Filomat 37:10 (2023), 3153–3166
https://doi.org/10.2298/FIL2310153C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The group inverse of certain block complex matrices

Huanyin Chen[a], Marjan Sheibani Abdolyousefi[b]

[a] *School of Mathematics, Hangzhou Normal University, Hangzhou, China*
[b] *Farzanegan Campus, Semnan University, Semnan, Iran*

Abstract. We present new additive results for the group inverse of block complex matrices. As an application, the representations for the group inverse of a block complex matrix are given. These extend the main results of Benítez, Liu and Zhu (Linear Multilinear Algebra, **59**(2011), 279–289).

1. Introduction

In a ring R, an element a is said to have group inverse x if a, x commute, $x = x^2a$ and $a = a^2x$. Such x is unique if exists, and denote it by $a^\#$. In this paper, the ring of interest is $\mathbb{C}^{n\times n}$, the ring of $n \times n$ complex matrices, and the main goal of this paper is to compute the group inverse of matrices of a certain block form. One motivation for considering this problem is the pursuit of a closed-form solution for systems of second-order linear differential equations which may be written in the following vector-valued form: $Ax''(t) + Bx'(t) + Cx(t) = 0$ where $A, B, C \in \mathbb{C}^{n\times n}$ (with A being potentially singular) and x is a C^n-valued function (see [3]).

Recall that $A \in \mathbb{C}^{n\times n}$ has Drazin inverse provided that there exists $X \in \mathbb{C}^{n\times n}$ such that $AX = XA, X = XAX$ and $A^k = A^{k+1}X$ for some $k \in \mathbb{N}$. Such X is unique if it exists, denoted by A^D. The smallest positive integer k such that the preceding conditions hold is called the Drazin index of A, denoted by ind(A). Evidently, A has group inverse if and only if it has Drazin index 1, if and only if $rank(A) = rank(A^2)$. Many authors have investigate group inverse from many different views, e.g., [1, 4–6, 9–12].

In Section 2, we present a new additive result for the group inverse of block complex matrices. Let $P, Q \in \mathbb{C}^{n\times n}$ have group inverses. If $PQ^iP = 0$ for $i = 1, \cdots, n$, then $P + Q$ has group inverse. The explicit formula for $(P + Q)^\#$ is given. This also extends [1, Theorem 2.1] to a wider case.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{m\times n}, C \in \mathbb{C}^{n\times m}, D \in \mathbb{C}^{n\times n}$. It is of interesting to find the group inverse of the block complex matrix M. This problem is quite complicated and was expensively studied by many authors, see for example [1, 5, 14]. In Section 3, we apply our additive results on group inverse to a block complex matrix. The existences and explicit representations for the group inverse of a block complex are thereby given. These extend the main results of Benítez, Liu and Zhu (see [1, Theorem 3.4, Theorem 3.5, Theorem 3.6 and Theorem 3.7]).

2020 *Mathematics Subject Classification*. 15A09, 65F20
Keywords. group inverse; additive result; spectral idempotent, block complex matrix
Received: 14 May 2022; Revised: 07 August 2022; Accepted: 23 September 2022
Communicated by Dijana Mosić
Research supported by the Natural Science Foundation of Zhejiang Province, China (No. LY21A010018).
Email addresses: huanyinchen@aliyun.com (Huanyin Chen), m.sheibani@semnan.ac.ir (Marjan Sheibani Abdolyousefi)

It is attractive to investigate the Drazin (group) inverse of the block matrix $\begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$, where I_n is the identity matrix. This special block matrix is closely connected to the solution of singular differential equation (see [3]). Finally, in Section 4, the existence and the computational formula for the group inverse of the perturbed anti-triangular block complex matrix $\begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$ are given. Evidently, a wider kind of singular differential equations posed by Campbell is thereby solved (see [4]).

Throughout the paper, we denote by $\mathbb{C}$ and $\mathbb{C}^{n\times n}$ the field of all complex numbers and the Banach algebra of all $n\times n$ complex matrices respectively. We use $\mathbb{N}$ to stand for the set of all natural numbers. Let $P \in \mathbb{C}^{n\times n}$. The spectral idempotent $I_n - PP^D$ is denoted by P^π.

2. Additive properties

In this section, we investigate the group inverse of the sum of two group invertible matrices. We may now state:

Theorem 2.1. *Let $P, Q \in \mathbb{C}^{n\times n}$ have group inverses. If $PQ^iP = 0$ for $i = 1, \cdots, n$, then $P + Q$ has group inverse. In this case,*

$$\begin{array}{rcl} (P+Q)^\# & = & Q^\pi P^\# + Q^\# P^\pi - P^\# Q Q^\# - PP^\# Q^\# \\ & + & QQ^\# P^\# QQ^\# + Q^\# PP^\# QQ^\# + QQ^\# PP^\# Q^\#. \end{array}$$

Proof. Let $p(\lambda) = \lambda^n - a_1\lambda^{n-1} - \cdots - a_{n-1}\lambda - a_n$ be the characteristic polynomial of Q. By using Caylay-Hamilton Theorem, $p(Q) = 0$, i.e., $Q^n = a_1Q^{n-1} + \cdots + a_{n-1}Q + a_nI_n$. Then $Q^{n+1} = a_1Q^n + \cdots + a_{n-1}Q^2 + a_nQ$. By hypothesis, $PQ^iP = 0$ for $i = 1, \cdots, n$, and so $PQ^{n+1}P = 0$. By induction, $PQ^iP = 0$ for any $i \in \mathbb{N}$. If Q is not nilpotent, there exists some $m \in \mathbb{N}$ such that $Q^{n+1} = c_nQ^n + \cdots + c_mQ^m (c_m \neq 0)$. Hence $Q^m = ZQ^{m+1}$ for some $Z \in \mathbb{C}[Q]$. This implies that $Q^D = Q^mZ^{m+1}$. Therefore we verify that

$$P(Q^D)^iP = 0, P(QQ^D)^iP$$

for any $i \in \mathbb{N}$. Let

$$\begin{array}{rcl} M & = & Q^\pi P^\# + Q^\# P^\pi - P^\# QQ^\# - PP^\# Q^\# \\ & + & QQ^\# P^\# QQ^\# + Q^\# PP^\# QQ^\# + QQ^\# PP^\# Q^\#. \end{array}$$

Since we have

$$\begin{array}{rcl} PM & = & PQ^\pi P^\# + PQ^\# P^\pi - PP^\# QQ^\# - PQ^\# \\ & + & PQQ^\# P^\# QQ^\# + PQ^\# PP^\# QQ^\# + PQQ^\# PP^\# Q^\# \\ & = & PP^\# Q^\pi, \\ QM & = & QQ^\# P^\pi - QP^\# QQ^\# - QPP^\# Q^\# + QP^\# QQ^\# \\ & + & QQ^\# PP^\# QQ^\# + QPP^\# Q^\# \\ & = & QQ^\# - QQ^\# PP^\# Q^\pi. \end{array}$$

Hence,

$$(P+Q)M = PP^\# Q^\pi + QQ^\# - QQ^\# PP^\# Q^\pi.$$

Moreover, we have

$$\begin{array}{rcl} MP & = & PP^\# - QQ^\# PP^\# - PP^\# Q^\# P + QQ^\# PP^\# Q^\# P \\ & = & Q^\pi PP^\#, \\ MQ & = & Q^\pi P^\# Q + Q^\# P^\pi Q - P^\# Q - PP^\# QQ^\# \\ & + & QQ^\# P^\# Q + Q^\# PP^\# Q + QQ^\# PP^\# QQ^\# \\ & = & P^\pi QQ^\# + QQ^\# PP^\# QQ^\#. \end{array}$$

Hence

$$M(P+Q) = Q^\pi PP^\# + P^\pi QQ^\# + QQ^\# PP^\# QQ^\#.$$

Accordingly,

$$\begin{aligned}(P+Q)M &= PP^{\#}Q^{\pi} + QQ^{\#} - QQ^{\#}PP^{\#}Q^{\pi}\\ &= Q^{\pi}PP^{\#} + P^{\pi}QQ^{\#} + QQ^{\#}PP^{\#}QQ^{\#}\\ &= M(P+Q).\end{aligned}$$

Thus we compute that

$$\begin{aligned}I_n - (P+Q)M &= P^{\pi}Q^{\pi} + QQ^{\#}PP^{\#}Q^{\pi}\\ &= Q^{\pi}P^{\pi} + Q^{\pi}PP^{\#}QQ^{\#}\\ &= I_n - M(P+Q).\end{aligned}$$

Therefore

$$\begin{aligned}&[I_n - (P+Q)M](P+Q)\\ =\;& [P^{\pi}Q^{\pi} + QQ^{\#}PP^{\#}Q^{\pi}](P+Q)\\ =\;& P^{\pi}Q^{\pi}P + QQ^{\#}PP^{\#}Q^{\pi}P\\ =\;& P^{\pi}(I_n - QQ^{\#})P + QQ^{\#}P\\ =\;& -(I_n - PP^{\#})QQ^{\#}P + QQ^{\#}P\\ =\;& 0.\end{aligned}$$

That is, $P+Q = (P+Q)M(P+Q)$.

Also we have

$$\begin{aligned}&M[I_n - M(P+Q)]\\ =\;& M[Q^{\pi}P^{\pi} + Q^{\pi}PP^{\#}QQ^{\#}]\\ =\;& [MQ^{\pi}][P^{\pi} + PP^{\#}QQ^{\#}]\\ =\;& [Q^{\pi}P^{\#}Q^{\pi} + Q^{\#}P^{\pi}Q^{\pi}][P^{\pi} + PP^{\#}QQ^{\#}]\\ =\;& [Q^{\pi}P^{\#}Q^{\pi} + Q^{\#}P^{\pi}Q^{\pi}][I - PP^{\#}Q^{\pi}]\\ =\;& Q^{\#}P^{\pi}Q^{\pi} - Q^{\#}P^{\pi}Q^{\pi}PP^{\#}Q^{\pi}\\ =\;& Q^{\#}P^{\pi}Q^{\pi} - Q^{\#}PP^{\#}Q^{\pi}\\ =\;& 0.\end{aligned}$$

Hence $M = M(P+Q)M$. Therefore $P+Q$ has group inverse M. That is,

$$\begin{aligned}(P+Q)^{\#} &= Q^{\pi}P^{\#} + Q^{\#}P^{\pi} - P^{\#}QQ^{\#} - PP^{\#}Q^{\#}\\ &+ QQ^{\#}P^{\#}QQ^{\#} + Q^{\#}PP^{\#}QQ^{\#} + QQ^{\#}PP^{\#}Q^{\#},\end{aligned}$$

as asserted. □

Corollary 2.2. *([1, Theorem 2.1]) Let $P, Q \in \mathbb{C}^{n\times n}$ have group inverses. If $PQ = 0$, then $P+Q$ has group inverse. In this case,*

$$(P+Q)^{\#} = Q^{\pi}P^{\#} + Q^{\#}P^{\pi}.$$

Proof. This is obvious by Theorem 2.1. □

In [2, Theorem 3.3], Bu investigated the existence and representation for the group inverse of $P+Q$ under $PQP = 0$ and additional conditions for $P, Q \in \mathbb{C}^{n\times n}$. For 2×2 complex matrices P, Q, we give an explicit result as follow.

Corollary 2.3. *Let $P, Q \in \mathbb{C}^{2\times 2}$ have group inverses. If $PQP = 0$, then $P+Q$ has group inverse. In this case,*

$$\begin{aligned}(P+Q)^{\#} &= Q^{\pi}P^{\#} + Q^{\#}P^{\pi} - P^{\#}QQ^{\#} - PP^{\#}Q^{\#}\\ &+ QQ^{\#}P^{\#}QQ^{\#} + Q^{\#}PP^{\#}QQ^{\#} + QQ^{\#}PP^{\#}Q^{\#}.\end{aligned}$$

Proof. This is obvious by Theorem 2.1. □

We demonstrate Theorem 2.1 by the following numerical example.

Example 2.4. *Let* $P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$. *Then* P, Q *are idempotents, and so they have group inverses. Clearly,* $PQ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$. *In this case,* $PQP = PQ^2P = 0$. *Then* $P+Q$ *has group inverse. In this case,*

$$(P+Q)^{\#} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

3. Block complex matrices

We now apply our preceding theorem to the group inverse of the block complex matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{m\times n}, C \in \mathbb{C}^{n\times m}, D \in \mathbb{C}^{n\times n}$. These also extend the main results of Benítez, Liu and Zhu (see [1, Theorem 3.4, Theorem 3.5, Theorem 3.6 and Theorem 3.7]) to wider cases. We can derive

Theorem 3.1. *Let* A *and* D *have group inverses. If* $BD^iCA = 0, BD^iCB = 0$ *for* $i = 0, \cdots, n-1$ *and* $A^{\pi}B = 0, D^{\pi}C = 0$, *then* M *has group inverse. In this case,*

$$M^{\#} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned}
\Gamma &= A^{\#} - (A^{\#})^2BD^{\#}C - A^{\#}B(D^{\#})^2C,\\
\Delta &= (A^{\#})^2B - (A^{\#})^2BDD^{\#} - A^{\#}BD^{\#},\\
\Lambda &= -D^{\#}CA^{\#} + (D^{\#})^2CA^{\pi} + D^{\#}C(A^{\#})^2BD^{\#}C + (D^{\#})^2CA^{\#}BD^{\#}C + D^{\#}CA^{\#}B(D^{\#})^2C,\\
\Xi &= -D^{\#}C(A^{\#})^2B - (D^{\#})^2CA^{\#}B + D^{\#}C(A^{\#})^2BDD^{\#} + (D^{\#})^2CA^{\#}BDD^{\#} + D^{\#}CA^{\#}BD^{\#}.
\end{aligned}$$

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}.$$

Since $A^{\pi}B = 0, D^{\pi}C = 0$, it follows by [9, Theorem 2.1] that P and Q have group inverses. Moreover, we have

$$P^{\#} = \begin{pmatrix} A^{\#} & (A^{\#})^2B \\ 0 & 0 \end{pmatrix}, P^{\pi} = \begin{pmatrix} A^{\pi} & -A^{\#}B \\ 0 & I_n \end{pmatrix};$$
$$Q^{\#} = \begin{pmatrix} 0 & 0 \\ (D^{\#})^2C & D^{\#} \end{pmatrix}, Q^{\pi} = \begin{pmatrix} I_m & 0 \\ -D^{\#}C & D^{\pi} \end{pmatrix}.$$

Moreover, we compute that

$$\begin{aligned}
Q^{\pi}P^{\#} &= \begin{pmatrix} A^{\#} & (A^{\#})^2B \\ -D^{\#}CA^{\#} & -D^{\#}C(A^{\#})^2B \end{pmatrix},\\
Q^{\#}P^{\pi} &= \begin{pmatrix} 0 & 0 \\ (D^{\#})^2CA^{\pi} & D^{\#} - (D^{\#})^2CA^{\#}B \end{pmatrix},\\
P^{\#}QQ^{\#} &= \begin{pmatrix} (A^{\#})^2BD^{\#}C & (A^{\#})^2BDD^{\#} \\ 0 & 0 \end{pmatrix},\\
PP^{\#}Q^{\#} &= \begin{pmatrix} A^{\#}B(D^{\#})^2C & A^{\#}BD^{\#} \\ 0 & 0 \end{pmatrix},\\
QQ^{\#}P^{\#}QQ^{\#} &= \begin{pmatrix} 0 & 0 \\ D^{\#}C(A^{\#})^2BD^{\#}C & D^{\#}C(A^{\#})^2BDD^{\#} \end{pmatrix},\\
Q^{\#}PP^{\#}QQ^{\#} &= \begin{pmatrix} 0 & 0 \\ (D^{\#})^2CA^{\#}BD^{\#}C & (D^{\#})^2CA^{\#}BDD^{\#} \end{pmatrix},\\
QQ^{\#}PP^{\#}Q^{\#} &= \begin{pmatrix} 0 & 0 \\ D^{\#}CA^{\#}B(D^{\#})^2C & D^{\#}CA^{\#}BD^{\#} \end{pmatrix}.
\end{aligned}$$

We easily check that

$$Q^i = \begin{pmatrix} 0 & 0 \\ D^{i-1}C & D^i \end{pmatrix}.$$

By using Caylay-Hamilton Theorem, we prove that $BD^{i-1}CA = 0, BD^{i-1}CB = 0$ for $i \in \mathbb{N}$. Therefore

$$\begin{aligned} PQ^iP &= \begin{pmatrix} BD^{i-1}C & BD^{i-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} BD^{i-1}CA & BD^{i-1}CB \\ 0 & 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

for any $i \in \mathbb{N}$. In light of Theorem 2.1, M has group inverse. In this case,

$$\begin{aligned} M^{\#} &= Q^{\pi}P^{\#} + Q^{\#}P^{\pi} - P^{\#}QQ^{\#} - PP^{\#}Q^{\#} \\ &+ QQ^{\#}P^{\#}QQ^{\#} + Q^{\#}PP^{\#}QQ^{\#} + QQ^{\#}PP^{\#}Q^{\#} \\ &= \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Gamma &= A^{\#} - (A^{\#})^2BD^{\#}C - A^{\#}B(D^{\#})^2C, \\ \Delta &= (A^{\#})^2B - (A^{\#})^2BDD^{\#} - A^{\#}BD^{\#}, \\ \Lambda &= -D^{\#}CA^{\#} + (D^{\#})^2CA^{\pi} + D^{\#}C(A^{\#})^2BD^{\#}C \\ &+ (D^{\#})^2CA^{\#}BD^{\#}C + D^{\#}CA^{\#}B(D^{\#})^2C, \\ \Xi &= -D^{\#}C(A^{\#})^2B - (D^{\#})^2CA^{\#}B + D^{\#}C(A^{\#})^2BDD^{\#} \\ &+ (D^{\#})^2CA^{\#}BDD^{\#} + D^{\#}CA^{\#}BD^{\#}. \end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let A and D have group inverses. If $CA^iBC = 0$, $CA^iBD = 0$ for $i = 0, \cdots, m-1$ and $A^{\pi}B = 0$, $D^{\pi}C = 0$, then M has group inverse. In this case,*

$$M^{\#} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned} \Gamma &= -A^{\#}B(D^{\#})^2C - (A^{\#})^2BD^{\#}C + A^{\#}B(D^{\#})^2CAA^{\#} \\ &+ (A^{\#})^2BD^{\#}CAA^{\#} + A^{\#}BD^{\#}CA^{\#}, \\ \Delta &= -A^{\#}BD^{\#} + (A^{\#})^2BD^{\pi} + A^{\#}B(D^{\#})^2CA^{\#}B \\ &+ (A^{\#})^2BD^{\#}CA^{\#}B + A^{\#}BD^{\#}C(A^{\#})^2B, \\ \Lambda &= (D^{\#})^2C - (D^{\#})^2CAA^{\#} - D^{\#}CA^{\#}, \\ \Xi &= D^{\#} - (D^{\#})^2CA^{\#}B - D^{\#}C(A^{\#})^2B. \end{aligned}$$

Proof. In view of Theorem 3.1, $N := \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ has group inverse. Moreover,

$$N^{\#} = \begin{pmatrix} \Xi & \Lambda \\ \Delta & \Gamma \end{pmatrix},$$

where

$$\begin{aligned} \Xi &= D^{\#} - (D^{\#})^2CA^{\#}B - D^{\#}C(A^{\#})^2B, \\ \Lambda &= (D^{\#})^2C - (D^{\#})^2CAA^{\#} - D^{\#}CA^{\#}, \\ \Delta &= -A^{\#}BD^{\#} + (A^{\#})^2BD^{\pi} + A^{\#}B(D^{\#})^2CA^{\#}B \\ &+ (A^{\#})^2BD^{\#}CA^{\#}B + A^{\#}BD^{\#}C(A^{\#})^2B, \\ \Gamma &= -A^{\#}B(D^{\#})^2C - (A^{\#})^2BD^{\#}C + A^{\#}B(D^{\#})^2CAA^{\#}. \end{aligned}$$

We easily see that

$$M = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix},$$

and so

$$M^{\#} = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix} N^{\#} \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix},$$

as required. □

We now turn to use the alternative splitting approach for a complex block matrix and derive the following.

Theorem 3.3. *Let A and D have group inverses. If $ABD^iC = 0$, $CBD^iC = 0$ for $i = 0, \cdots, n-1$ and $CA^{\pi} = 0$, $BD^{\pi} = 0$, then M has group inverse. In this case,*

$$M^{\#} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned} \Gamma &= A^{\#} - BD^{\#}C(A^{\#})^2 + B(D^{\#})^2CA^{\#}, \\ \Delta &= -A^{\#}BD^{\#} - AA^{\#}B(D^{\#})^2 + BD^{\#}C(A^{\#})^2BD^{\#} \\ &+ B(D^{\#})^2CA^{\#}BD^{\#} + BD^{\#}CA^{\#}B(D^{\#})^2, \\ \Lambda &= D^{\pi}C(A^{\#})^2 + D^{\#}CA^{\#}, \\ \Xi &= -C(A^{\#})^2BD^{\#} - CA^{\#}B(D^{\#})^2 + DD^{\#}C(A^{\#})^2BD^{\#} \\ &+ D^{\#}CA^{\#}BD^{\#} + DD^{\#}CA^{\#}B(D^{\#})^2. \end{aligned}$$

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}.$$

Since $CA^{\pi} = 0, BD^{\pi} = 0$, it follows by [9, Theorem 2.1] that P and Q have group inverses. Moreover, we have

$$P^{\#} = \begin{pmatrix} A^{\#} & 0 \\ C(A^{\#})^2 & 0 \end{pmatrix}, P^{\pi} = \begin{pmatrix} A^{\pi} & 0 \\ -CA^{\#} & I_n \end{pmatrix};$$
$$Q^{\#} = \begin{pmatrix} 0 & B(D^{\#})^2 \\ 0 & D^{\#} \end{pmatrix}, Q^{\pi} = \begin{pmatrix} I_m & -BD^{\#} \\ 0 & D^{\pi} \end{pmatrix}.$$

We easily check that

$$Q^i = \begin{pmatrix} 0 & BD^{i-1} \\ 0 & BD^i \end{pmatrix},$$

and so

$$\begin{aligned} PQ^iP &= \begin{pmatrix} 0 & ABD^{i-1} \\ 0 & CBD^{i-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \\ &= \begin{pmatrix} ABD^{i-1}C & 0 \\ CBD^{i-1}C & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

In light of Theorem 2.1, M has group inverse. We compute that

$$\begin{array}{rcl} Q^{\pi}P^{\#} & = & \begin{pmatrix} A^{\#}-BD^{\#}C(A^{\#})^2 & 0 \\ D^{\pi}C(A^{\#})^2 & 0 \end{pmatrix}, \\ Q^{\#}P^{\pi} & = & \begin{pmatrix} B(D^{\#})^2CA^{\#} & 0 \\ D^{\#}CA^{\#} & 0 \end{pmatrix}, \\ P^{\#}QQ^{\#} & = & \begin{pmatrix} 0 & A^{\#}BD^{\#} \\ 0 & C(A^{\#})^2BD^{\#} \end{pmatrix}, \\ PP^{\#}Q^{\#} & = & \begin{pmatrix} 0 & AA^{\#}B(D^{\#})^2 \\ 0 & CA^{\#}B(D^{\#})^2 \end{pmatrix}, \\ QQ^{\#}P^{\#}QQ^{\#} & = & \begin{pmatrix} 0 & BD^{\#}C(A^{\#})^2BD^{\#} \\ 0 & DD^{\#}C(A^{\#})^2BD^{\#} \end{pmatrix}, \\ Q^{\#}PP^{\#}QQ^{\#} & = & \begin{pmatrix} 0 & B(D^{\#})^2CA^{\#}BD^{\#} \\ 0 & D^{\#}CA^{\#}BD^{\#} \end{pmatrix}, \\ QQ^{\#}PP^{\#}Q^{\#} & = & \begin{pmatrix} 0 & BD^{\#}CA^{\#}B(D^{\#})^2 \\ 0 & DD^{\#}CA^{\#}B(D^{\#})^2 \end{pmatrix}. \end{array}$$

We easily check that

$$Q^i = \begin{pmatrix} 0 & 0 \\ D^{i-1}C & D^i \end{pmatrix},$$

and so

$$\begin{array}{rcl} PQ^iP & = & \begin{pmatrix} BD^{i-1}C & BD^{i-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\ & = & \begin{pmatrix} BD^{i-1}CA & BD^{i-1}CB \\ 0 & 0 \end{pmatrix} \\ & = & 0 \end{array}$$

for any $i \in \mathbb{N}$. Therefore we have

$$\begin{array}{rcl} M^{\#} & = & Q^{\pi}P^{\#}+Q^{\#}P^{\pi}-P^{\#}QQ^{\#}-PP^{\#}Q^{\#} \\ & + & QQ^{\#}P^{\#}QQ^{\#}+Q^{\#}PP^{\#}QQ^{\#}+QQ^{\#}PP^{\#}Q^{\#} \\ & = & \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix}, \end{array}$$

where

$$\begin{array}{rcl} \Gamma & = & A^{\#}-BD^{\#}C(A^{\#})^2+B(D^{\#})^2CA^{\#}, \\ \Delta & = & -A^{\#}BD^{\#}-AA^{\#}B(D^{\#})^2+BD^{\#}C(A^{\#})^2BD^{\#} \\ & + & B(D^{\#})^2CA^{\#}BD^{\#}+BD^{\#}CA^{\#}B(D^{\#})^2, \\ \Lambda & = & D^{\pi}C(A^{\#})^2+D^{\#}CA^{\#}, \\ \Xi & = & -C(A^{\#})^2BD^{\#}-CA^{\#}B(D^{\#})^2+DD^{\#}C(A^{\#})^2BD^{\#} \\ & + & D^{\#}CA^{\#}BD^{\#}+DD^{\#}CA^{\#}B(D^{\#})^2. \end{array}$$

This completes the proof. □

Corollary 3.4. *Let A and D have group inverses. If $DCA^iB = 0$, $BCA^iB = 0$ for $i = 0, 1, \cdots, m-1$ and $BD^{\pi} = 0$, $CA^{\pi} = 0$, then M has group inverse. In this case,*

$$M^{\#} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{array}{rcl} \Gamma & = & -B(D^\#)^2CA^\# - BD^\#C(A^\#)^2 + AA^\#B(D^\#)^2CA^\# \\ & + & A^\#BD^\#CA^\# + AA^\#BD^\#C(A^\#)^2, \\ \Delta & = & A^\pi B(D^\#)^2 + A^\#BD^\#, \\ \Lambda & = & -D^\#CA^\# - DD^\#C(A^\#)^2 + CA^\#B(D^\#)^2CA^\# \\ & + & C(A^\#)^2BD^\#CA^\# + CA^\#BD^\#C(A^\#)^2, \\ \Xi & = & D^\# - CA^\#B(D^\#)^2 + C(A^\#)^2BD^\#. \end{array}$$

Proof. In view of Theorem 3.3, $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$ has group inverse. We easily check that

$$M = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix}\begin{pmatrix} D & C \\ B & A \end{pmatrix}\begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}.$$

Analogously to Corollary 3.2, we obtain the result. □

4. Special block matrices

The aim of this section is to present existences and computational formulas for the group inverse of the anti-triangular block complex matrix $M = \begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix}$ under the weaker perturbation condition. These also provide algebraic method to find all function solutions of a new class of singular differential equations posed by Campbell (see [1]). In [13, Theorem 2.10], Zou et al. investigated the group inverse of the preceding M under the condition $EF = 0$, We now extend their result to a wider case.

Theorem 4.1. *Let $E, F \in \mathbb{C}^{n\times n}$ have group inverses. If $EF^iE = 0$ for $i = 1, \cdots, n$, then $M = \begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix}$ has group inverse if and only if $F^\pi E^\pi F^\pi = 0$. In this case,*

$$M^\# = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{array}{rcl} \Gamma & = & E^\# + E^\#FF^\# + 2EF^\# + FF^\#E^\# - FF^\#E^\#FF^\# \\ & - & 2FF^\#EF^\#, \\ \Delta & = & (E^\#)^2 + (E^\#)^2FF^\# + EE^\#F^\# + FF^\#(E^\#)^2 + F^\#E^\pi \\ & - & FF^\#(E^\#)^2FF^\# + F^\#EE^\#FF^\# - FF^\#EE^\#F^\#, \\ \Lambda & = & FF^\#E^\pi + F(E^\#)^2FF^\# + F(E^\#)^2FF^\# + FF^\#EE^\#FF^\# \\ & + & 2FEE^\#F^\#, \\ \Xi & = & -FF^\#E^\# + F(E^\#)^3FF^\# + F(E^\#)^3FF^\# \\ & + & FF^\#E^\#FF^\# + 2FE^\#F^\#. \end{array}$$

Proof. Clearly, we have

$$M^2 = \begin{pmatrix} E^2 + F & E \\ FE & F \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} E^2 & E \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} F & 0 \\ FE & F \end{pmatrix}.$$

Since E, F have group inverses, it follows by [9, Theorem 2.1] that P and Q have group inverses. Since $EF^iE = 0$ for $i = 1, \cdots, n-1$, By using Caylay-Hamilton Theorem, we have $EF^iE = 0$ for any $i \in \mathbb{N}$. Therefore

$$\begin{aligned} PQ^iP &= \begin{pmatrix} E^2F^i & E^2F^{i-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E^2 & E \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} E^2F^iE^2 & E^2F^iE \\ 0 & 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

for any $i \in \mathbb{N}$. In light of Theorem 2.1, M^2 has group inverse. Moreover, we have

$$\begin{aligned} (M^2)^\# &= Q^\pi P^\# + Q^\# P^\pi - P^\# QQ^\# - PP^\# Q^\# \\ &+ QQ^\# P^\# QQ^\# + Q^\# PP^\# QQ^\# + QQ^\# PP^\# Q^\#. \end{aligned}$$

Hence,

$$\begin{aligned} MM^D &= (P+Q)(M^2)^\# \\ &= PP^\# + QQ^\# - PP^\# QQ^\# - QQ^\# PP^\# Q^\pi. \end{aligned}$$

In light of [9, Theorem 2.1], we have

$$P^\# = \begin{pmatrix} (E^\#)^2 & (E^\#)^3 \\ 0 & 0 \end{pmatrix}, P^\pi = \begin{pmatrix} E^\pi & -E^\# \\ 0 & I_n \end{pmatrix};$$
$$Q^\# = \begin{pmatrix} F^\# & 0 \\ X & F^\# \end{pmatrix}; Q^\pi = \begin{pmatrix} F^\pi & 0 \\ XF + FF^\# E & F^\pi \end{pmatrix},$$

where

$$X = F^\# EF^\pi - F^\# FEF^\#.$$

Clearly,

$$XF + FF^\# E = FF^\# EF^\pi.$$

We easily check that

$$\begin{aligned} PP^\# &= \begin{pmatrix} EE^\# & E^\# \\ 0 & 0 \end{pmatrix}, \\ QQ^\# &= \begin{pmatrix} FF^\# & 0 \\ -FF^\# EF^\pi & FF^\# \end{pmatrix}, \\ PP^\# QQ^\# &= \begin{pmatrix} EE^\# FF^\# & E^\# FF^\# \\ 0 & 0 \end{pmatrix}, \\ QQ^\# PP^\# Q^\pi &= \begin{pmatrix} FF^\# EE^\# F^\pi & FF^\# E^\# F^\pi \\ -FF^\# EF^\pi & -FF^\# EE^\# F^\pi \end{pmatrix}. \end{aligned}$$

Hence,

$$MM^D = \begin{pmatrix} FF^\# + F^\pi EE^\# F^\pi & F^\pi E^\# F^\pi \\ 0 & FF^\# + FF^\# EE^\# F^\pi \end{pmatrix}.$$

Hence,

$$I_{2n} - MM^D = \begin{pmatrix} F^\pi E^\pi F^\pi & -F^\pi E^\# F^\pi \\ 0 & (I_n - FF^\# EE^\#)F^\pi \end{pmatrix}.$$

Accordingly,

$$\begin{aligned} (I_{2n} - MM^D)M &= \begin{pmatrix} F^\pi E^\pi F^\pi & -F^\pi E^\# F^\pi \\ 0 & (I_n - FF^\# EE^\#)F^\pi \end{pmatrix}\begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix} \\ &= \begin{pmatrix} F^\pi E^\pi F^\pi E & F^\pi E^\pi F^\pi \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore M has group inverse if and only if $F^\pi E^\pi F^\pi = 0$.

Moreover, we compute that

$$\begin{aligned}
Q^{\pi}P^{\#} &= \begin{pmatrix} F^{\pi}(E^{\#})^2 & F^{\pi}(E^{\#})^3 \\ FF^{\#}E^{\#} & FF^{\#}(E^{\#})^2 \end{pmatrix}, \\
Q^{\#}P^{\pi} &= \begin{pmatrix} F^{\#}E^{\pi} & -F^{\#}E^{\#} \\ -F^{\#}EFF^{\#} - F^{\#}FEF^{\#} & F^{\#}E^{\pi} \end{pmatrix}, \\
P^{\#}QQ^{\#} &= \begin{pmatrix} (E^{\#})^2FF^{\#} & (E^{\#})^3FF^{\#} \\ 0 & 0 \end{pmatrix}, \\
PP^{\#}Q^{\#} &= \begin{pmatrix} EE^{\#}F^{\#} & E^{\#}F^{\#} \\ 0 & 0 \end{pmatrix}, \\
QQ^{\#}P^{\#}QQ^{\#} &= \begin{pmatrix} FF^{\#}(E^{\#})^2FF^{\#} & FF^{\#}(E^{\#})^3FF^{\#} \\ -FF^{\#}E^{\#}FF^{\#} & -FF^{\#}(E^{\#})^2FF^{\#} \end{pmatrix}, \\
Q^{\#}PP^{\#}QQ^{\#} &= \begin{pmatrix} F^{\#}EE^{\#}FF^{\#} & F^{\#}E^{\#}FF^{\#} \\ F^{\#}EFF^{\#} & F^{\#}EE^{\#}FF^{\#} \end{pmatrix}, \\
QQ^{\#}PP^{\#}Q^{\#} &= \begin{pmatrix} FF^{\#}EE^{\#}F^{\#} & FF^{\#}E^{\#}F^{\#} \\ -FF^{\#}EF^{\pi}EE^{\#}F^{\#} & -FF^{\#}EF^{\pi}E^{\#}F^{\#} \end{pmatrix}.
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
M^{\#} &= M(M^2)^{\#} \\
&= \begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix}\begin{pmatrix} \gamma & \delta \\ \lambda & \xi \end{pmatrix} \\
&= \begin{pmatrix} E\gamma + \lambda & E\delta + \xi \\ F\gamma & F\delta \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= F^{\pi}(E^{\#})^2 + F^{\#}E^{\pi} + (E^{\#})^2FF^{\#} + EE^{\#}F^{\#} + FF^{\#}(E^{\#})^2FF^{\#} \\
&+ F^{\#}EE^{\#}FF^{\#} + FF^{\#}EE^{\#}F^{\#}, \\
\delta &= F^{\pi}(E^{\#})^3 - F^{\#}E^{\#} + (E^{\#})^3FF^{\#} + E^{\#}F^{\#} + FF^{\#}(E^{\#})^3FF^{\#} \\
&+ F^{\#}E^{\#}FF^{\#} + FF^{\#}E^{\#}F^{\#}, \\
\lambda &= FF^{\#}E^{\#} - F^{\#}EFF^{\#} - F^{\#}FEF^{\#} - FF^{\#}E^{\#}FF^{\#} \\
&+ F^{\#}EFF^{\#} - FF^{\#}EF^{\pi}EE^{\#}F^{\#}, \\
\xi &= FF^{\#}(E^{\#})^2 + F^{\#}E^{\pi} - FF^{\#}(E^{\#})^2FF^{\#} + F^{\#}EE^{\#}FF^{\#} \\
&- FF^{\#}EF^{\pi}E^{\#}F^{\#}.
\end{aligned}$$

Let $\Gamma = E\gamma + \lambda, \Delta = E\delta + \xi, \Lambda = F\gamma$ and $\Xi = F\delta$. Then we complete the proof. □

Corollary 4.2. *Let $E, F \in \mathbb{C}^{n\times n}$ have group inverses. If $EF^iE = 0$ for $i = 1, \cdots, n$, then $M = \begin{pmatrix} E & F \\ I_n & 0 \end{pmatrix}$ has group inverse if and only if $F^{\pi}E^{\pi}F^{\pi} = 0$. In this case,*

$$M^{\#} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned}
\Gamma &= -E^{\#}FF^{\#} + 2FF^{\#}(E^{\#})^3F + FF^{\#}E^{\#}FF^{\#} + 2F^{\#}E^{\#}F, \\
\Delta &= E^{\pi}FF^{\#} + FF^{\#}(E^{\#})^2F + FF^{\#}(E^{\#})^2F + FF^{\#}EE^{\#}FF^{\#} \\
&+ 2F^{\#}EE^{\#}F, \\
\Lambda &= (E^{\#})^2 + FF^{\#}(E^{\#})^2 + F^{\#}EE^{\#} + (E^{\#})^2FF^{\#} + E^{\pi}F^{\#} \\
&- FF^{\#}(E^{\#})^2FF^{\#} + FF^{\#}EE^{\#}F^{\#} - F^{\#}EE^{\#}FF^{\#}, \\
\Xi &= E^{\#} + FF^{\#}E^{\#} + 2F^{\#}E + E^{\#}FF^{\#} - FF^{\#}E^{\#}FF^{\#} \\
&- 2F^{\#}EFF^{\#}.
\end{aligned}$$

Proof. Clearly, $M^{\#} = [(M^T)^{\#}]^T$, where $M^T = \begin{pmatrix} E^T & I_n \\ F^T & 0 \end{pmatrix}$. Applying Theorem 4.1 to the transpose M^T of M, we obtain the result. □

We are now ready to prove the following.

Theorem 4.3. *Let $E, F \in \mathbb{C}^{n\times n}$ have group inverses. If $FE^iF = 0$ for $i = 1, \cdots, n$, then $M = \begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix}$ has group inverse if and only if $E^\pi F^\pi E^\pi = 0$. In this case,*

$$M^\# = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned}
\Gamma &= E^\#F^\pi + EF^\#EE^\# + EFF^\#(E^\#)^2 + EF^\#EE^\# \\
&+ EE^\#F^\#E + E^\#FF^\#EE^\# - (E^\#)^2FF^\#E \\
&+ EFF^\#(E^\#)^2 - EE^\#FF^\#E^\# + 2F^\#E \\
&- FF^\#E^\#, \\
\Delta &= -EE^\#F^\# + (E^\#)^2F^\pi + EF^\#E^\# + 2EFF^\#(E^\#)^3 \\
&+ EF^\#E^\# + EE^\#F^\#EE^\# + E^\#FF^\#E^\# \\
&- (E^\#)^2FF^\#EE^\# - EE^\#FF^\#(E^\#)^2 \\
&+ F^\# + F^\#EE^\# - FF^\#(E^\#)^2, \\
\Lambda &= FF^\# + FF^\#EE^\# + 2F(E^\#)^2 \\
\Xi &= FF^\#E^\# + 2F(E^\#)^3.
\end{aligned}$$

Proof. Clearly, we have

$$M^2 = \begin{pmatrix} E^2 + F & E \\ FE & F \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} F & 0 \\ FE & F \end{pmatrix}, Q = \begin{pmatrix} E^2 & E \\ 0 & 0 \end{pmatrix}.$$

Since E, F have group inverses. By virtue of [9, Theorem 2.1], P and Q have group inverses. Since $FE^iF = 0$, we see that

$$Q^i = \begin{pmatrix} E^{2i} & E^{2i-1} \\ 0 & 0 \end{pmatrix},$$

and so

$$PQ^iP = \begin{pmatrix} FE^{2i} & FE^{2i-1} \\ FE^{2i+1} & FE^{2i} \end{pmatrix}\begin{pmatrix} F & 0 \\ FE & F \end{pmatrix} = 0$$

for $i = 1, \cdots, n$.

Moreover, we have

$$\begin{aligned}
(M^2)^\# &= Q^\pi P^\# + Q^\# P^\pi - P^\# QQ^\# - PP^\# Q^\# \\
&+ QQ^\# P^\# QQ^\# + Q^\# PP^\# QQ^\# + QQ^\# PP^\# Q^\#.
\end{aligned}$$

Hence,

$$MM^D = PP^\# + QQ^\# - PP^\#QQ^\# - QQ^\#PP^\#Q^\pi.$$

As in the proof of Theorem 4.1, we have

$$P^\# = \begin{pmatrix} F^\# & 0 \\ F^\#E - F^\#FEF^\# & F^\# \end{pmatrix}; P^\pi = \begin{pmatrix} F^\pi & 0 \\ FF^\#EF^\pi & F^\pi \end{pmatrix}$$
$$Q^\# = \begin{pmatrix} (E^\#)^2 & (E^\#)^3 \\ 0 & 0 \end{pmatrix}, Q^\pi = \begin{pmatrix} E^\pi & -E^\# \\ 0 & I_n \end{pmatrix}.$$

We easily check that

$$\begin{aligned} PP^{\#} &= \begin{pmatrix} FF^{\#} & 0 \\ -FF^{\#}E & FF^{\#} \end{pmatrix}, \\ QQ^{\#} &= \begin{pmatrix} EE^{\#} & E^{\#} \\ 0 & 0 \end{pmatrix}, \\ PP^{\#}QQ^{\#} &= \begin{pmatrix} FF^{\#}EE^{\#} & FF^{\#}E^{\#} \\ -FF^{\#}E & -FF^{\#}EE^{\#} \end{pmatrix}, \end{aligned}$$

$$QQ^{\#}PP^{\#}Q^{\pi} = \begin{pmatrix} EE^{\#}FF^{\#}E^{\pi} & -EE^{\#}FF^{\#}E^{\#} + E^{\#}FF^{\#}EE^{\#} + E^{\#}FF^{\#} \\ 0 & 0 \end{pmatrix}.$$

Then

$$MM^D = \begin{pmatrix} EE^{\#} + E^{\pi}FF^{\#}E^{\pi} & E^{\#}F^{\pi} - E^{\pi}FF^{\#}E^{\#} - E^{\#}FF^{\#}EE^{\#} \\ 0 & FF^{\#} + FF^{\#}EE^{\#} \end{pmatrix}.$$

Hence,

$$I_{2n} - MM^D = \begin{pmatrix} E^{\pi}F^{\pi}E^{\pi} & -E^{\#}F^{\pi} + E^{\pi}FF^{\#}E^{\#} + E^{\#}FF^{\#}EE^{\#} \\ 0 & F^{\pi} - FF^{\#}EE^{\#} \end{pmatrix}.$$

Accordingly,

$$\begin{aligned} &(I_{2n} - MM^D)M = \\ &\begin{pmatrix} E^{\pi}F^{\pi}E^{\pi} & -E^{\#}F^{\pi} + E^{\pi}FF^{\#}E^{\#} + E^{\#}FF^{\#}EE^{\#} \\ 0 & F^{\pi} - FF^{\#}EE^{\#} \end{pmatrix} \begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix} \\ &= \begin{pmatrix} E^{\pi}F^{\pi}E^{\pi}F & E^{\pi}F^{\pi}E^{\pi} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore M has group inverse if and only if $E^{\pi}F^{\pi}E^{\pi} = 0$.

We compute that

$$\begin{aligned} Q^{\pi}P^{\#} &= \begin{pmatrix} E^{\pi}F^{\#} & -E^{\#}F^{\#} \\ F^{\#}E & F^{\#} \end{pmatrix}, \\ Q^{\#}P^{\pi} &= \begin{pmatrix} (E^{\#})^2F^{\pi} & (E^{\#})^3F^{\pi} \\ 0 & 0 \end{pmatrix}, \\ P^{\#}QQ^{\#} &= \begin{pmatrix} F^{\#}EE^{\#} & F^{\#}E^{\#} \\ F^{\#}E & F^{\#}EE^{\#} \end{pmatrix}, \\ PP^{\#}Q^{\#} &= \begin{pmatrix} FF^{\#}(E^{\#})^2 & FF^{\#}(E^{\#})^3 \\ -FF^{\#}E^{\#} & -FF^{\#}(E^{\#})^2 \end{pmatrix}, \\ QQ^{\#}P^{\#}QQ^{\#} &= \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, \\ A_1 &= EE^{\#}F^{\#}EE^{\#} + E^{\#}F^{\#}E, \\ A_2 &= EE^{\#}F^{\#}E^{\#} + E^{\#}F^{\#}EE^{\#}; \\ Q^{\#}PP^{\#}QQ^{\#} &= \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}, \\ B_1 &= (E^{\#})^2FF^{\#}EE^{\#} - (E^{\#})^3FF^{\#}E, \\ B_2 &= (E^{\#})^2FF^{\#}E^{\#} - (E^{\#})^3FF^{\#}EE^{\#}; \\ QQ^{\#}PP^{\#}Q^{\#} &= \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}, \\ C_1 &= EE^{\#}FF^{\#}(E^{\#})^2 - E^{\#}FF^{\#}E^{\#}, \\ C_2 &= EE^{\#}FF^{\#}(E^{\#})^3 - E^{\#}FF^{\#}(E^{\#})^2. \end{aligned}$$

Accordingly, we have

$$\begin{array}{rcl} M^{\#} & = & M(M^2)^{\#} \\ & = & \begin{pmatrix} E & I_n \\ F & 0 \end{pmatrix}\begin{pmatrix} \gamma & \delta \\ \lambda & \xi \end{pmatrix} \\ & = & \begin{pmatrix} E\gamma+\lambda & E\delta+\xi \\ F\gamma & F\delta \end{pmatrix}, \end{array}$$

where

$$\begin{array}{rcl} \gamma & = & E^{\pi}F^{\#}+(E^{\#})^2F^{\pi}+F^{\#}EE^{\#}+FF^{\#}(E^{\#})^2 \\ & + & EE^{\#}F^{\#}EE^{\#}+E^{\#}F^{\#}E+(E^{\#})^2FF^{\#}EE^{\#} \\ & - & (E^{\#})^3FF^{\#}E+EE^{\#}FF^{\#}(E^{\#})^2-E^{\#}FF^{\#}E^{\#}, \\ \delta & = & -E^{\#}F^{\#}+(E^{\#})^3F^{\pi}+F^{\#}E^{\#}+FF^{\#}(E^{\#})^3 \\ & + & EE^{\#}F^{\#}E^{\#}+E^{\#}F^{\#}EE^{\#}+(E^{\#})^2FF^{\#}E^{\#} \\ & - & (E^{\#})^3FF^{\#}EE^{\#}+EE^{\#}FF^{\#}(E^{\#})^3-E^{\#}FF^{\#}(E^{\#})^2, \\ \lambda & = & F^{\#}E+F^{\#}E-FF^{\#}E^{\#}, \\ \xi & = & F^{\#}+F^{\#}EE^{\#}-FF^{\#}(E^{\#})^2. \end{array}$$

Let $\Gamma=E\gamma+\lambda, \Delta=E\delta+\xi, \Lambda=F\gamma$ and $\Xi=F\delta$. This completes the proof. □

Corollary 4.4. *Let $E,F\in\mathbb{C}^{n\times n}$ have group inverses. If $FE^iF=0$ for $i=1,\cdots,n$, then $M=\begin{pmatrix} E & F \\ I_n & 0 \end{pmatrix}$ has group inverse if and only if $E^{\pi}F^{\pi}E^{\pi}=0$. In this case,*

$$M^{\#} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{array}{rcl} \Gamma & = & E^{\#}FF^{\#}+2(E^{\#})^3F, \\ \Delta & = & FF^{\#}+EE^{\#}FF^{\#}+2(E^{\#})^2F, \\ \Lambda & = & -F^{\#}EE^{\#}+F^{\pi}(E^{\#})^2+E^{\#}F^{\#}E+2(E^{\#})^3FF^{\#}E \\ & + & E^{\#}F^{\#}E+EE^{\#}F^{\#}EE^{\#}+E^{\#}FF^{\#}E^{\#} \\ & - & EE^{\#}FF^{\#}(E^{\#})^2-(E^{\#})^2FF^{\#}EE^{\#}+F^{\#}+EE^{\#}F^{\#} \\ & - & (E^{\#})^2FF^{\#}, \\ \Xi & = & F^{\pi}E^{\#}+EE^{\#}F^{\#}E+(E^{\#})^2FF^{\#}E+EE^{\#}F^{\#}E \\ & + & EF^{\#}EE^{\#}+EE^{\#}FF^{\#}E^{\#}-EFF^{\#}(E^{\#})^2 \\ & + & (E^{\#})^2FF^{\#}E-E^{\#}FF^{\#}EE^{\#}+2EF^{\#} \\ & - & E^{\#}FF^{\#}. \end{array}$$

Proof. Applying Theorem 4.3 to the transpose of M, we complete the proof as in Corollary 4.2. □

Acknowledgement

The authors would like to thank the referee for his/her valuable and helpful suggestions which led to improvement of the paper.

References

[1] J. Benítez; X. Liu and T. Zhu, Additive results for the group inverse in an algebra with applications to block operators, *Linear Multilinear Algebra*, **59**(2011), 279–289.

[2] C. Bu; X. Zhou; L. Ma and J. Zhou, On the group inverse for the sum of matrices, *J. Aust. Math. Soc.*, **96**(2014), 36–43.

[3] S.L. Campbell, The Drazin inverse and systems of second order linear differential equations, *Linear & Multilinear Algebra*, **14**(1983), 195–198.

[4] N. Castro-González; J. Robles and J.Y. Vélez-Cerrada, The group inverse of 2×2 matrices over a ring, *Linear Algebra Appl.*, **438**(2013), 3600–3609.

[5] C. Cao; Y. Wang and Y. Sheng, Group inverses for some 2×2 block matrices over rings, *Front. Math. China*, **11**(2016), 521–538.

[6] C.Y. Deng, On the group invertibility of operators, *Electronic J. Linear Algebra*, **31**(2016), 492–510.
[7] L. Guo; J. Chen and H. Zou, Representations for the Drazin inverse of the sum of two matrices and its applications, *Bull. Iranian Math. Soc.*,**45**(2019), 683–699.
[8] L. Guo and X. Du, Representations for the Drazin inverses of 2×2 block matrices, *Applied Math. Comput.*, **217**(2010), 2833–2842.
[9] N. Mihajlović and D.S. Djordjević, On group invrtibility in rings, *Filomat*, **33**(2019), 6141–6150.
[10] D. Mosić, Group inverse and generalized Drazin inverse of block matrices in a Banach algebra, *Bull. Korean Math. Soc.*, **51**(2014), 765–771.
[11] X. Song and B. Zheng, The group invertibility of the product of two matrices over ring and its applications, *Comput. Applied Math.*, **33**(2014), 821–829.
[12] M. Zhou; J. Chen and X. Zhu, The group inverse and core inverse of sum of two elements in a ring, *Comm. Algebra*, **48**(2020), 676-690.
[13] H. Zou; J. Chen and D. Mosić, The Drazin invertibility of an anti-triangular matrix over a ring, *Studia Scient. Math. Hungar.*, **54**(2017), 489–508.
[14] H. Zou; D. Mosic and Y. Chen, The existence and representation of the Drazin inverse of a 2×2 block matrix over a ring, *J. Algebra Appl.*, **18**(2019), 1950212 (23 pages), DOI: 10.1142/S0219498819502128.