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Hom-symmetric spaces and Hom-Jordan Hom-symmetric spaces

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Abstract. In this paper, we introduce and study the notions of Hom-reflection space and Hom-symmetric space. We provide some examples of Hom-reflection spaces (resp. Hom-symmetric spaces) by using ordinary reflection (resp. symmetric) spaces. Also, we associate a Hom-reflection (resp. Hom-symmetric) space to a Hom-Lie group. Finally, we provide some properties of a Hom-Jordan algebra and show that there is a relationship between Hom-Jordan algebras and Hom-symmetric spaces.

Introduction

Symmetric spaces, which were introduced by O. Loos in [23] have many similarities to Lie groups (only the multiplication is changed to a group multiplication). There is a linear structure on the tangent bundle of a symmetric space, which is called Lie triple system. In fact, this system plays a role similar to Lie algebra for a Lie group. For an impulsive description of physical problems in terms of a theatrical group, one can use the theory of symmetric spaces (see [29] for more details).

The authors of [13] have introduced Hom-Lie algebras, as a generalization of Lie algebras, in the study of quantum deformations of Witt and Virasoro algebras. Because Hom-algebra structures are closely related to the discrete and deformed vector fields, many researchers became interested to this field [13, 20].

The notion of a Hom-group, as a non-associative analogue of a group, appeared in the study of the universal enveloping algebra and elements of group-like type in [21]. Then, M. Hassanzadeh studied some concepts on Hom-groups [14, 15]. Next, J. Jiang, S. K. Mishra and Y. Sheng by adding a smooth manifold structure on a Hom-group introduced the notion of Hom-Lie group in [18]. Also, by defining the left-invariant sections of the pullback bundle of a Hom-Lie group, they associated a Hom-Lie algebra to a Hom-Lie group. Recently, in [28] the authors studied the Kähler-Norden geometry on Hom-Lie groups. Hom-Jordan algebras are commutative algebras where the identities defining the structure are twisted by a homomorphism. They are first introduced by A. Makhlouf in his paper [24].

In this paper, we introduce Hom-reflection spaces and Hom-symmetric spaces which are obtained by twisting the usual identities by a map. We provide some examples of Hom-reflection spaces (resp. Hom-symmetric spaces) by using ordinary reflection (resp. symmetric) spaces. Also, we associate a Hom-reflection (resp. Hom-symmetric) space to a Hom-Lie group. Later on, we extend certain properties of a Jordan algebra to the Hom-version, more precisely the fundamental identity and the notion of inverses

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for a Jordan algebra. Finally, we show that there is a relationship between Hom-Jordan algebras and Hom-symmetric spaces.

The paper is organized as follows. In Section 1, we recall some basic definitions and results concerning reflection spaces, symmetric spaces, Jordan symmetric algebras and Hom-Lie groups. Definitions and examples of Hom-reflection spaces and Hom-symmetric spaces are provided in section 2. In section 3, a Hom-version of the fundamental identity and the notion of inverses for a Jordan algebra has been given. We also show that there is a relationship between Hom-Jordan algebras and Hom-symmetric spaces.

1. Preliminaries

We recall the notion and some basic results of symmetric spaces, Jordan symmetric spaces, Hom-groups and Hom-Lie groups (see [5–7, 11, 17, 22, 23], for more details).

Definition 1.1. *A pair (M, m) is called reflection space if M is a smooth manifold and $m : M \times M \to M,\ (x, y) \mapsto m(x, y) = xy = \sigma_x(y)$ is a smooth product map such that*

(i) $xx = x$,

(ii) $x(xy) = y$, *i.e.*, $\sigma_x^2 = id_M$,

(iii) $x(yz) = (xy)(xz)$, *i.e.*, $\sigma_x \in Aut(M, m)$,

for all $x,\ y,\ z \in M$. Moreover, if 2 is invertible in $\mathbb{K}$ and $T_x(\sigma_x) = -id_{T_xM}$, where $T_x(\sigma_x)$ is the differential of σ_x and id_{T_xM} is the identity of T_xM, then (M, m) is said to be symmetric space (over $\mathbb{K}$).

Remark 1.2. *From (ii) and (iii) of the above definition we conclude that the left multiplication operator σ_x is an automorphism of order two fixing x, which is called the symmetry around x. Also, in the real finite-dimensional case, $T_x(\sigma_x) = -id_{T_xM}$ if and only if for all $x \in M$, the fixed point x of σ_x is isolated.*

Proposition 1.3. *[22, 23] Let (M, m) be a symmetric space. Then the tangent bundle (TM, Tm) of a reflection (resp. symmetric) space is again a reflection (resp. symmetric) space, where*

$$\begin{aligned} Tm(X, Y)(f)(a, b) &= (X, Y)(f \circ m(a, b)) \\ &= X(f \circ R_b)(a) + Y(f \circ L_a)(b), \end{aligned} \tag{1}$$

for all $f \in C^\infty(M)$ and $a, b \in M$ ($L_a(b) = m(a, b) = R_b(a)$ are the left and right translations).

Definition 1.4. *A Jordan algebra is a commutative algebra A with a multiplication $\mu : A \times A \to A$, $(a, b) \mapsto \mu(a, b) = ab$ such that the multiplication satisfies*

$$a(a^2 b) = a^2(ab), \quad \forall a, b \in A. \tag{2}$$

A Jordan algebra is said to be unital if it admits an element e, called the unit, satisfying $ea = a$, for all $a \in A$.

Definition 1.5. *Let A be a unital Jordan algebra over $\mathbb{K}$. An element a in a unital Jordan algebra A is said to be invertible if there is an element b such that $ab = e$ and $a^2 b = a$.*

Theorem 1.6. *Let A be a unital Jordan algebra such that A is a finite dimensional real vector space. Then the set $I(A)$ of invertible elements is an open in A. Endowed with the multiplication $\mu : I(A) \times I(A) \to I(A)$ defined by*

$$\mu(a, b) = P(a)b^{-1}, \quad \forall a, b \in I(A), \tag{3}$$

where $P(a)b = 2a(ab) - a^2 b$, $I(A)$ is a symmetric space, which is called the Jordan symmetric space.

Definition 1.7. *[14] A (regular) Hom-group is a quadruplet (G, μ, e, α) consisting of a set G with a distinguished member e (is called unit) of G, a map $\mu : G \times G \to G$ (multiplication map) and a map $\alpha : G \to G$ (bijection), such that*

1. $\mu(\alpha(g), \mu(h, k)) = \mu(\mu(g, h), \alpha(k))$, $\forall g, h, k \in G$ *(Hom-associativity property),*
2. $\alpha(\mu(g, h)) = \mu(\alpha(g), \alpha(h))$ *(α is multiplicative),*
3. $\mu(g, e) = \mu(e, g) = \alpha(g)$, $\alpha(e) = e$ *(Hom-unitarity condition),*
4. $\forall\, g \in G$, $\exists\, g^{-1} \in G$ *satisfying* $\mu(g, g^{-1}) = \mu(g^{-1}, g) = e$,
5. $\mu(g, h)^{-1} = \mu(h^{-1}, g^{-1})$ *(the inverse map $g \longmapsto g^{-1}$ is an antimorphism).*

Moreover, if G is a smooth manifold, α is a diffeomorphism and the multiplication and inverse maps are smooth, then (G, μ, e, α) is called Hom-Lie group.

2. Hom-reflection space and Hom-symmetric space

Definition 2.1. *An α-manifold is a triple (M, μ, α) in which M is a smooth manifold, $\mu : M \times M \to M$ is a smooth binary map and $\alpha : M \to M$ is a diffeomorphism such that*

$$(\alpha \circ \mu)(a, b) = \mu(\alpha(a), \alpha(b)), \ \text{or} \ \ l_{\alpha(a)} \circ \alpha = \alpha \circ l_a, \ r_{\alpha(b)} \circ \alpha = \alpha \circ r_b, \ \forall a, b \in M,$$

where $l_a(b) = \mu(a, b) = r_b(a)$.

Example 2.2. *Let M be a smooth manifold with a multiplication $\mu : M \times M \to M$. We recall that a map $\alpha : M \to M$ is an automorphism of (M, μ) if it is a diffeomorphism of M and satisfies $(\alpha \circ \mu)(a, b) = \mu(\alpha(a), \alpha(b))$. So, if $\alpha : M \to M$ is an automorphism of (M, μ), the binary operation $a \diamond b = \alpha(\mu(a, b))$ gives an α-manifold structure on M.*

Definition 2.3. *A Hom-reflection space is an α-manifold (M, μ, α) satisfying*

(i) $\mu(a, a) = \alpha(a)$ *or* $l_a(a) = \alpha(a)$,

(ii) $\mu(\alpha(a), \mu(a, b)) = \alpha^2(b)$ *(where $\alpha^2 = \alpha \circ \alpha$) or* $l_{\alpha(a)} \circ l_a \circ \alpha^{-1} = \alpha$,

(iii) $\mu(\alpha(a), \mu(b, c)) = \mu(\mu(a, b), \mu(a, c))$ *or* $l_{\alpha(a)} \circ l_b = l_{l_a(b)} \circ l_a$,

for all a, b, $c \in M$.

Let $\triangle : M \to M \times M$ be the diagonal map on M defined by $\triangle(a) = (a, a)$, $\tau : M \times M \to M \times M$ be the flip map defined by $\tau(a, b) = (b, a)$ and $pr_2 : M \times M \to M$ be the second projection on $M \times M$ defined by $pr_2(a, b) = b$. We can present an equivalent and axiomatic definition of the notion of Hom-reflection space. A Hom-reflection space is a triple (M, μ, α), where M is a smooth manifold, $\mu : M \times M \to M$ is a smooth multiplication and $\alpha : M \to M$ is a diffeomorphism satisfying the following axioms:

(i)

$$\begin{array}{ccc} M \times M & \xrightarrow{\mu} & M \\ \big\downarrow{\scriptstyle \alpha\times\alpha} & & \big\downarrow{\scriptstyle \alpha} \\ M \times M & \xrightarrow{\mu} & M \end{array},$$

(ii)

$$\begin{array}{ccc} M & \xrightarrow{\quad\triangle\quad} & M \times M \\ & {\scriptstyle \alpha}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & M & \end{array},$$

(iii)

$$\begin{array}{ccccc} M \times M & & \xrightarrow{\quad\triangle\times id\quad} & & M \times M \times M \\ \big\downarrow{\scriptstyle pr_2} & & & & \big\downarrow{\scriptstyle \alpha\times\mu} \\ M & \xrightarrow{\alpha^2} & M & \xleftarrow[\mu]{} & M \end{array},$$

(iv)

$$\begin{array}{ccccc} M\times M\times M & \xrightarrow{\triangle\times id\times id} & M\times M\times M\times M & \xrightarrow{id\times\tau\times id} & M\times M\times M\times M \\ \downarrow{\scriptstyle \alpha\times\mu} & & & & \downarrow{\scriptstyle \mu\times\mu} \\ M\times M & \xrightarrow{\quad\mu\quad} & M & \xleftarrow[\mu]{\quad\quad} & M\times M \end{array}.$$

Example 2.4. *Let (M, m) be a reflection space and $\alpha : M \to M$ be an automorphism. Then the triple (M, μ, α) is a Hom-reflection space, where the product μ is given by*

$$\mu(a, b) = m(\alpha(a), \alpha(b)), \quad \forall a,\ b \in M. \tag{4}$$

Remark 2.5. *Let (M, μ, α) be a Hom-reflection space. Then, we get a reflection space structure (M, m) equipped with the product $m : M \times M \to M$ defined by $m(a, b) = \alpha^{-1}(\mu(a, b))$, for all $a,\ b \in M$, which is called the compatible reflection space of (M, μ, α).*

Example 2.6. *Considering a manifold M and a diffeomorphism $\alpha : M \to M$, the operator $\mu(a, b) = \alpha(b)$ for any $a, b \in M$, gives a Hom-reflection space.*

Example 2.7. *Let $(G, \cdot)$ be a Lie group and $\alpha : G \to G$ be a diffeomorphism. Then (G, μ, α) is a Hom-reflection space, where $\mu(g, h) = \alpha(g) \cdot \alpha^2(h^{-1} \cdot g)$, for all $g, h \in G$.*

Example 2.8. *Take $M = \mathbb{R}^n$ with a reflection space structure defined by $m(v, w) = 2v - w$ and let f be the automorphism of $\mathbb{R}^n$ defined by $f(v) = av + b$, for some invertible a and some b. Then the binary map $\mu(v, w) = f(m(v, w)) = 2av - aw + b$ gives a Hom-reflection space structure on M.*

Proposition 2.9. *Let $(G, \diamond, e_\alpha, \alpha)$ be a Hom-Lie group. Defining $\cdot : G \times G \to G$ by*

$$g \cdot h = g \diamond \alpha^{-1}(h^{-1} \diamond g), \tag{5}$$

the triple $(G, \cdot, \alpha)$ is a Hom-reflection space.

Proof. For all $g \in G$, we have

$$g \cdot g = g \diamond \alpha^{-1}(g^{-1} \diamond g) = g \diamond e_\alpha = \alpha(g),$$

i.e., (i) of Definition 2.3 holds. Next, by using the Hom-associativity of α and $\diamond$, one can show that

$$\begin{aligned} \alpha(g) \cdot (g \cdot h) &= \alpha(g) \cdot (g \diamond \alpha^{-1}(h^{-1} \diamond g)) \\ &= \alpha(g) \diamond \Big((\alpha^{-2}(g^{-1} \diamond h) \diamond \alpha^{-1}(g^{-1})) \diamond g\Big) \\ &= \alpha(g) \diamond \alpha^{-1}\Big((g^{-1} \diamond h) \diamond (g^{-1} \diamond g)\Big) \\ &= \alpha(g) \diamond \alpha^{-1}\Big((g^{-1} \diamond h) \diamond e_\alpha\Big) \\ &= \alpha(g) \diamond (g^{-1} \diamond h) \\ &= (g \diamond g^{-1}) \diamond \alpha(h) \\ &= e_\alpha \diamond \alpha(h) \\ &= \alpha^2(h). \end{aligned}$$

So (*ii*) holds. Finally, for all $g,\ h,\ k \in G$, by using the Hom-associativity of α and $\diamond$, we have

$$\begin{aligned}
(g\cdot h)\cdot(g\cdot k) &= (g\cdot h)\diamond\alpha^{-1}\Big((g\cdot k)^{-1}\diamond(g\cdot h)\Big)\\
&= \Big(g\diamond\alpha^{-1}(h^{-1}\diamond g)\Big)\diamond\alpha^{-1}\Big((\alpha^{-1}(g^{-1}\diamond k)\diamond g^{-1})\diamond(g\diamond\alpha^{-1}(h^{-1}\diamond g))\Big)\\
&= \Big(g\diamond\alpha^{-1}(h^{-1}\diamond g)\Big)\diamond\alpha^{-1}\Big((g^{-1}\diamond k)\diamond(g^{-1}\diamond\alpha^{-1}(g\diamond\alpha^{-1}(h^{-1}\diamond g))\Big)\\
&= \Big(g\diamond\alpha^{-1}(h^{-1}\diamond g)\Big)\diamond\alpha^{-1}\Big((g^{-1}\diamond k)\diamond(\alpha^{-1}(g^{-1}\diamond g)\diamond\alpha^{-1}(h^{-1}\diamond g))\Big)\\
&= \Big(g\diamond\alpha^{-1}(h^{-1}\diamond g)\Big)\diamond\alpha^{-1}\Big((g^{-1}\diamond k)\diamond(e_\alpha\diamond\alpha^{-1}(h^{-1}\diamond g))\Big)\\
&= \Big(g\diamond\alpha^{-1}(h^{-1}\diamond g)\Big)\diamond\alpha^{-1}\Big((g^{-1}\diamond k)\diamond(h^{-1}\diamond g)\Big)\\
&= \alpha(g)\diamond\Big(\alpha^{-1}(h^{-1}\diamond g)\diamond\alpha^{-2}((g^{-1}\diamond k)\diamond(h^{-1}\diamond g))\Big)\\
&= \alpha(g)\diamond\Big(\alpha^{-2}((h^{-1}\diamond g)\diamond(g^{-1}\diamond k))\diamond\alpha^{-1}(h^{-1}\diamond g)\Big).
\end{aligned}$$

But

$$\begin{aligned}
(h^{-1}\diamond g)\diamond(g^{-1}\diamond k) &= \alpha(h^{-1})\diamond\Big(g\diamond\alpha^{-1}(g^{-1}\diamond k)\Big)\\
&= \alpha(h^{-1})\diamond\Big(\alpha^{-1}(g\diamond g^{-1})\diamond k\Big)\\
&= \alpha(h^{-1})\diamond(e_\alpha\diamond k)\\
&= \alpha(h^{-1}\diamond k).
\end{aligned}$$

Then

$$\begin{aligned}
(g\cdot h)\cdot(g\cdot k) &= \alpha(g)\diamond\alpha^{-1}\Big((h^{-1}\diamond k)\diamond(h^{-1}\diamond g)\Big)\\
&= \alpha(g)\diamond\Big((\alpha^{-2}(h^{-1}\diamond k)\diamond\alpha^{-1}(h^{-1}))\diamond g\Big)\\
&= \alpha(g)\diamond\alpha^{-1}\Big((\alpha^{-1}(h^{-1}\diamond k)\diamond h^{-1})\diamond\alpha(g)\Big)\\
&= \alpha(g)\diamond\alpha^{-1}\Big((h\diamond\alpha^{-1}(k^{-1}\diamond h))^{-1}\diamond\alpha(g)\Big)\\
&= \alpha(g)\diamond\alpha^{-1}\Big((h\cdot k)^{-1}\diamond\alpha(g)\Big)\\
&= \alpha(g)\cdot(h\cdot k).
\end{aligned}$$

□

Let (M,μ,α) be a Hom-reflection space. The pullback map $\alpha^*: C^\infty(M)\to C^\infty(M)$ is a morphism of the function ring $C^\infty(M)$, i.e.,

$$\alpha^*(fg)=\alpha^*(f)\alpha^*(g),\ \ \forall f,g\in C^\infty(M).$$

A Hom-bundle is a vector bundle $A\to M$ with a smooth map $\alpha: M\to M$ and an algebra morphism $\alpha_A:\Gamma(A)\to\Gamma(A)$ satisfying

$$\alpha_A(fx)=\alpha^*(f)\alpha_A(x),\ \ \forall x\in\Gamma(A),\ f\in C^\infty(M),$$

where $\Gamma(A)$ is the $C^\infty(M)$-module of sections of $A\to M$ and it is denoted by $(A\to M,\alpha,\alpha_A)$. The triple $(\alpha^!TM,\alpha,Ad_{\alpha^*})$ is a famous example of a Hom-bundle, where $\alpha^!TM$ is the pullback bundle of TM along the diffeomorphism $\alpha: M\to M$ and $Ad_{\alpha^*}(x)=\alpha^*\circ x\circ(\alpha^*)^{-1}$, for any $x\in\Gamma(\alpha^!TM)$. Considering

$$x(f)=X(f)\circ\alpha,\ \ \forall f\in C^\infty(M),$$

where $X\in\Gamma(TM)$ and $x\in\Gamma(\alpha^!TM)$, it is seen that there is a one-to-one correspondence between $\Gamma(TM)$ and $\Gamma(\alpha^!TM)$ (see [18], for more details).

Proposition 2.10. *Let (M,μ,α) be a Hom-reflection space and (M,m) be its compatible reflection space. Then $(\Gamma(\alpha^!TM),\alpha^!\mu,Ad_{(\alpha^{-1})^*})$ is a Hom-reflection space, where the product $\alpha^!\mu$ is defined by*

$$\alpha^!\mu(x,y)(f)(a,b)=x(f\circ r_b\circ\alpha^{-1})(a)+y(f\circ l_a\circ\alpha^{-1})(b), \tag{6}$$

for all $f\in C^\infty(M)$ and for all $a,b\in M$.

Proof. Let $x,y\in\Gamma(\alpha^!TM)$ and let X,Y be the corresponding sections of TM, i.e., $x=X\circ\alpha$ and $y=Y\circ\alpha$. Then we obtain

$$\begin{aligned}\alpha^!\mu(x,y)(f)(a,b)&=x(f\circ r_b\circ\alpha^{-1})(a)+y(f\circ l_a\circ\alpha^{-1})(b)\\&=X(f\circ R_{\alpha(b)})(\alpha(a))+Y(f\circ L_{\alpha(a)})(\alpha(b)),\quad\forall f\in C^\infty(M),\ \forall a,b\in M,\end{aligned} \tag{7}$$

where $R_{\alpha(b)}(a)=m(a,\alpha(b))=\mu(\alpha^{-1}(a),b)=r_b\circ\alpha^{-1}(a)$ and $L_{\alpha(a)}(b)=m(\alpha(a),b)=\mu(a,\alpha^{-1}(b))=l_a\circ\alpha^{-1}(b)$. But

$$\begin{aligned}X(f\circ R_{\alpha(b)})(\alpha(a))+Y(f\circ L_{\alpha(a)})(\alpha(b))&=X(f\circ R_{\alpha(b)}\circ\alpha\circ\alpha^{-1})(\alpha(a))+Y(f\circ L_{\alpha(a)}\circ\alpha\circ\alpha^{-1})(\alpha(b))\\&=X((f\circ\alpha)\circ R_b\circ\alpha^{-1})(\alpha(a))+Y((f\circ\alpha)\circ L_a\circ\alpha^{-1})(\alpha(b))\\&=Ad_{(\alpha^{-1})^*}(X((f\circ\alpha)\circ R_b)(a))+Ad_{(\alpha^{-1})^*}(Y((f\circ\alpha)\circ L_a)(b))\\&=Ad_{(\alpha^{-1})^*}(X((f\circ\alpha)\circ R_b)(a)+Y((f\circ\alpha)\circ L_a)(b))\\&=Ad_{(\alpha^{-1})^*}((X,Y)((f\circ\alpha)\circ m(a,b)))\\&=Ad_{(\alpha^{-1})^*}((X,Y)(f\circ m(\alpha(a),\alpha(b))))\\&=Ad_{(\alpha^{-1})^*}(Tm(X,Y)(f)(\alpha(a),\alpha(b))).\end{aligned}$$

So $\alpha^!\mu(x,y)(f)(a,b)=Ad_{(\alpha^{-1})^*}(Tm(X,Y)(f)(\alpha(a),\alpha(b)))$. Therefore, according to Proposition 1.3 and Example2.4, the proposition is proved. □

Definition 2.11. *A Hom-symmetric space is a Hom-reflection space (M,μ,α) satisfying*

$$x(f\circ l_a\circ\alpha^{-1})(a)=-Ad_{\alpha^*}(x)(f)(a),\quad\forall a\in M,\ \forall x\in\Gamma(\alpha^!TM). \tag{8}$$

The examples of Hom-reflection spaces in the previous subsection are also Hom-symmetric spaces.

Let $x\in\Gamma(\alpha^!TM)$ and X be the corresponding sections of TM, i.e., $x=X\circ\alpha$. Then we have

$$X(f\circ L_{\alpha(a)})(\alpha(a))=-X(f)(\alpha(a)).$$

So, $T_{\alpha(a)}L_{\alpha(a)}=-id_{T_{\alpha(a)}M}$ and consequently $T_al_a=T_a(L_{\alpha(a)}\circ\alpha)=(T_{\alpha(a)}L_{\alpha(a)})T_a\alpha=-T_a\alpha$.

Lemma 2.12. *Let (M,μ,α) be a Hom-symmetric space. Then,*

$$x(f\circ r_a\circ\alpha^{-1})(a)=2Ad_{\alpha^*}(x)(f)(a),\ \ \forall a\in M,\ \ \forall x\in\Gamma(\alpha^!TM).$$

Proof. By Proposition 2.10 and (i) of Definition 2.3, we have

$$x(f\circ r_a\circ\alpha^{-1})(a)+x(f\circ l_a\circ\alpha^{-1})(a)=Ad_{\alpha^*}(x)(f)(a).$$

But $x(f\circ l_a\circ\alpha^{-1})(a)=-Ad_{\alpha^*}(x)(f)(a)$. Then

$$x(f\circ r_a\circ\alpha^{-1})(a)=2Ad_{\alpha^*}(x)(f)(a).$$

□

Proposition 2.13. *Let (M,μ,α) be a Hom-symmetric space and (M,m) be its compatible symmetric space. Then $(\Gamma(\alpha^!TM),\alpha^!\mu,Ad_{\alpha^*})$ is a Hom-symmetric space, where*

$$\alpha^!\mu(x,y)(f)(a,b)=x(f\circ r_b\circ\alpha^{-1})(a)+y(f\circ l_a\circ\alpha^{-1})(b),\quad\forall f\in C^\infty(M),\ \ \forall a,b\in M. \tag{9}$$

Moreover

$$\alpha^!\mu(x,y)(f)(a,b)=2Ad_{\alpha^*}(x)(f)(a)-Ad_{\alpha^*}(y)(f)(b). \tag{10}$$

Proof. Similar to Proposition 2.10 we can obtain (9). Also, Lemma 2.12 gives us (10). □

3. Hom-Jordan Hom-symmetric space

Definition 3.1. *A triple (V, μ, α) consisting of a $\mathbb{K}$-linear space V, a bilinear map $\mu : V \times V \to V$ (called multiplication or product) and a linear map $\alpha : V \to V$ is called a Hom-Jordan algebra if μ is commutative and satisfies the Hom-Jordan identity*

$$\mu(\alpha^2(x), \mu(\mu(x, x), y)) = \mu(\alpha(\mu(x, x)), \mu(\alpha(x), y)), \quad \forall x, y \in V, \tag{11}$$

where $\alpha^2 = \alpha \circ \alpha$.

Example 3.2. *If (V, μ, α) is a Hom-associative algebra over $\mathbb{K}$ with product $(x, y) \mapsto \mu(x, y) = xy$, then $V^+ = (V, \mu', \alpha)$ is a Hom-Jordan algebra, where*

$$\mu'(x, y) = \frac{xy + yx}{2}, \quad \forall x, y \in V.$$

Recall that a Hom-associative algebra is a triple (V, μ, α) consisting of a $\mathbb{K}$-linear space V, a linear map $\alpha : V \to V$ and a multiplication $\mu : V \otimes V \to V$ satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)), \quad \forall x, y, z \in V.$$

Definition 3.3. *A Hom-Jordan algebra (V, μ, α) is called multiplicative (resp. regular) if for any $x, y \in V$, $\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y))$ (resp. α is invertible). Also, it is called unital if there exists an element $e \in V$ (called the Hom-unit of V) such that $\alpha(e) = e$ and for all $x \in V$, $\mu(e, x) = \alpha(x)$ (it is denoted by (V, μ, e, α)).*

Proposition 3.4. *If (V, μ, α) is a multiplicative Hom-Jordan algebra, then $(\widehat{V}, \bullet, e, \widehat{\alpha})$ is a unital Hom-Jordan algebra, where*

1. *$\widehat{V} = \mathbb{K} \oplus V$ (any element of $\widehat{V}$ can be written as (a, x), $a \in \mathbb{K}$ and $x \in V$),*
2. *the Hom-unit is $e = (1, 0)$,*
3. *$\widehat{\alpha} = id \times \alpha$,*
4. *For all (a, x), $(b, y) \in \widehat{V}$,*

$$(a, x) \bullet (b, y) = (ab, a\alpha(y) + b\alpha(x) + \mu(x, y)). \tag{12}$$

Proof. For all $x, y \in V$, we write $\mu(x, y) = xy$ and $\mu(x, x) = x^2$. It is evident that the commutativity of the product "$\bullet$" is a consequence of the commutativity of the multiplication μ and the commutativity of the multiplication on $\mathbb{K}$. Let $A = (a, x)$ and $B = (b, y)$ be two elements of $\widehat{V}$, then we have

$$\begin{aligned}
\widehat{\alpha}^2(A) \bullet ((A \bullet A) \bullet B) &= (a, \alpha^2(x)) \bullet ((a^2, 2a\alpha(x) + x^2) \bullet (b, y)) \\
&= (a, \alpha^2(x)) \bullet (a^2b, a^2\alpha(y) + 2ab\alpha^2(x) + b\alpha(x^2) + 2a\alpha(x)y + x^2y) \\
&= (a^3b, a^3\alpha^2(y) + 2a^2b\alpha^3(x) + ab\alpha^2(x^2) + 2a^2\alpha^2(x)\alpha(y) + a\alpha(x^2y) \\
&+ \alpha^2(x)(a^2b\alpha^3(x) + a^2\alpha(y) + 2ab\alpha^2(x) + b\alpha(x^2) + 2a\alpha(x)y + x^2y)) \\
&= (a^3b, a^3\alpha^2(y) + 2a^2b\alpha^3(x) + ab\alpha^2(x^2) + 2a^2\alpha^2(x)\alpha(y) + a\alpha(x^2y) + a^2b\alpha^3(x) \\
&\quad + a^2\alpha^2(x)\alpha(y) + 2ab\alpha^2(x)\alpha^2(x) + b\alpha^2(x)\alpha(x^2) + 2a\alpha^2(x)(\alpha(x)y) + \alpha^2(x)(x^2y)) \\
&= (a^3b, a^3\alpha^2(y) + 3a^2b\alpha^3(x) + 3ab\alpha^2(x^2) + 3a^2\alpha^2(x)\alpha(y) + a\alpha(x^2)\alpha(y) \\
&\quad + b\alpha^2(x)\alpha(x^2) + 2a\alpha^2(x)(\alpha(x)y) + \alpha^2(x)(x^2y)),
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\alpha}(A \bullet A) \bullet (\widehat{\alpha}(A) \bullet B) &= (a^2, 2a\alpha^2(x) + \alpha(x^2)) \bullet ((a, \alpha(x)) \bullet (b, y)) \\
&= (a^2, 2a\alpha^2(x) + \alpha(x^2)) \bullet (ab, a\alpha(y) + b\alpha^2(x) + \alpha(x)y) \\
&= (a^3b, a^3\alpha^2(y) + a^2b\alpha^3(x) + a^2\alpha(\alpha(x)y) + 2a^2b\alpha^3(x) + ab\alpha^2(x^2) \\
&+ (2a\alpha^2(x) + \alpha(x^2))(a\alpha(y) + b\alpha^2(x) + \alpha(x)y)) \\
&= (a^3b, a^3\alpha^2(y) + a^2b\alpha^3(x) + a^2\alpha(\alpha(x)y) + 2a^2b\alpha^3(x) + ab\alpha^2(x^2) + 2a^2\alpha^2(x)\alpha(y) \\
&+ 2ab\alpha^2(x)\alpha^2(x) + 2a\alpha^2(x)(\alpha(x)y) + a\alpha(x^2)\alpha(y) + b\alpha(x^2)\alpha^2(x) + \alpha(x^2)(\alpha(x)y)) \\
&= (a^3b, a^3\alpha^2(y) + 3a^2b\alpha^3(x) + 3ab\alpha^2(x^2) + 3a^2\alpha^2(x)\alpha(y) + a\alpha(x^2)\alpha(y) \\
&+ b\alpha^2(x)\alpha(x^2) + 2a\alpha^2(x)(\alpha(x)y) + \alpha(x^2)(\alpha(x)y)).
\end{aligned}$$

But $\alpha^2(x)(x^2y) = \alpha(x^2)(\alpha(x)y)$. Hence the product "$\bullet$" satisfies the Hom-Jordan Identity, i.e.,

$$\widehat{\alpha}^2(A) \bullet ((A \bullet A) \bullet B) = \widehat{\alpha}(A \bullet A) \bullet (\widehat{\alpha}(A) \bullet B),$$

for all $A = (a, x),\ B = (b, y) \in \widehat{V}$. For all $A = (a, x) \in \widehat{V}$, it follows that

$$e \bullet A = (1, 0) \bullet (a, x) = (a, \alpha(x)).$$

Thus $(\widehat{V}, \bullet, e, \widehat{\alpha})$ is a unital Hom-Jordan algebra. □

3.1. Fundamental identities

Let (V, μ, α) be a multiplicative Hom-Jordan algebra. In terms of left and right multiplications $l(x)y = xy$ and $r(x)y = yx$ the definition of a Hom-Jordan algebra may be written as

$$l(x) = r(x), \tag{13}$$

$$l(\alpha^2(x))l(x^2) = l(\alpha(x^2)l(\alpha(x)), \tag{14}$$

where $l(x)l(y) = l(x) \circ l(y)$. Replace x by $x + \lambda y$ ($\lambda \neq 0 \in \mathbb{K}$) in (14) to obtain $T = l(\alpha(x + \lambda y)^2)l(\alpha(x + \lambda y)) - l(\alpha^2(x + \lambda y))l((x + \lambda y)) = 0$. Using the linearity of $l(x)$, we may write $T = T_0 + \lambda T_1 + \lambda^2 T_2 + \lambda^3 T_3$, where $T_0 = l(\alpha(x^2))l(\alpha(x)) - l(\alpha^2(x))l(x^2)$, $T_1 = 2\Big[l(\alpha(xy))l(\alpha(x)) - l(\alpha^2(x))l(xy)\Big] + l(\alpha(x^2))l(\alpha(y)) - l(\alpha^2(y))l(x^2)$, $T_2 = 2\Big[l(\alpha(xy))l(\alpha(y)) - l(\alpha^2(y))l(xy)\Big] + l(\alpha(y^2))l(\alpha(y)) - l(\alpha^2(x))l(y^2)$ and $T_3 = l(\alpha(y^2))l(\alpha(y)) - l(\alpha^2(y))l(y^2)$. From (14), we have $T = T_0 = T_3 = 0$. So, for every $\lambda \neq 0 \in \mathbb{K}$, we obtain $\lambda T_1 + \lambda^2 T_2 = 0$. Hence $T_1 + \lambda T_2 = 0$, for every $0 \neq \lambda \in \mathbb{K}$.

Lemma 3.5. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then for all $x, y \in V$, we have*

$$2\Big[l(\alpha(xy))l(\alpha(x)) - l(\alpha^2(x))l(xy)\Big] + l(\alpha(x^2))l(\alpha(y)) - l(\alpha^2(y))l(x^2) = 0. \tag{15}$$

Proof. From the above, we have $T_1 + \lambda T_2 = 0$ (for every $\lambda \neq 0 \in \mathbb{K}$), where $T_1 = 2\Big[l(\alpha(xy))l(\alpha(x)) - l(\alpha^2(x))l(xy)\Big] + l(\alpha(x^2))l(\alpha(y)) - l(\alpha^2(y))l(x^2)$, and $T_2 = 2\Big[l(\alpha(xy))l(\alpha(y)) - l(\alpha^2(y))l(xy)\Big] + l(\alpha(y^2))l(\alpha(y)) - l(\alpha^2(x))l(y^2)$. Let λ_1 and λ_2 be two non-zero scalars in $\mathbb{K}$. We see that $T_1 + \lambda_1 T_2 = T_1 + \lambda_2 T_2 = 0$, $(\lambda_1 - \lambda_2)T_2 = 0$, $T_1 = T_2 = 0$. Hence the lemma is proven. □

We next replace x by $x + \lambda z$ in (15) to obtain $U = 2\Big[l(\alpha(xy + \lambda zy))l(\alpha(x) + \lambda\alpha(z)) - l(\alpha^2(x) + \lambda\alpha^2(z))l(xy + \lambda zy)\Big] + l(\alpha(x^2) + 2\lambda\alpha(xz) + \lambda^2\alpha(z^2))l(\alpha(y)) - l(\alpha^2(y))l(x^2 + 2\lambda xz + \lambda^2 z^2) = U_0 + 2\lambda U_1 + \lambda^2 U_2 = 0$. Here $U_0 = T_1 = 0$ and $U_2 = 2\Big[l(\alpha(zy))l(\alpha(z)) - l(\alpha^2(z))l(zy)\Big] + l(\alpha(z^2))l(\alpha(y)) - l(\alpha^2(y))l(z^2) = 0$. So, we have the following lemma:

Lemma 3.6. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then for all $x, y, z \in V$, we have*

$$l(\alpha(xy))l(\alpha(z)) + l(\alpha(yz))l(\alpha(x)) + l(\alpha(xz))l(\alpha(y)) = l(\alpha^2(x))l(yz) + l(\alpha^2(y))l(xz) + l(\alpha^2(z))l(xy). \tag{16}$$

Proof. Computing U_1, we find

$$U_1 = l(\alpha(xy))l(\alpha(z)) + l(\alpha(yz))l(\alpha(x)) + l(\alpha(xz))l(\alpha(y)) - l(\alpha^2(x))l(yz) - l(\alpha^2(y))l(xz) - l(\alpha^2(z))l(xy).$$

From the above, we have $U_1 = 0$. Thus the lemma holds. □

Now, applying the identity (16) to a quantity w and using the commutativity of the product, we obtain

$$\alpha(xy)(w\alpha(z)) + \alpha(xz)(w\alpha(y)) + \alpha(yz)(w\alpha(x)) = ((xy)w)\alpha^2(z) + ((xz)w)\alpha^2(y) + ((yz)w)\alpha^2(x). \tag{17}$$

We write $as_\alpha(x, y, z)$ for the α-associator $(xy)\alpha(z) - \alpha(x)(yz)$. Then equation (17) can be rewritten as

$$as_\alpha(xy, w, \alpha(z)) + as_\alpha(xz, w, \alpha(y)) + as_\alpha(yz, w, \alpha(x)) = 0. \tag{18}$$

By replacing w by $\alpha(w)$, the equation (17) becomes

$$\alpha(xy)\alpha(wz) + \alpha(xz)\alpha(wy) + \alpha(yz)\alpha(wx) = ((xy)\alpha(w))\alpha^2(z) + ((xz)\alpha(w))\alpha^2(y) + ((yz)\alpha(w))\alpha^2(x). \tag{19}$$

Interchange z and w and use the commutativity of the product to write

$$\alpha(xy)\alpha(zw) + \alpha(yz)\alpha(xw) + \alpha(xz)\alpha(yw) = ((xy)\alpha(z))\alpha^2(w) + \alpha^2(y)(\alpha(z)(xw)) + \alpha^2(x)(\alpha(z)(yw)). \tag{20}$$

Then, we have the following lemma:

Lemma 3.7. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then,*

$$\begin{aligned} l(\alpha(xy))l(\alpha(z))\alpha + l(\alpha(yz))l(\alpha(x))\alpha + l(\alpha(xz))l(\alpha(y))\alpha = l((xy)\alpha(z))\alpha^2 + l(\alpha^2(y))l(\alpha(z))l(x) \\ + l(\alpha^2(x))l(\alpha(z))l(y), \end{aligned} \tag{21}$$

for all $x, y, z \in V$.

Lemma 3.8. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then,*

$$\begin{aligned} l((xz)\alpha(y) - (yz)\alpha(x))\alpha^2 = l(\alpha^2(z))\Big[l(\alpha(x))l(y) - l(\alpha(y))l(x)\Big] \\ - \Big[l(\alpha^2(x))l(\alpha(y)) - l(\alpha^2(y))l(\alpha(x))\Big]l(z), \end{aligned} \tag{22}$$

for all $x, y, z \in V$.

Proof. Since the left hand side of (21) is symmetric in y and z, we get

$$\begin{aligned} l(\alpha(xz))l(\alpha(y))\alpha + l(\alpha(yz))l(\alpha(x))\alpha + l(\alpha(xy))l(\alpha(z))\alpha = l((xz)\alpha(y))\alpha^2 + l(\alpha^2(z))l(\alpha(y))l(x) \\ + l(\alpha^2(x))l(\alpha(y))l(z). \end{aligned}$$

Also, the left hand side of the above equation is symmetric in x and y. This shows the lemma. □

An important role in the theory of Jordan algebras plays the so-called quadratic representation. In the sequel, we will extend this notion for Hom-Jordan algebras.

Definition 3.9. *Let (V, μ, α) be a multiplicative Hom-Jordan. The Hom-quadratic representation of V is the map $Q : V \to End(V)$ defined by*

$$Q(x) = 2l(\alpha(x))l(x) - l(x^2)\alpha. \tag{23}$$

Example 3.10. *If (A, μ, α) is a Hom-associative algebra, then the Hom-quadratic representation of A^+ (see example 3.2) is given by $Q(x)y = \alpha(x)(yx) = (xy)\alpha(x)$, for all $x, y \in A$.*

Lemma 3.11. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then the Hom-quadratic representation Q satisfies*

$$l(\alpha^2(x))Q(x) - Q(\alpha(x))l(x) = 0, \quad \forall x \in v. \tag{24}$$

Proof. Let $x \in V$. Considering $\alpha l(x) = l(\alpha(x))\alpha$ we get

$$\begin{aligned} l(\alpha^2(x))Q(x) - Q(\alpha(x))l(x) &= 2l(\alpha^2(x))l(\alpha(x))l(x) - l(\alpha^2(x))l(x^2)\alpha \\ &\quad - 2l(\alpha^2(x))l(\alpha(x))l(x) + l(\alpha(x^2))\alpha l(x) \\ &= -l(\alpha^2(x))l(x^2)\alpha + l(\alpha(x^2))l(\alpha(x))\alpha \\ &= (l(\alpha(x^2))l(\alpha(x)) - l(\alpha^2(x))l(x^2))\alpha \\ &= 0. \end{aligned}$$

Hence the lemma holds. □

Now, set the map $Q(x, y)$ given by

$$Q(x, y) = Q(x + y) - Q(x) - Q(y), \quad \forall x, y \in V. \tag{25}$$

Using the definition of the map Q given by (23), we obtain easily

$$Q(x, y) = 2(l(\alpha(x))l(y) + l(\alpha(y))l(x) - l(xy)\alpha) \text{ and } Q(x, x) = 2Q(x). \tag{26}$$

Lemma 3.12. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then, we have*

$$Q(xy, \alpha(x))\alpha - Q(\alpha(x))l(y) - l(\alpha^2(y))Q(x) = 0, \quad \forall x, y \in V. \tag{27}$$

Proof. Let $x, y \in V$. Using (26), we have

$$\begin{aligned} Q(xy, \alpha(x))\alpha - Q(\alpha(x))l(y) - l(\alpha^2(y))Q(x) &= 2(l(\alpha(xy))l(\alpha(x)) + l(\alpha(\alpha(x)))l(xy) \\ &\quad - l((xy)\alpha(x))\alpha)\alpha - 2l(\alpha^2(x))l(\alpha(x))l(y) + l(\alpha(x^2))l(\alpha(y))\alpha \\ &\quad - 2l(\alpha^2(y))l(\alpha(x))l(x) + l(\alpha^2(y))l(x^2)\alpha \\ &= 2(l(\alpha(xy))l(\alpha(x))\alpha + l(\alpha^2(x))l(xy)\alpha - l((xy)\alpha(x))\alpha^2) \\ &\quad - 2l(\alpha^2(x))l(\alpha(x))l(y) + l(\alpha(x^2))l(\alpha(y))\alpha \\ &\quad - 2l(\alpha^2(y))l(\alpha(x))l(x) + l(\alpha^2(y))l(x^2)\alpha. \end{aligned}$$

But, replacing z by x in (21), we find

$$\begin{aligned} l(\alpha(xy))l(\alpha(x))\alpha - l((xy)\alpha(x))\alpha^2 &= l(\alpha^2(x))l(\alpha(x))l(y) + l(\alpha^2(y))l(\alpha(x))l(x) \\ &\quad - l(\alpha(xy))l(\alpha(x))\alpha - l(\alpha(x^2))l(\alpha(y))\alpha. \end{aligned}$$

Hence,

$$\begin{aligned} Q(xy, \alpha(x))\alpha - Q(\alpha(x))l(y) - l(\alpha^2(y))Q(x) &= 2(l(\alpha^2(x))l(xy) - l(\alpha(xy))l(\alpha(x)))\alpha \\ &\quad + (l(\alpha^2(y))l(x^2) - l(\alpha(x^2))l(\alpha(y))). \end{aligned}$$

So, by (15), $Q(xy, \alpha(x))\alpha - Q(\alpha(x))l(y) - l(\alpha^2(y))Q(x) = 0$. □

Lemma 3.13. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then,*

$$l(\alpha^2(x))Q(x, y) - Q(\alpha(x), \alpha(y))l(x) = Q(\alpha(x))l(y) - l(\alpha^2(y))Q(x), \quad \forall x, y \in V. \tag{28}$$

Proof. This identity is obtained by replacing x by $x + \lambda y$ in (24). □

Next, we set $l(x, y)$ defined by

$$l(x, y)(z) = \{xyz\} = Q(x, z)y, \quad \forall z \in V. \tag{29}$$

Using (26), we observe

$$l(x, y) = 2(l(\alpha(x))l(y) - l(\alpha(y))l(x) + l(xy)\alpha). \tag{30}$$

Remark 3.14. *For all x, y and $z \in V$, $xyz = Q(x, z)y = Q(z, x)y = zyx$.*

Lemma 3.15. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then*

$$Q(\alpha^2(x))l(y, x) = l(\alpha^2(x), \alpha^2(y))Q(x), \quad \forall x, y \in V. \tag{31}$$

Proof. We have

$$\frac{1}{2}Q(\alpha^2(x))l(y, x) = Q(\alpha^2(x))l(\alpha(y))l(x) - Q(\alpha^2(x))l(\alpha(x))l(y) + Q(\alpha^2(x))l(xy)\alpha, \ \ x, \ y \in V.$$

But $Q(\alpha^2(x))l(\alpha(x)) = l(\alpha^3(x))Q(\alpha(x))$ and $Q(\alpha^2(x))l(\alpha(y)) = Q(\alpha(xy), \alpha^2(x))\alpha - l(\alpha^3(y))Q(\alpha(x))$. Hence

$$\begin{aligned}
\frac{1}{2}Q(\alpha^2(x))l(y, x) &= \Big(Q(\alpha(xy), \alpha^2(x))\alpha - l(\alpha^3(y))Q(\alpha(x)\Big)l(x) - l(\alpha^3(x))Q(\alpha(x))l(y) + Q(\alpha^2(x))l(xy)\alpha \\
&= \Big(Q(\alpha(xy), \alpha^2(x))\alpha - l(\alpha^3(y))Q(\alpha(x)\Big)l(x) + Q(\alpha^2(x))l(xy)\alpha \\
&\quad - l(\alpha^3(x))\Big(Q(xy, \alpha(x))\alpha - l(\alpha^2(y))Q(x)\Big) \\
&= Q(\alpha(xy), \alpha^2(x))l(\alpha(x))\alpha - l(\alpha^3(y))l(\alpha^2(x))Q(x) - l(\alpha^3(x))Q(xy, \alpha(x))\alpha \\
&\quad + l(\alpha^3(x))l(\alpha^2(y))Q(x) + Q(\alpha^2(x))l(xy)\alpha \\
&= \Big(Q(\alpha(xy), \alpha^2(x))l(\alpha(x))\alpha - l(\alpha^3(x))Q(xy, \alpha(x))\alpha\Big) \\
&\quad + \Big(l(\alpha^3(x))l(\alpha^2(y)) - l(\alpha^3(y))l(\alpha^2(x))\Big)Q(x) + Q(\alpha^2(x))l(xy)\alpha.
\end{aligned}$$

But (28) implies

$$Q(\alpha(xy), \alpha^2(x))l(\alpha(x)) - l(\alpha^3(x))Q(xy, \alpha(x)) = l(\alpha^2(xy))Q(\alpha(x)) - Q(\alpha^2(x))l(xy).$$

So

$$\begin{aligned}
\frac{1}{2}Q(\alpha^2(x))l(y, x) &= \Big(l(\alpha^3(x))l(\alpha^2(y)) - l(\alpha^3(y))l(\alpha^2(x))\Big)Q(x) + l(\alpha^2(xy))Q(\alpha(x))\alpha \\
&= \frac{1}{2}l(\alpha^2(x), \alpha^2(y))Q(x),
\end{aligned}$$

because $Q(\alpha(x))\alpha = \alpha Q(x)$. □

As a consequence of this lemma, we have the following:

Corollary 3.16. *Let (V, μ, α) be a multiplicative Hom-Jordan. Then*

$$Q(\alpha^2(x))l(y, x) = l(\alpha^2(x), \alpha^2(y))Q(x) = Q(Q(x)y, \alpha^2(x))\alpha^2, \quad \forall x, y \in V, \tag{32}$$

which is called the Homotopy Hom-formula.

Proof. Using (29) and (31) we get

$$Q(\alpha^2(x))l(y,x)z = l(\alpha^2(x),\alpha^2(y))Q(x)z = Q(Q(x)z,\alpha^2(x))\alpha^2(y). \tag{33}$$

Since $Q(\alpha^2(x))l(y,x)z = Q(\alpha^2(x))Q(y,z)x$ and $Q(y,z)$ is symmetric, then $Q(\alpha^2(x))l(y,x)z$ is symmetric with respect to y and z. Therefore (33) implies

$$Q(\alpha^2(x))l(y,x)z = Q(Q(x)y,\alpha^2(x))\alpha^2(z),$$

which conclude the proof. □

So, using the notation in (29), one can show

Proposition 3.17. *Let (V,μ,α) be a multiplicative Hom-Jordan. Then, for all $x, y, z \in V$, we have*

$$\begin{aligned}\{\alpha^4(x)\{\alpha^2(y)\alpha^2(x)\{zxy\}\}\alpha^4(x)\} &= \{\alpha^4(x)\alpha^4(y)\{\alpha^2(x)\{zxy\}\alpha^2(x)\}\} \\ &= \{\alpha^4(x)\alpha^4(y)\{\alpha^2(x)\alpha^2(z)\{xyx\}\}\}.\end{aligned} \tag{34}$$

Proof. Considering (32) and $Q(x,x) = 2Q(x)$, we have

$$\{\alpha^2(x)\{yxz\}\alpha^2(x)\} = \{\alpha^2(x)\alpha^2(y)\{xzx\}\},$$

and so the proposition holds. □

Next, replace x by $x + \lambda z$ in (27), we have the following:

Lemma 3.18. *Let (V,μ,α) be a multiplicative Hom-Jordan. Then, we have*

$$Q(yz,\alpha(x))\alpha + Q(xy,\alpha(z))\alpha = Q(\alpha(x),\alpha(z))l(y) + l(\alpha^2(y))Q(x,z), \quad \forall x,y,z \in V. \tag{35}$$

Corollary 3.19. *Let (V,μ,α) be a multiplicative Hom-Jordan. Then, we have*

$$l(\alpha(x),\alpha(u))l(y) + l(xy,\alpha(u))\alpha = l(\alpha(x),yu)\alpha + l(\alpha^2(y))l(x,u)), \quad \forall x,y,z \in V. \tag{36}$$

Proof. Applying (35) to $u \in V$, we obtain

$$Q(yz,\alpha(x))\alpha(u) + Q(xy,\alpha(z))\alpha(u) = Q(\alpha(x),\alpha(z))l(y)u + l(\alpha^2(y))Q(x,z)u.$$

But

$$Q(yz,\alpha(x))\alpha(u) + Q(xy,\alpha(z))\alpha(u) = l(yz,\alpha(u))\alpha(x) + l(xy,\alpha(u))\alpha(z),$$

and

$$Q(\alpha(x),\alpha(z))l(y)u + l(\alpha^2(y))Q(x,z)u = l(\alpha(x),yu)\alpha(z) + l(\alpha^2(y))l(x,u)z.$$

Computing $l(yz,\alpha(u))\alpha(x)$, we find

$$l(yz,\alpha(u))\alpha(x) = l(\alpha(x),\alpha(u))l(y)z.$$

Hence the corollary holds. □

Using the notation in (29), we obtain by the identity (36)

$$\alpha^2(y)\{uvw\} = \{(yu)\alpha(v)\alpha(w)\} - \{\alpha(u)(yv)\alpha(w)\} + \{\alpha(u)\alpha(v)(yw)\}. \tag{37}$$

Using (22), we get

$$D_2\{uvw\} = \{D_1(u)\alpha^2(v)\alpha^2(w)\} + \{\alpha^2(u)D_1(v)\alpha^2(w)\} + \{\alpha^2(u)\alpha^2(v)D_1(w)\}, \tag{38}$$

where $D_1 = l(\alpha(x))l(y) - l(\alpha(y))l(x)$ and $D_2 = l(\alpha^3(x))l(\alpha^2(y)) - l(\alpha^3(y))l(\alpha^2(x))$. Using (37) and (38) (replace y by xy), we find

$$\{\alpha^2(x)\alpha^2(y)\{uvw\}\} - \{\alpha^2(u)\alpha^2(v)\{xyw\}\} = \{\{xyu\}\alpha^2(v)\alpha^2(w)\} - \{\alpha^2(u)\{yxv\}\alpha^2(w)\}. \tag{39}$$

Also, we have

$$\begin{aligned} l(\alpha^2(x),\alpha^2(y))l(u,v) - l(\alpha^2(u),\alpha^2(v))l(x,y) &= l(\{xyu\},\alpha^2(v))\alpha^2 - l(\alpha^2(u),\{yxv\})\alpha^2 \\ &= l(l(x,y)u,\alpha^2(v))\alpha^2 - l(\alpha^2(u),l(y,x)v)\alpha^2 \\ &= l(Q(x,u)y,\alpha^2(v))\alpha^2 - l(\alpha^2(u),Q(y,v)x)\alpha^2. \end{aligned} \tag{40}$$

A particular case of this equation is (setting $u = x$, $v = y$ and α invertible)

$$l(Q(x)y,\alpha^2(v)) = l(\alpha^2(x),Q(y)x). \tag{41}$$

Furthermore we observe that the left hand side of (39) is skew-symmetric in the pairs (x,y), (u,v), hence

$$\{\{xyu\}\alpha^2(v)\alpha^2(w)\} - \{\alpha^2(u)\{yxv\}\alpha^2(w)\} = \{\alpha^2(x)\{vuy\}\alpha^2(w)\} - \{\{uvx\}\alpha^2(y)\alpha^2(w)\}. \tag{42}$$

Replace u by y , y by u, x by v, v by u and w by v in (42) to write

$$\{\alpha^2(y)\{uvu\}\alpha^2(v)\} = 2\{\{vuy\}\alpha^2(u)\alpha^2(v)\} - \{\alpha^2(v)\{uyu\}\alpha^2(v)\}. \tag{43}$$

Theorem 3.20. *Let (V,μ,α) be a multiplicative Hom-Jordan. Then, for all $u,v \in V$, we have*

$$Q(Q(\alpha^2(u))\alpha^2(v))\alpha^4 = Q(\alpha^4(u))Q(\alpha^2(v))Q(u). \tag{44}$$

If the map α is invertible, the formula (44) becomes

$$Q(Q(u)v)\alpha^2 = Q(\alpha^2(u))Q(v)Q(\alpha^{-2}(u))\alpha^{-2}. \tag{45}$$

This identity is said to be the Hom-fundamental formula.

Proof. Note first of all that we have $\alpha^2(Q(u)v) = Q(\alpha^2(u))\alpha^2(v)$, for all u,v. Next, we substitute x by $\{uvu\} = 2Q(u)v$ and w by u in (39) and rewrite (39) for $x,\alpha^2(y)$, $\alpha^2(u)$, $\alpha^2(v)$ and $\alpha^2(w)$, we find

$$\begin{aligned} 8Q(Q(\alpha^2(u)\alpha^2(v))\alpha^4(y) &= 8Q(\alpha^2(Q(u)v))\alpha^4(y) \\ &= 4\{\alpha^2(Q(u)v)\alpha^4(y)\alpha^2(Q(u)v)\} \\ &= \{\alpha^2(x)\alpha^4(y)\{\alpha^2(u)\alpha^2(v)\alpha^2(u)\}\} \\ &= 2\{\alpha^4(u)\alpha^4(v)\{\alpha^2(u)\alpha^2(y)\{uvu\}\}\} - \{\alpha^4(u)\{\alpha^2(y)\{uvu\}\alpha^2(v)\}\alpha^4(u)\}. \end{aligned}$$

But

$$\{\alpha^2(y)\{uvu\}\alpha^2(v)\} = 2\{\{vuy\}\alpha^2(u)\alpha^2(v)\} - \{\alpha^2(v)\{uyu\}\alpha^2(v)\}.$$

Hence

$$\begin{aligned} 8Q(Q(\alpha^2(u)\alpha^2(v))\alpha^4(y) &= 2\{\alpha^4(u)\alpha^4(v)\{\alpha^2(u)\alpha^2(y)\{uvu\}\}\} - 2\{\alpha^4(u)\{\{vuy\}\alpha^2(u)\alpha^2(v)\}\alpha^4(u)\} \\ &\quad + \{\alpha^4(u)\{\alpha^2(v)\{uyu\}\alpha^2(v)\}\alpha^4(u)\}. \end{aligned}$$

By Proposition 3.17, we have $\{\alpha^4(u)\alpha^4(v)\{\alpha^2(u)\alpha^2(y)\{uvu\}\}\} = \{\alpha^4(u)\{\{vuy\}\alpha^2(u)\alpha^2(v)\}\alpha^4(u)\}$. So

$$\begin{aligned} 8Q(Q(\alpha^2(u)\alpha^2(v))\alpha^4(y) &= \{\alpha^4(u)\{\alpha^2(v)\{uyu\}\alpha^2(v)\}\alpha^4(u)\} \\ &= 8Q(\alpha^4(u))Q(\alpha^2(v))Q(u)y. \end{aligned}$$

Suppose α is invertible. Using $\alpha^2(Q(u)v) = Q(\alpha^2(u))\alpha^2(v)$ and $\alpha^{-2}(Q(u)v) = Q(\alpha^{-2}(u))\alpha^{-2}(v)$, the above identity can be written as $\alpha^2(Q(Q(u)v)\alpha^2(y)) = \alpha^2(Q(\alpha^2(u))Q(v)Q(\alpha^{-2}(u))\alpha^{-2}(y)))$. So $Q(Q(u)v)\alpha^2(y) = Q(\alpha^2(u))Q(v)Q(\alpha^{-2}(u))\alpha^{-2}($
This ends the proof. □

Remark 3.21. *When the twisting map α is equal to the identity map, we recover the usual fundamental formula [19][26][16].*

3.2. Inverses

Definition 3.22. *Let (V, μ, e, α) be a unital multiplicative Hom-Jordan algebra. An element x in V is said to be invertible if there is an element y such that $\mu(x, y) = e$ and $\mu(\mu(x, x), \alpha(y)) = \alpha^2(x)$. In this case y is called the inverse of x and we write $y = x^{-1}$.*

Proposition 3.23. *Let (V, μ, e, α) be a unital multiplicative Hom-Jordan algebra and $x \in V$. If x is invertible with inverse y, then $Q(x)y = \alpha^2(x)$ and $Q(\alpha^3(x))\alpha^2(\mu(y, y)) = e$. If α is a bijection, then x is invertible with the inverse y if and only if $Q(x)y = \alpha^2(x)$ and $Q(\alpha(x))\mu(y, y) = e$.*

Proof. We write $\mu(x, y) = xy$ and $\mu(x, x) = x^2$, for all $x, y \in V$. Let x be an invertible element in V with the inverse y. so, we have

$$Q(x)y = (2l(\alpha(x))l(x) - l(x^2)\alpha)y = 2\alpha(x)(xy) - x^2\alpha(y) = 2\alpha(x)e - \alpha^2(x) = 2\alpha^2(x) - \alpha^2(x) = \alpha^2(x).$$

From the identity (15), we have $2\Big[l(\alpha(xy))l(\alpha(x)) - l(\alpha^2(x))l(xy)\Big] + l(\alpha(x^2))l(\alpha(y)) - l(\alpha^2(y))l(x^2) = 0$. But

$$\begin{aligned}(l(\alpha(xy))l(\alpha(x)) - l(\alpha^2(x))l(xy))u &= (l(e)l(\alpha(x) - l(\alpha^2(x))l(e))u \\ &= e(\alpha(x)u) - \alpha^2(x)(eu) \\ &= \alpha^2(x)\alpha(u) - \alpha^2(x)\alpha(u) \\ &= 0, \ \ \forall u \in V.\end{aligned}$$

So,

$$l(\alpha(x^2))l(\alpha(y)) - l(\alpha^2(y))l(x^2) = 0. \tag{46}$$

Similarly, using (15), one can show

$$l(\alpha(y^2))l(\alpha(x)) - l(\alpha^2(x))l(y^2) = 0. \tag{47}$$

Applying (46) to $\alpha(y)$, we find $\alpha(x^2y^2) = e$. Therefore

$$\begin{aligned}Q(\alpha^3(x))\alpha^2(y^2) &= (2l(\alpha^4(x))l(\alpha^3(x)) - l(\alpha^3(x^2))\alpha)\alpha^2(y^2) \\ &= 2\alpha^4(x)(\alpha^3(x)\alpha^2(y^2)) - \alpha^3(x^2y^2) \\ &= 2\alpha^4(x)\alpha^4(y) - e \\ &= e.\end{aligned}$$

If α is invertible, using $\alpha^2(Q(u)v) = Q(\alpha^2(u))\alpha^2(v)$, we obtain $Q(\alpha(x))y^2 = e$. Now we will show the other way for the invertibility of α, i.e., suppose that $Q(x)y = \alpha^2(x)$ and $Q(\alpha(x))y^2 = e$ and show that x is invertible. The equation $Q(\alpha(x))y^2 = e$ gives $Q(Q(\alpha(x))y^2) = Q(e)$. So $Q(Q(\alpha(x))y^2)\alpha^2 = Q(e)\alpha^2$. Using $Q(e) = \alpha^2$ and (45), we obtain $Q(\alpha^3(x))Q(y^2)Q(\alpha^{-1}(x))\alpha^{-2} = \alpha^4$. Applying α^{-2} from the left to the both sides of this equation and using $\alpha^{-2}Q(u) = Q(\alpha^{-2}(u))\alpha^{-2}$, we get

$$(Q(\alpha(x))\alpha^{-2})(Q(y^2)\alpha^{-2}))Q(\alpha(x)) = \alpha^2. \tag{48}$$

But, from (45), $y^2 = Q(\alpha^{-1}(y))e$ and $\alpha^2(Q(u)v) = Q(\alpha^2(u))\alpha^2(v)$, we have

$$Q(y^2)\alpha^2 = Q(Q(\alpha^{-1}(y))e)\alpha^2 = Q(\alpha(y))Q(e)Q(\alpha^{-3}(y))\alpha^{-2} = Q(\alpha(y))\alpha^2Q(\alpha^{-3}(y))\alpha^{-2} = Q(\alpha(y))Q(\alpha^{-1}(y)).$$

Hence

$$Q(y^2) = Q(\alpha(y))Q(\alpha^{-1}(y))\alpha^{-2} = (Q(\alpha(y))\alpha^{-2})Q(\alpha(y)). \tag{49}$$

Therefore the equation (48) becomes

$$(Q(\alpha(x))\alpha^{-2})(Q(\alpha(y))\alpha^{-2})(Q(\alpha(y))\alpha^{-2})Q(\alpha(x)) = \alpha^2.$$

Applying α^{-2} from the left to both sides of the above equation, we obtain

$$(Q(\alpha(x))\alpha^{-2})(Q(\alpha(y))\alpha^{-2})(Q(\alpha(y))\alpha^{-2})(Q(\alpha(x))\alpha^{-2}) = id, \tag{50}$$

where *id* is the identity map. So both $(Q(\alpha(x))\alpha^{-2})$ and $(Q(\alpha(y))\alpha^{-2})$ are invertible. Applying α to the identity $Q(x)y = \alpha^2(x)$, we obtain $Q(\alpha(x)\alpha(y)) = \alpha^3(x)$. This gives $Q(Q(\alpha(x))\alpha(y))\alpha^2 = Q(\alpha^3(x))\alpha^2 = \alpha^2 Q(\alpha(x))$. So, from (45), we obtain

$$Q(\alpha^3(x))Q(\alpha(y))Q(\alpha^{-1}(x))\alpha^{-2} = \alpha^2 Q(\alpha(x)). \tag{51}$$

Next, from $\alpha^2(Q(u)v) = Q(\alpha^2(u))\alpha^2(v)$ and $\alpha^{-2}(Q(u)v) = Q(\alpha^{-2}(u))\alpha^{-2}(v)$, the identity (51) becomes

$$\alpha^2(Q(\alpha(x))\alpha^{-2})(Q(\alpha(y))\alpha^{-2})Q(\alpha(x)) = \alpha^2 Q(\alpha(x)).$$

Applying α^{-2} from the left to both sides of this equation, we obtain

$$(Q(\alpha(x))\alpha^{-2})(Q(\alpha(y))\alpha^{-2})Q(\alpha(x)) = Q(\alpha(x)).$$

Therefore, by applying α^{-2} from the right to both sides of this equation, we have

$$(Q(\alpha(x))\alpha^{-2})(Q(\alpha(y))\alpha^{-2})(Q(\alpha(x))\alpha^{-2}) = (Q(\alpha(x))\alpha^{-2}). \tag{52}$$

So $Q(\alpha(x))\alpha^{-2}$ and $Q(\alpha(y))\alpha^{-2}$ are each other's inverses. Thus $Q(\alpha(y))$ is invertible and we have $Q(\alpha(y))^{-1} = \alpha^{-2}Q(\alpha(x))\alpha^{-2}$. But, from $\alpha Q(y) = Q(\alpha(y))\alpha$, we have $Q(y) = \alpha^{-1}Q(\alpha(y))\alpha$. So $Q(y)$ is invertible and $Q(y)^{-1} = \alpha^{-2}Q(x)\alpha^{-2}$.
Moreover, we have $Q(\alpha(y))l(y) = l(\alpha^2(y))Q(y)$. Applying on the left $Q(\alpha(y))^{-1}$ for both sides of this equation, we obtain $l(y) = Q(\alpha(y))^{-1}l(\alpha^2(y))Q(y)$. Therefore, by applying on the right $Q(y)^{-1}$, we have $Q(y)^{-1}l(y) = Q(\alpha(y))^{-1}l(\alpha^2(y))$. Hence, from $\alpha^{-2}l(\alpha^2(y) = l(y)\alpha^{-2}$ and by applying α to the above identity, one can show that

$$l(\alpha(y)\alpha^{-1}Q(x)\alpha^{-2} = \alpha^{-1}Q(\alpha(x))l(y)\alpha^{-2}. \tag{53}$$

This acting to e gives $\alpha(y)x^2 = Q(x)y = \alpha^2(x)$. From (53), we have $l(\alpha(y)\alpha^{-1}Q(x) = \alpha^{-1}Q(\alpha(x))l(y)$. Applying α to both sides and using $\alpha l(u) = l(\alpha(u))\alpha$, we find $l(\alpha^2(y))Q(x) = Q(\alpha(x))l(y)$. Acting this identity on y we get $l(\alpha^2(y))Q(x)y = Q(\alpha(x))l(y)y = Q(\alpha(x))y^2$. But $Q(x)y = \alpha^2(x)$ and $Q(\alpha(x))y^2 = e$. Hence $\alpha^2(y)\alpha^2(x) = e$. Thus the proposition holds. □

Proposition 3.24. *Let (V, μ, e, α) be a regular unital multiplicative Hom-Jordan algebra and $x \in V$. x is invertible if and only if $Q(x)$ defines a bijection on V. Moreover $x^{-1} = (Q(x))^{-1}\alpha^2(x)$ and $Q(x)^{-1} = \alpha^{-2}Q(x^{-1})\alpha^{-2}$.*

Proof. If x is invertible then the above implies $Q(x)$ is invertible with inverse $Q(x)^{-1} = \alpha^{-2}Q(x^{-1})\alpha^{-2}$. Any inverse y of x satisfies $Q(x)y = \alpha^2(x)$. So $y = Q(x)^{-1}\alpha^2(x)$. Conversely if $Q(x)$ invertible, then $Q(x)y = \alpha^2(x)$, where $y = Q(x)^{-1}\alpha^2(x)$. So (45) implies $Q(\alpha(x))y^2 = e$. □

Proposition 3.25. *Let (V, μ, e, α) be a regular unital multiplicative Hom-Jordan algebra and $x, y \in V$. $Q(x)y$ is invertible if and only if x and y are invertible, in which case $(Q(x)y)^{-1} = Q(x^{-1})y^{-1}$.*

Proof. We have $Q(x)y$ is invertible if and only if $Q(Q(x)y)$ is invertible. But

$$Q(Q(x)y) = Q(\alpha^2(x))Q(y)Q(\alpha^{-2}(x))\alpha^{-4} = \alpha^2 Q(x)\alpha^{-2}Q(y)\alpha^{-2}Q(x)\alpha^{-2}.$$

Since α is invertible, this implies $Q(x)$ and $Q(y)$ are invertible. Thus x and y are invertible. A simple calculation allows us to prove that the inverse of $Q(x)y$ is $Q(x^{-1})y^{-1}$:

$$\begin{aligned} Q(Q(x)y)Q(x^{-1})y^{-1} &= Q(Q(x)y)\alpha^2\alpha^{-2}Q(x^{-1})y^{-1} \\ &= Q(\alpha^2(x))Q(y)Q(\alpha^{-2}(x))\alpha^{-4}Q(x^{-1})y^{-1} \\ &= Q(\alpha^2(x))Q(y)\alpha^{-2}Q(x)\alpha^{-2}Q(x^{-1})y^{-1}. \end{aligned}$$

But $\alpha^{-2}Q(x)\alpha^{-2} = Q(x^{-1})^{-1}$. Hence

$$Q(Q(x)y)Q(x^{-1})y^{-1} = Q(\alpha^2(x))Q(y)y^{-1} = Q(\alpha^2(x))\alpha^2(y) = \alpha^2(Q(x)y).$$

Similarly, one can show $Q(\alpha(Q(x)y))(Q(x^{-1})y^{-1})^2 = e$. □

Proposition 3.26. *Let (V, μ, e, α) be a regular unital multiplicative Hom-Jordan algebra and $x \in V$ be an invertible element. Then, we have*

$$l(x^{-1}) = Q(\alpha(x))^{-1}l(\alpha^2(x))\alpha^2, \tag{54}$$

and

$$l(\alpha(x^{-1}))l(x) = l(\alpha(x))l(x^{-1}). \tag{55}$$

Proof. Let $x \in V$ be an invertible element. From (32), we have $Q(\alpha^3(a))l(b, \alpha(a)) = Q(Q(\alpha(a))b, \alpha^3(a))\alpha^2$, for all $a, b \in V$. Set $b = y^2$ with $y = x^{-1}$ and $a = x$. Then $Q(\alpha(x))y^2 = e$ and $Q(e, \alpha^3(x)) = 2l(\alpha^4(x))\alpha$. Since $l(\alpha(y^2))l(\alpha(x)) - l(\alpha^2(x))l(y^2) = 0$ (from 47), $l(y, \alpha(x)) = 2l(\alpha^2(y))\alpha$. So $Q(\alpha^3(x))l(\alpha^2(y))\alpha = l(\alpha^4(x))\alpha^3$. Applying α^{-2} for both sides, we obtain $Q(\alpha(x))l(y) = l(\alpha^2(x))\alpha^2$. Thus, $l(y) = Q(\alpha(x))^{-1}l(\alpha^2(x))\alpha^2$. From (53), one can show $l(b)Q(a)^{-1} = Q(\alpha(a))^{-1}l(\alpha^2(b))$, for two invertibles elements $a, b \in V$. Taking $a = b = \alpha(x)$, we have $l(\alpha(x))Q(\alpha(x))^{-1} = Q(\alpha^2(x))^{-1}l(\alpha^3(x))$. Therefore

$$\begin{aligned} l(\alpha(x))l(x^{-1}) &= l(\alpha(x))Q(\alpha(x))^{-1}l(\alpha^2(x))\alpha^2 \\ &= Q(\alpha^2(x))^{-1}l(\alpha^3(x))l(\alpha^2(x))\alpha^2 \\ &= Q(\alpha^2(x))^{-1}l(\alpha^3(x))\alpha^2 l(x) \\ &= l(\alpha(x^{-1}))l(x). \end{aligned}$$

□

Theorem 3.27. *Let (V, μ, e, α) be a regular unital multiplicative Hom-Jordan algebra of finite dimension over $\mathbb{R}$. The set M of invertible elements of V is open in V and becomes a Hom-symmetric space with the product*

$$x \star y = Q(x)y^{-1}, \tag{56}$$

and the twist map α^2, which is called the Hom-Jordan Hom-symmetric space of V.

Proof. Let x be an element of M, i.e., x is invertible. By the Proposition 3.24 this is equivalent to $Q(x)$ is invertible. In finite dimension $Q(x)$ is invertible if and only if $det(Q(x)) \neq 0$. Hence, M is an open. Next, let $x \in M$, we have

$$x \star x = Q(x)x^{-1} = Q(x)Q(x)^{-1}\alpha^2(x) = \alpha^2(x). \tag{57}$$

Hence (i) of Definition 2.3 holds. For all $x, y \in M$, it easily follows that

$$\begin{aligned} \alpha^2(x) \star (x \star y) &= Q(\alpha^2(x))(x \star y)^{-1} \\ &= Q(\alpha^2(x))(Q(x)y^{-1})^{-1} \\ &= Q(\alpha^2(x))Q(x^{-1})y \quad \text{(from Proposition (3.25)} \\ &= (Q(\alpha^2(x))\alpha^2 Q(x)^{-1})\alpha^2(y) \\ &= (\alpha^2 Q(x)Q(x)^{-1})\alpha^2(y) \\ &= \alpha^4(y), \end{aligned}$$

which implies that the product $\star$ satisfies (ii) of Definition 2.3. Next, we have

$$\alpha^2(x) \star (y \star z) = Q(\alpha^2(x))(y \star z)^{-1} = Q(\alpha^2(x))(Q(y)z^{-1})^{-1} = Q(\alpha^2(x))Q(y^{-1})z,$$

and

$$\begin{aligned}(x \star y) \star (x \star z) &= Q(x \star y)(x \star z)^{-1}\\ &= Q(Q(x)y^{-1})Q(x^{-1})z\\ &= Q(Q(x)y^{-1})\alpha^2\alpha^{-2}Q(x^{-1})z\\ &= Q(\alpha^2(x))Q(y^{-1})Q(\alpha^{-2}(x))\alpha^{-4}Q(x^{-1})z\\ &= Q(\alpha^2(x))Q(y^{-1})\alpha^{-2}Q(x)\alpha^{-2}Q(x^{-1})z, \quad \forall x, y, z \in V.\end{aligned}$$

But $\alpha^{-2}Q(x)\alpha^{-2}Q(x^{-1}) = id$. So $(x \star y) \star (x \star z) = Q(\alpha^2(x))Q(y^{-1})z$, and the product $\star$ satisfies (iii) of Definition 2.3. Now, to complete the proof, we must prove that the differential of the left multiplication $s(x)(y) = Q(x) \star y^{-1}$ at a point h is equal to $-d_h\alpha^2$, where $d_h\alpha^2$ is the differential of α^2 at h. But α^2 is a linear map, thus the differential of α^2 is itself. Let $j : M \to V$, $x \to j(x) = x^{-1} = Q(x)^{-1}\alpha^2(x)$. For all $x, h \in M$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}j(x + th) - j(x) &= Q(x + th)^{-1}\alpha^2(x + th) - Q(x)^{-1}\alpha^2(x)\\ &= tQ(x + th)^{-1}\alpha^2(h) + (Q(x + th)^{-1} - Q(x)^{-1})\alpha^2(x)\\ &= tQ(x + th)^{-1}\alpha^2(h) - Q(x)^{-1}(Q(x + th) - Q(x))Q(x + th)^{-1}\alpha^2(x).\end{aligned}$$

But $Q(x + th) = Q(x) + tQ(x, h) + t^2Q(h)$. Thus

$$j(x + th) - j(x) = t\Big[Q(x + th)^{-1}\alpha^2(h) - Q(x)^{-1}(Q(x, h) + tQ(h))Q(x + th)^{-1}\alpha^2(x).$$

Therefore

$$\frac{d}{dt_{|t=0}}(j(x + th) - j(x)) = Q(x)^{-1}\alpha^2(h) - Q(x)^{-1}Q(x, h)Q(x)^{-1}\alpha^2(x).$$

Now, $Q(x, h)Q(x)^{-1}\alpha^2(x) = Q(x, h)x^{-1} = 2\alpha^2(h)$, since $l(\alpha(x^{-1}))l(x) = l(\alpha(x))l(x^{-1})$. It follows that

$$\frac{d}{dt_{|t=0}}(j(x + th) - j(x)) = -Q(x)^{-1}\alpha^2(h).$$

As a result

$$\frac{d}{dt_{|t=0}}Q(x)(j(x + th) - j(x)) = -Q(x)Q(x)^{-1}\alpha^2(h) = -\alpha^2(h).$$

Hence (8) holds and $(M, \star, \alpha^2)$ is a Hom-symmetric space. □

3.3. Examples of Hom-Jordan Hom-symmetric spaces

We construct examples of Hom-Jordan Hom-symmetric spaces using theorem 3.27.

Example 3.28. *(Hom-Jordan Hom-symmetric space of dimension 2)*

Let $E = \{e_1, e_2\}$ be a basis of of a 2−dimensional linear space V over $\mathbb{K}$. The following multiplication μ and linear map α on V define a unital Hom-Jordan algebra, with unit e_1, over $\mathbb{K}^2$:

$$\mu(e_1, e_1) = e_1, \mu(e_1, e_2) = \mu(e_2, e_1) = -e_2, \mu(e_2, e_2) = e_1$$

$\alpha(e_1) = e_1, \alpha(e_2) = -e_2.$

Therefore each element x in the Hom-Jordan algebra (V, μ, e_1, α) is given by its coordinate vector in the base E;

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1, x_2 \in \mathbb{K}.$$

Therefore, for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in (V, μ, e_1, α), the associated matrix of the Hom-quadratic representation $Q(x)$ (definition 3.9) in the base E is of the form

$$Q(x) = \begin{pmatrix} x_1^2 + x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_1^2 + x_2^2 \end{pmatrix}.$$

The determinant of this matrix is equal $det(Q(x)) = (x_1 - x_2)^2(x_1 + x_2)^2$. Then, $Q(x)$ is invertible if and only if $x_1 \neq \pm x_2$. Moreover, the inverse of an element $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in (V, μ, e_1, α) is presented in the base E by the vector $x^{-1} = \frac{1}{x_1^2 - x_2^2}\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$. The set of invertible elements in (V, μ, e_1, α) is $M_2 = \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (V, \mu, e_1, \alpha) \mid x_1 \neq \pm x_2\}$. Therefore, according to the theorem 3.27, $(M_2, \triangleright, id)$ is Hom-symmetric space, where the product $\triangleright$ is defined by:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \triangleright \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{y_1^2 - y_2^2} \begin{pmatrix} y_1(x_1^2 + x_2^2) - 2x_1x_2y_2 \\ 2x_1x_2y_1 - y_2(x_1^2 + x_2^2) \end{pmatrix}.$$

Example 3.29. *(Hom-Jordan Hom-symmetric space of dimension 3)*
Let W be a 3−dimensional linear space over $\mathbb{K}$ with a basis $\mathcal{B} = \{w_1, w_2, w_3\}$. The tuple (W, μ, w_1, β) is a unital Hom-Jordan algebra, with the unit w_1, where the product is given by the following table

W	w_1	w_2	w_3
w_1	w_1	aw_2	bw_3
w_2	aw_2	0	0
w_3	bw_3	0	0

and the linear map $\beta : W \to W$ is defined by

$$\beta(w_1) = w_1, \beta(w_2) = aw_2), \beta(w_3) = bw_3,$$

such that a and b are two non-zero scalars in $\mathbb{K}$.
In the basis $\mathcal{B}$, an element x in W is given by the column vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.
By definition 3.9,the associated matrix of the Hom-quadratic representation $Q(x)$ in the base $\mathcal{B}$ is of the form

$$Q(x) = \begin{pmatrix} x_1^2 & 0 & 0 \\ 2a^2x_1x_2 & a^2x_1^2 & 0 \\ b^2x_1x_3 & 0 & b^2x_1^2 \end{pmatrix}.$$

So, $Q(x)$ is invertible if and only if $x_1 \neq 0$. By the Proposition 3.24, the inverse of invertible element $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in (W, \mu, w_1, \beta)$ is given by

$$x^{-1} = \frac{1}{x_1^2} \begin{pmatrix} x_1 \\ -x_2 \\ -x_3 \end{pmatrix}.$$

Denote the set of invertible elements in (W,μ,w_1,β) *by* M_3. *Thus,* $M_3 = \{x = \begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix} \in (V,\mu,e_1,\alpha) \mid x_1 \neq 0\}$. *According to Theorem 3.27,* $(M_3,\bullet,\varphi)$ *is a Hom-symmetric space, where the product* $\bullet$ *is given by:*

$$\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix} \bullet \begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = \frac{x_1}{y_1^2}\begin{pmatrix} x_1y_1\\ a^2(2x_2y_1 - x_1y_2)\\ b^2(2x_3y_1 - x_1y_3)\end{pmatrix}$$

and the map $\varphi : M_3 \to M_3$ *is defined by*

$$\varphi\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix} x_1\\ a^2x_2\\ b^2x_3\end{pmatrix}.$$

References

[1] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.
[2] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
[3] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.
[4] N. Aizawa and H. Sato, *q-deformation of the Virasoro algebra with central extension*, Phys. Lett. B, **256** (1991), 185–190.
[5] W. Bertram and M. Didry, *Symmetric bundles and representations of Lie triple systems*, J. Gen. Lie Theory Appl., **3**(4) (2009), 261–284.
[6] W. Bertram and K.-H. Neeb, *Projective completions of Jordan pairs. Part II: Manifold structures and symmetric spaces*, Geom. Dedicata, **112**(2) (2005), 73–113.
[7] H. Braun and M. Koecher, *Jordan-Algebren*, Springer-Verlag, Berlin , 1966.
[8] M. Chaichian, P. Kulish and J. Lukierski, *q-deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras,* Phys. Lett. B, **237** (1990), 401–406.
[9] S. Chouaibi, A. Makhlouf, E. Peyghan and I. Basdouri, *Free Hom-groups, Hom-rings and Semisimple modules*, arXiv:2101.03333.
[10] T. L. Curtright and C. K. Zachos, *Deforming maps for quantum algebras*, Phys. Lett. B, **243** (1990), 237–244.
[11] M. Didry, Structures algébriques associéees aux espaces syméetriques, thesis, Institut Elie Cartan, Nancy 2006 (see http://www.iecn.u-nancy.fr/ e didrym/).
[12] Y. Fregier and A. Gohr, *On Hom-type algebras*, J. Gen. Lie Theory Appl., **4** (2010), 16 pages.
[13] J. T. Hartwig, D. Larsson and S. D. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra, **295**(2) (2006), 314–361.
[14] M. Hassanzadeh, *Hom-groups, representations and homological algebra*, Colloquium Mathematicum **158** (2019), 21–38.
[15] M. Hassanzadeh, *Lagrange's theorem For Hom-Groups*, Rocky Mountain J. Math., **49**(3) (2019), 773–787.
[16] N. Jacobson, *Lectures on quadratic Jordan algebras,* Tata Institute of Fundamental Research Lectures on Mathematics, 45, Bombay: Tata Institute of Fundamental Research, MR 0325715 (1969).
[17] N. Jacobson, *Structure and Representations of Jordan Algebras,* American Mathematical Society Colloquium XXXIX, Providence, Rhode Island , 1968.
[18] J. Jiang, S. K. Mishra and Y. Sheng, *Hom-Lie algebras and Hom-Lie groups, integration and differentiation*, Sigma, **16** (2020), 22 pages.
[19] M. Koecher, *The Minnesota notes on Jordan algebras and their applications, Lecture Notes in Mathematics,* 1710, Springer, ISBN 3-540-66360-6, Zbl 1072.17513(1999).
[20] D. Larsson and S. D. Silvestrov, *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*, J. Algebra, **288**(2) (2005), 321–344.
[21] C. Laurent-Gengoux, A. Makhlouf and J. Teles, *Universal algebra of a Hom-Lie algebra and group-like elements*, J. Pure Appl. Algebra, **222**(5), (2018), 1139–1163.
[22] O. Loos, *Spiegelungsräume und homogene symmetrische Räume*, Math. Z., **99** (1967), 141–170.
[23] O. Loos, *Symmetric Spaces I,* Benjamin, New York 1969.
[24] A. Makhlouf, *Hom-alternative and Hom-Jordan algebras,* Int. Electron. J. Algebra, **8** (2010), 177–190.
[25] A. Makhlouf and S.D. Silvestrov, *Hom-algebra structures,* J. Gen. Lie Theory Appl., **2**(2) (2008), 51–64.
[26] K. Meyberg, *Lectures on algebras and triple systems,* University of Virginia (1972).
[27] E. Peyghan and L. Nourmohammadifar, *Almost contact Hom-Lie algebras and Sasakian Hom-Lie Algebras*, J. Algebra Appl., (2022) 2250005 (27 pages).
[28] E. Peyghan, L. Nourmohammadifar, A. Makhlouf and A. Gezer, *Kähler-Norden structures on Hom-Lie groups and Hom-Lie algebras*, arXiv:2002.03436.
[29] H. Tilgner, *Symmetric Spaces in Relativity and Quantum Theories*, Group Theory in Non-Linear Problems, (1974), 143–184.
[30] D. Yau, *Hom-Maltsev, Hom-alternative, and Hom-Jordan algebras*, Int. E. J. Alg., **11** (2012), 177–217.