume that the modular space X_ρ satisfies property (R). Let C be a ρ-closed ρ-bounded convex nonempty subset of X_ρ . Let K be a nonempty ρ-closed convex subset of C. Let $\{f_n\}$ be a sequence in C and consider the ρ-type function $\tau : K \to [0, +\infty]$ defined by*

$$\tau(g) = \limsup_{n\to+\infty} \rho(f_n - g).$$

Then τ has a unique minimum point in K.

A subset $P \subset X_\rho$ is called a pointed ρ-closed convex cone if P is a nonempty ρ-closed subset of X_ρ satisfying the following properties:
(i) $P + P \subset P$,
(ii) $\lambda P \subset P$ for all $\lambda \in R^+$,
(iii)$P \cap (-P) = 0$.
Using P, we define an ordering on X_ρ by $f \leq g$ if and only if $g - f \in P$. We further suppose that the modular space X_ρ is equipped with the partial order defined by P.

3. Main Results

Before we state our main results, let us recall the definition of a monotone semigroups.

Definition 3.1. *[6] Let C be a nonempty subset of a modular space X_ρ. A one-parameter family $S = \{T_t : t \geq 0\}$ of mappings from C into C is called monotone semigroup on C if it satisfies the following assumptions:*
(i) $T_0(f) = f$ for all $f \in C$,
(ii) $T_{s+t}(f) = T_s\{T_t(f)\}$ for all $s, t \geq 0$,
(iii) T_t is monotone, that is, $T_t(f) \leq T_t(g)$ for all $t \geq 0,\ f, g \in C$ such that $f \leq g$.

Next, we present slight modification in the definitions of monotone ρ-Lipschitz semigroup (see, Definition 3.17 in [6]) and asymptotic pointwise nonexpansive semigroups (see, Definition 2.6 in [5]) as follows.

Definition 3.2. *(a) A semigroup S is called monotone ρ-Lipschitz semigroup if S is monotone and $t \geq 0$, there exists $k \geq 0$ such that*

$$\rho(T_t(f) - T_t(g)) \leq k\rho(f - g),$$

for all $f, g \in C$ with $f <> g$. For $k < 1$, S is a monotone ρ-contraction semigroup and if $k = 1$, then it is called a monotone ρ-nonexpansive semigroup.
(b) A semigroup S is called monotone asymptotic pointwise ρ-nonexpansive if S is monotone and, for each $t \geq 0,\ f \in C$, there exists a function $(t, f) \to \alpha(t, f) := \alpha_t(f)$ from $[0, \infty[\times C$ to $[0, \infty]$ with $\limsup_{t\to+\infty} \alpha_t(f) \leq 1$ such that

$$\rho(T_t(f) - T_t(g)) \leq \alpha_t(f)\rho(f - g),$$

for all $g \in C$ with $f <> g$.
The set of all common fixed points of S is defined by

$$Fix(S) = \{f \in C : T_t(f) = f \text{ for all } t \geq 0 \text{ and } f <> T_t(f)\} = \bigcap_{t\geq 0} Fix(T_t).$$

It is important to note that the above assumptions (a) and (b) of monotone semigroups are required to hold on only those elements which are comparable under the underlying partial ordering.
Also note that, for assumption (b) without loss of generality, we may assume $\alpha_t(f) \geq 1$ for all $t \geq 0,\ f \in C$, and $\limsup_{t\to+\infty} \alpha_t(f) = \lim_{t\to+\infty} \alpha_t(f) = 1$.
The following lemma generalizes the minimizing sequence property for type functions generated by a sequence to the case of type functions defined by a one-parameter family $\{u_t : t \geq 0\}$. Also, the method of proof of the lemma is technically connected with the proof of Lemma 7.11 in [9].

Lemma 3.3. *[6] Let ρ be a convex modular satisfying the Fatou property and (UUC1), and let X_ρ be a ρ-complete modular space. Let C be a nonempty ρ-closed convex subset of X_ρ. Let S be a monotone ρ-nonexpansive semigroup on C. Fix $f_0 \in C$ and consider the function $\varphi : C \to R^+$ given by*

$$\varphi(g) = \limsup_{t\to+\infty} \rho(T_t(f_0) - g) = \inf_{s\geq 0} \sup_{t\geq s} \rho(T_t(f_0) - g).$$

Then every minimizing sequence of φ, ρ-converges to the same limit.

Theorem 3.4. *Let ρ be a convex modular satisfying (UUC1) and the Fatou property. Let C be a nonempty ρ-closed convex ρ-bounded subset of a ρ-complete modular space X_ρ . Let $S = \{T_t : t \geq 0\}$ be a monotone asymptotic pointwise ρ-nonexpansive semigroup such that T_t is ρ-continuous. Assume that for all $t \geq 0$ there exists $f_0 \in C$ such that $f_0 <> T_t(f_0)$. Then S has a common fixed point, i.e., $h \in Fix(S)$ such that $f_0 <> h$.*

Proof. In pursuance of the assumption $f_0 <> T_t(f_0)$ the condition $\rho(T_t(f) - T_t(g)) \leq \alpha_t(f)\rho(f - g)$ is satisfied under two cases: Either $f_0 \preceq T_t(f_0)$ or $f_0 \succeq T_t(f_0)$. If it holds for the first case, then by the symmetry of modular ρ, it must hold for the second case too and the same is true for converse consideration. Therefore applying the given contractive condition on these two cases are the same. So, we consider only the first to explore our further investigations.
Also, in light of the same assumption, we ensure the existence of an element $f_0 \in C$, then $f_0 \preceq T_t(f_0) \in C$. If $f_0 = T_t(f_0)$ then f_0 is a common fixed point of semigroup $\mathcal{S}$. So the proof is accomplished.
However, if $f_0 \neq T_t(f_0)$, then in view of partial order relation and Proposition 2.12, we have

$$K = \bigcap_{t\geq 0}[T_t(f_0), \rightarrow) \cap C$$

is nonempty. In fact, utilizing the Proposition 2.12, it suffices to prove that

$$\bigcap_{t\in F}[T_t(f_0), \rightarrow) \cap C$$

is nonempty for any finite subset $F = \{t_0, t_1, ..., t_n\}$ of R^+, where t_i are arbitrarily chosen in R^+.
Let us consider $f = T_{t_0+t_1+...+t_n}(f_0) \in C$. Since S is a monotone semigroup and $f_0 \preceq T_t(f_0)$ for all $t \geq 0$, we have $T_s(f_0) \preceq T_{s+t}(f_0)$ for all $s, t \geq 0$. Hence $T_{t_i}(f_0) \preceq f$ for all $i \in \{1, 2, ..., n\}$, that is, $f \in [T_{t_i}(f_0), \rightarrow) \cap C$.
Thus $\bigcap_{t_i\in F}[T_{t_i}(f_0), \rightarrow) \cap C$ is nonempty for all $n \geq 0$. Moreover, K is ρ-closed convex.
Further, as K is invariant by S. In fact, let $f \in K$ and $t, s \geq 0$ such that $t \geq s$, then $t - s \geq 0$. So, $T_{t-s}(f_0) \preceq f$ implies $T_t(f_0) \preceq T_s(f)$. If $t < s$, then $\epsilon = s - t > 0$. Since $f_0 \preceq f$, we have

$$f_0 \preceq T_\epsilon(f_0) \preceq T_\epsilon(f) \implies T_t(f_0) \preceq T_{t+\epsilon}(f) = T_s(f).$$

Thus, $T_t(f_0) \preceq T_s(f)$ for all $t, s \geq 0$. Then $T_s(f) \in K$ for all $s \geq 0$. Therefore $S(K) \subset K$. Consider the function $\varphi : K \to [0, +\infty)$ defined by

$$\varphi(g) = \limsup_{t\to+\infty} \rho(T_t(f_0) - g) = \inf_{s\geq 0} \sup_{t\geq s} \rho(T_t(f_0) - g).$$

As K is ρ-bounded, $\varphi_0 = \inf_{g\in C} \varphi(g) < +\infty$. So, for any $n \geq 1$, there exists $h_n \in K$ such that

$$\varphi_0 \leq \varphi(h_n) \leq \varphi_0 + \frac{1}{n}.$$

Then in the light of Lemma 3.3, $\{h_n\}$ is a minimizing sequence of φ and ρ-converges to $h \in K$. To prove that $h \in Fix(S)$, it suffices to show that $\{T_t(h_n)\}$ is also a minimizing sequence of φ for any $t \geq 0$.
Fix $s, \eta \geq 0$, and let $t \geq s + \eta$, $g \in K$ then $T_{t-s}(f_0) \preceq g$. Since S is a monotone asymptotic pointwise ρ-nonexpansive semigroup, we have

$$\begin{aligned}
\rho(T_s(T_{t-s}(f_0)) - T_s(g)) &= \rho(T_t(f_0) - T_s(g)) \\
&\leq \alpha_t(f_0)\rho(T_{t-s}(f_0) - g) \\
&\leq \sup_{t\geq\eta} \alpha_t(f_0)\rho(T_t(f_0) - g),
\end{aligned}$$

$$\sup_{t\geq\eta} \rho(T_t(f_0) - T_s(h)) \leq \sup_{t\geq s+\eta} \rho(T_t(f_0) - T_s(h)) \leq \sup_{t\geq\eta} \alpha_t(f_0)\rho(T_t(f_0) - h).$$

Taking the $\inf_{\eta\geq 0}$ over the foregoing inequality, we have

$$\inf_{\eta\geq 0}\sup_{t\geq\eta} \rho(T_t(f_0) - T_s(h)) \leq \inf_{\eta\geq 0}\sup_{t\geq\eta} \alpha_t(f_0)\rho(T_t(f_0) - h).$$

Since η is arbitrary positive, we obtain

$$\begin{aligned}\varphi(T_s(g)) &\leq \inf_{\eta\geq 0}\sup_{t\geq\eta}\alpha_t(f_0)\rho(T_t(f_0)-h)\\ &= \limsup_{t\to+\infty}\alpha_t(f_0)\rho(T_t(f_0)-h)\\ &\leq \limsup_{t\to+\infty}\alpha_t(f_0)\limsup_{t\to+\infty}\rho(T_t(f_0)-h)\\ &= 1\cdot\varphi(g).\end{aligned}$$

Hence, for any $s \geq 0$, we have

$$\varphi(T_s(g)) \leq \varphi(g).$$

Thus for all $s \geq 0$, $\{T_s(h_n)\}$ is a minimizing sequence of φ.
Owing to Lemma 3.3, the sequence $\{T_s(h_n)\}$ ρ-converges to h for all $s \geq 0$. Since T_s is ρ-continuous for all $n, s \geq 0$, $\{T_s(h_n)\}$ ρ-converges to $T_s(h)$. Finally, in light of the uniqueness of the limit for all $s \geq 0$, we conclude that $h = T_s(h)$, that is, h is a common fixed point of the semigroup S. □

Remark 3.5. *Theorem 3.4 remains true and reduces to the theorem presented in [6] (see, Theorem 3.19), if we replace the assumption, monotone asymptotic pointwise ρ-nonexpansive semigroup by monotone ρ-nonexpansive semigroup besides retaining rest of the assumptions.*

The next lemma is a generalization of Lemma 2.16 for ρ-type functions defined by a given one-parameter family of mappings.

Lemma 3.6. *[6] Let ρ be a convex modular uniformly continuous and (UUCED), and let X_ρ be a modular space satisfying property (R). Let C be a nonempty ρ-closed convex ρ-bounded subset of X_ρ, let S be a monotone ρ-nonexpansive semigroup on C, and let K be a ρ-closed convex subset of C. Fix $f_0 \in C$ and consider the function $\varphi : C \to R^+$ given by*

$$\varphi(y) = \limsup_{t\to+\infty}\rho(T_t(f_0)-g) = \inf_{s\geq 0}\sup_{t\geq s}\rho(T_t(f_0)-g).$$

Then there exists a unique $h \in K$ such that $\varphi(h) = \inf_{g\in K}\varphi(g)$.

Bachar et al. [4] generalized Kozlowski's work in Banach spaces in the case of monotone nonexpansive semigroups, and proved the following common fixed point result.

Theorem 3.7. *[4] Let $(X, \|.\|)$ be a Banach space uniformly convex in every direction. Let C be a weakly compact convex nonempty subset of X, and $S = T_t : t \geq 0$ be a monotone nonexpansive semigroup defined on C. Assume that for all $t \geq 0$ there exists $f_0 \in C$ such that $f_0 <> T_t(f_0)$. Then S has a common fixed point, i.e., $h \in Fix(S)$ such that $f_0 <> h$.*

The next result is an extension of the Theorem 3.7 for asymptotic pointwise ρ-nonexpansive semigroups in uniformly convex in every direction (UCED) modular spaces.

Theorem 3.8. *Let ρ be a convex modular uniformly continuous and (UUCED), and let X_ρ be a modular space satisfying property (R). Let C be a nonempty ρ-closed convex ρ-bounded subset of X_ρ . Let S be a monotone asymptotic pointwise ρ-nonexpansive semigroup on C. Assume that for all $t \geq 0$ there exists $f_0 \in C$ such that $f_0 <> T_t(f_0)$. Then S has a common fixed point, i.e., $h \in Fix(S)$ such that $f_0 <> h$.*

Proof. Without loss of assumption, we consider that $f_0 \preceq T_t(f_0)$ for all $t > 0$. Let $\{s_n\}$ be a nondecreasing sequence in R^+ such that $s_0 = 0$ and $\lim_{n\to+\infty} s_n = +\infty$. For all $n \geq 0$, set

$$K_n = \bigcap_{t\geq s_n}[T_t(f_0),\to)\cap C.$$

Indeed, for all $u \geq 0$, $f_0 \preceq T_u(f_0)$, $\{K_n\}$ is a decreasing sequence of ρ-closed convex and ρ-bounded subsets of C. In particular, for $u = s_n$, we have $f_0 \preceq T_{s_n}(f_0)$. Let $t \geq s_n$, then we have $T_t(f_0) \preceq T_{t+s_n}(f_0) \preceq T_{2s_n}(f_0) = f$. So that $f = T_{2s_n}(f_0) \in K_n$. Hence K_n is nonempty for all $n \geq 0$.
For all $n \geq 0$, K_n is ρ-closed. In fact, let g_p be a sequence in K_n that ρ-converges to $g \in C$. For all $p \geq 0$, $g_p \in K_n$, that is, $T_t(f_0) \preceq g_p$ for all $t \geq s_n$ and $p \geq 0$. Then in light of the partial ordering defined on ρ-closed convex cone $P \subset X_\rho$, we have $g_p - T_t(f_0) \in P$ for all $t \geq s_n$ and $p \geq 0$. Since

$$\lim_{p\to+\infty} \rho(g_p - T_t(f_0) - g + T_t(f_0)) = \lim_{p\to+\infty} \rho(g_p - g) = 0,$$

and P is ρ-closed, we have $g - T_t(f_0) \in P$ for all $t \geq s_n$, that is, $g \in K_n$.
As P is convex and $\{K_n\} \subset C$, therefore $\{K_n\}$ is convex and ρ-bounded subset of C. Moreover, $\{K_n\}$ is decreasing because $\{s_n\}$ is an increasing sequence.
In light of the property (R), the set $K = \bigcap_{n\geq 0} K_n$ is nonempty ρ-closed and convex subset of C.
K is invariant by S. In fact, let $f \in K$, then $T_t(f_0) \preceq f$ for all $n \geq 0$ and $t \geq s_n$. Letting $\eta \geq 0$, let us prove that $T_t(f_0) \preceq T_\eta(f)$ for all $t \geq s_n$.
If $\eta \leq t$, then $t - \eta \geq 0$, where $t \geq s_n$, which implies that $T_{t-\eta}(f_0) \preceq f$. As $f \in K_0$ then in light of monotonicity of semigroup, we have $T_t(f_0) \preceq T_\eta(f)$. Hence, $T_\eta(f) \in K_n$. In all cases, we obtain $S(K) \subset K$.
Consider the function $\varphi : K \to R^+$ defined by

$$\varphi(g) = \limsup_{t\to+\infty} \rho(T_t(f_0) - g).$$

Owing to Lemma 3.6, φ has a unique minimum point $h \in K$.
Fix $s, \eta \geq 0$ and let $t \geq s + \eta$. As S is a monotone asymptotic pointwise ρ-nonexpansive semigroup, we have

$$\begin{aligned} \rho(T_t(f_0) - T_s(h)) &= \rho(T_s(T_{t-s}(f_0)) - T_s(h)) \\ &\leq \alpha_t(f_0)\rho(T_{t-s}(f_0) - h) \\ &\leq \sup_{t\geq\eta} \alpha_t(f_0)\rho(T_t(f_0) - h), \end{aligned}$$

which implies

$$\varphi(T_s(g)) \leq \inf_{\eta\geq 0} \sup_{t\geq\eta} \alpha_t(f_0)\rho(T_t(f_0) - g).$$

Hence, for all $s \geq 0$, we have $\varphi(T_s(h)) \leq \varphi(h)$. Thus, for all $s \geq 0$, $\{T_s(h_n)\}$ is a minimizing sequence of φ. In view of the uniqueness of h, we obtain $h = T_s(h)$, that is, h is a common fixed point of the semigroup S. □

Remark 3.9. *Theorem 3.8 remains true and reduces to the theorem presented in [6] (see, Theorem 3.22), if we replace the assumption, monotone asymptotic pointwise ρ-nonexpansive semigroup by monotone ρ-nonexpansive semigroup besides retaining rest of the assumptions.*

We highlight the role of asymptotic pointwise assumption by the following remark.

Remark 3.10. *We point out the fact that examples of monotone asymptotic pointwise nonexpansive mappings are not easy to explore. As it was highlighted by Kirk and Xu [12] that the example presented by Goebel and Kirk [7] may be modified to produce an example of a monotone asymptotic nonexpansive mapping. In fact, let C be the positive part of the unit ball in the Hilbert space l^2 and i.e.,*

$$C = \{f_n \in B^+ : f_n \geq 0 \text{ for all } n \geq 1\}.$$

As we know that Hilbert space l^2 is a modular function space, too. Define a transformation $T : C \to C$ by

$$T : (f_1, f_2, f_3, ...) \to (0, f_1^2, a_2 f_2, a_3 f_3, ...).$$

Assume that $a_i \in (0, 1)$ for all $n \geq 1$ and $\prod_{i=2}^{+\infty} a_i = \frac{1}{2}$, then it can be easily shown that T is a monotone asymptotic pointwise nonexpansive which is not nonexpansive. It indicates the utility of asymptotic pointwise assumption and usefulness of our findings.

4. Conclusion

We established two existence theorems for common fixed points under asymptotic pointwise ρ-nonexpansive assumption of monotone semigroups in modular function spaces. Moreover, we also highlight the usefulness of asymptotic pointwise assumption for monotone semigroups by the above remark. In this way, the existence results presented in this paper are generalizations and extension over the several well known recent results of the existing literature.

References

[1] A.A.N. Abdou, M.A. Khamsi, Fixed point theorems in modular vector spaces, J. Nonlinear Sci. Appl., 10 (2017), 4046-4057.
[2] A. Alam, M. Imdad, Comparable linear contractions in ordered metric spaces, Fixed Point Theory, 18 (2017), 415-432.
[3] M. Bachar, M.A. Khamsi, Recent contributions to fixed point theory of monotone mappings, J. Fixed Point Theory Appl., 19 (2015), 1953-1976.
[4] M. Bachar, M.A. Khamsi, W.M. Kozlowski, M. Bounkhel, Common fixed points of monotone Lipschitzian semigroups in Banach spaces, J. Nonlinear Sci. Appl., 11 (2017), 73-79.
[5] B.A. Bin Dehaish, M.A. Khamsi, W.M. Kozlowski, Common fixed points for pointwise Lipschitzian semigroups in modular function spaces, Fixed Point Theory Appl., 2013 (2013), 214.
[6] N. El Harmouchi, K. Chaira, E.M. Marhrani, Common fixed points of monotone ρ-nonexpansive semigroup in modular spaces, Fixed Point Theory Appl., 2020 (2020), 17.
[7] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Am. Math. Soc., 35 (1972), 171-174.
[8] M.A. Khamsi, Nonlinear semigroups in modular function spaces. Math. Jpn., 37 (1992), 1-9.
[9] M.A. Khamsi, W.M. Kozlowski, Fixed Point Theory in Modular Function Spaces, Birkhäuser, Basel (2015).
[10] M.A. Khamsi, W.M. Kozlowski, On asymptotic pointwise contractions in modular function spaces, Nonlinear Anal., 73 (2010), 2957-2967.
[11] M.A. Khamsi, W.M. Kozlowski, On asymptotic pointwise nonexpansive mappings in modular function spaces, J. Math. Anal. Appl., 380 (2011), 697-708.
[12] W.A. Kirk, H.K. Xu, Asymptotic pointwise contractions, Nonlinear Anal., 69 (2008), 4706-4712.
[13] W.M. Kozlowski, Advancements in fixed point theory in modular function spaces, Arab. J. Math., 1 (2012), 477-494.
[14] W.M. Kozlowski, Monotone Lipschitzian semigroups in Banach spaces, J. Aust. Math. Soc., 105 (2018), 417-428.
[15] W.M. Kozlowski, On the existence of common fixed points for semigroups of nonlinear mappings in modular function spaces, Comment. Math., 51 (2011), 81-98.
[16] P. Kumam, Fixed point theorems for nonexpansive mappings in modular spaces, Arch. Math., 40 (2004), 345-353.
[17] J.J. Nieto, R. Rodriguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 22 (2005), 223-239.
[18] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, Springer, New York (1983).
[19] G. Prasad, Fixed point theorems via w-distance in relational metric spaces with an application, Filomat, 34 (2020), 1889-1898.
[20] G. Prasad, R.C. Dimri, Fixed point theorems via comparable mappings in ordered metric spaces, J. Anal., 27 (2019), 1139-1150.
[21] G. Prasad, R.C. Dimri, Coincidence theorems in new generalized metric spaces under locally g-transitive binary relation, J. Indian Math. Soc., 85 (2018), 396-410.
[22] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., 132 (2003), 1435-1443.
[23] M. Turinici, Fixed points for monotone iteratively local contractions, Demonstr. Math., 19 (1986), 171-180.
[24] M. Turinici, Ran-Reurings fixed point results in ordered metric spaces, Libertas Math., 31 (2011), 49-55.