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Nonlinear mixed Jordan triple ∗-derivations on Standard operator algebras

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Abstract. Let $\mathfrak{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$ containing identity operator I, which is closed under the adjoint operation. Suppose that $\delta : \mathfrak{A} \to \mathfrak{A}$ is the nonlinear mixed Jordan triple ∗- derivation. Then δ is an additive ∗-derivation.

1. Introduction

Let $\mathfrak{A}$ be an ∗-algebra over the complex field $\mathbb{C}$. For $S, T \in \mathfrak{A}$, $[S, T]_* = ST - TS^*$ and $S \bullet T = ST + TS^*$ denotes the skew Lie product and Jordan ∗- product of S and T respectively. In several research domains, the skew Lie product and Jordan ∗- product are becoming increasingly relevant, and its study has attracted several author's attention, see [1–4, 6, 8–15]. An additive map $\delta : \mathfrak{A} \to \mathfrak{A}$ is called an additive derivation if $\delta(ST) = \delta(S)T + S\delta(T)$ for all $S, T \in \mathfrak{A}$. If $\delta(S^*) = \delta(S)^*$ for all $S \in \mathfrak{A}$ then δ is additive ∗-derivation. Let $\delta : \mathfrak{A} \to \mathfrak{A}$ be a mapping (without the additivity assumption). We say ψ is a nonlinear ∗-Lie derivation or nonlinear Jordan ∗- derivation if

$$\delta([S, T]_*) = [\delta(S), T]_* + [S, \delta(T)]_*$$

or

$$\delta(S \bullet T) = \delta(S) \bullet T + S \bullet \delta(T)$$

holds for all $S, T \in \mathfrak{A}$ respectively. With the nonlinear Jordan ∗- derivation and nonlinear skew Lie derivations in mind, we can continue to grow them in a natural manner. A map $\delta : \mathfrak{A} \to \mathfrak{A}$ is said to be a nonlinear Jordan triple ∗-derivation or skew Lie triple derivation if

$$\delta(S \bullet T \bullet U) = \delta(S) \bullet T \bullet U + S \bullet \delta(T) \bullet U + S \bullet T \bullet \delta(U)$$

or

$$\delta([[S, T]_*, U]_*) = [[\delta(S), T]_*, U]_* + [[S, \delta(T)]_*, U]_* + [[S, T]_*, \delta(U)]_*$$

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for all $S, T, U \in \mathfrak{A}$ respectively. In this paper, we will look into nonlinear mixed Jordan triple $*$-derivations on standard operator algebras. A map $\delta : \mathfrak{A} \rightarrow \mathfrak{A}$ is said to be a nonlinear mixed Jordan triple $*$-derivation if

$$\delta([S,T]_* \bullet U) = [\delta(S), T]_* \bullet U + [S, \delta(T)]_* \bullet U + [S,T]_* \bullet \delta(U)$$

for all $S, T, U \in \mathfrak{A}$. We prove that δ is a nonlinear mixed Jordan triple $*$- derivation on standard operator algebras if and only if δ is an additive $*$-derivation.

2. Notation and Preliminaries

Throughout this paper, $\mathcal{H}$ represents a Banach space over $\mathbb{F}$, where $\mathbb{F}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. $\mathcal{B}(\mathcal{H})$ represents the algebra of all bounded linear operators on $\mathcal{H}$. By $\mathcal{F}(\mathcal{H})$ we mean the subalgebra of bounded finite rank operators. It is to be noted that $\mathcal{F}(\mathcal{H})$ forms a $*$-closed ideal in $\mathcal{B}(\mathcal{H})$. An algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is said to be standard operator algebra in case $\mathcal{F}(\mathcal{H}) \subset \mathfrak{A}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be a projection provided $P^* = P$ and $P^2 = P$. An algebra $\mathfrak{A}$ is said to be prime if $A\mathfrak{A}B = 0$ implies either $A = 0$ or $B = 0$. It is to be noted that every standard operator algebra is prime and its centre is $\mathbb{F}I$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Any operator $S \in \mathcal{B}(\mathcal{H})$ can be expressed as $S = \Re(S) + i\Im(S)$, where $\Re(S) = \frac{S+S^*}{2}$ and $\Im(S) = \frac{S-S^*}{2i}$. Both $\Re(S)$ and $\Im(S)$ are self disjoint.

The following known results will help us in our proof:

Lemma 2.1. *[7, Lemma 2.1] Let $\mathfrak{A}$ be a standard operator algebra with the identity operator I on a complex Hilbert space which is closed under the adjoint operation. If $ST = TS^*$ holds true for all $T \in \mathfrak{A}$, then $S \in \mathbb{R}I$.*

Lemma 2.2. *[5, Problem 230] Suppose $\mathfrak{A}$ is a Banach algebra with the identity I. For any $S, T \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, if $[S,T] = \lambda I$, then $\lambda = 0$.*

3. Main Result

Now take a projection $P_1 \in \mathfrak{A}$ and let $P_2 = I - P_1$. We write $\mathfrak{A}_{jk} = P_j\mathfrak{A}P_k$ for $j,k = 1,2$. Then by the Peirce decomposition of $\mathfrak{A}$, we have $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22}$. Note that any operator $S \in \mathfrak{A}$ can be expressed as $S = S_{11} + S_{12} + S_{21} + S_{22}$ and $S^*_{jk} \in \mathfrak{A}_{kj}$ for any $S_{jk} \in \mathfrak{A}_{jk}$.

Theorem 3.1. *Let $\mathfrak{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$ containing identity operator I, which is closed under the adjoint operation. Suppose that $\delta : \mathfrak{A} \rightarrow \mathfrak{A}$ satisfies $\delta([S,T]_* \bullet U) = [\delta(S), T]_* \bullet U + [S, \delta(T)]_* \bullet U + [S,T]_* \bullet \delta(U)$ for all $S, T, U \in \mathfrak{A}$. Then δ is an additive $*$-derivation.*

This section's major aim is to prove our main theorem by proving several lemmas.

Lemma 3.2. $\delta(0) = 0$.

Proof. It is obvious that
$\delta(0) = \delta([0,0]_* \bullet 0) = [\delta(0), 0]_* \bullet 0 + [0, \delta(0)]_* \bullet 0 + [0,0]_* \bullet \delta(0) = 0.$
□

Lemma 3.3. *For every $S_{11} \in \mathfrak{A}_{11}, T_{12} \in \mathfrak{A}_{12}, U_{21} \in \mathfrak{A}_{21}, V_{22} \in \mathfrak{A}_{22}$, we have*

$$\delta(S_{11} + T_{12} + U_{21} + V_{22}) = \delta(S_{11}) + \delta(T_{12}) + \delta(U_{21}) + \delta(V_{22}).$$

Proof. Let $M = \delta(S_{11} + T_{12} + U_{21} + V_{22}) - (\delta(S_{11}) + \delta(T_{12}) + \delta(U_{21}) + \delta(V_{22}))$. We have

$$\begin{aligned}\delta([P_j, S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet P_i) \quad &= \quad [\delta(P_j), S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet P_i \\ &\quad +[P_j, \delta(S_{11} + T_{12} + U_{21} + V_{22})]_* \bullet P_i \\ &\quad +[P_j, S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet \delta(P_i).\end{aligned}$$

On the other hand, we have $[P_j, S_{11}]_* \bullet P_i = [P_j, V_{22}]_* \bullet P_i = 0$. Also, $[P_j, T_{12}]_* \bullet P_i = 0$ or $[P_j, U_{21}]_* \bullet P_i = 0$ for $i, j = 1, 2$ and $i \neq j$. Then

$$\begin{aligned}\delta([P_j, S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet P_i) &= \delta([P_j, S_{11}]_* \bullet P_i) + \delta([P_j, T_{12}]_* \bullet P_i) \\ &\quad + \delta([P_j, U_{21}]_* \bullet P_i) + \delta([P_j, V_{22}]_* \bullet P_i) \\ &= [\delta(P_j), S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet P_i \\ &\quad + [P_j, \delta(S_{11}) + \delta(T_{12}) + \delta(U_{21}) + \delta(V_{22})]_* \bullet P_i \\ &\quad + [P_j, S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet \delta(P_i).\end{aligned}$$

By comparing the above two equations, we find $[P_j, M]_* \bullet P_i = 0$. This implies that $P_jMP_i + P_iM^*P_j = 0$. Multiplying both sides with P_j from the left, we obtain $P_jMP_i = 0$ with $i \neq j$. Hence, $M = M_{11} + M_{22}$. Again for every $B_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}\delta([B_{12}, S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet P_2) &= [\delta(B_{12}), S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet P_2 \\ &\quad + [B_{12}, \delta(S_{11} + T_{12} + U_{21} + V_{22})]_* \bullet P_2 \\ &\quad + [B_{12}, S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet \delta(P_2).\end{aligned}$$

On the other hand, by using Lemma 3.2, we have

$$\begin{aligned}\delta([B_{12}, S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet P_2) &= \delta([B_{12}, S_{11}]_* \bullet P_2) + \delta([B_{12}, T_{12}]_* \bullet P_2) \\ &\quad + \delta([B_{12}, U_{21}]_* \bullet P_2) + \delta([B_{12}, V_{22}]_* \bullet P_2) \\ &= [\delta(B_{12}), S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet P_2 \\ &\quad + [B_{12}, \delta(S_{11}) + \delta(T_{12}) + \delta(U_{21}) + \delta(V_{22})]_* \bullet P_2 \\ &\quad + [B_{12}, S_{11} + T_{12} + U_{21} + V_{22}]_* \bullet \delta(P_2).\end{aligned}$$

By comparing the last two expressions, we find $[B_{12}, M]_* \bullet P_2 = 0$. That means $B_{12}MP_2 + P_2M^*B_{12}^* = 0$. Multiplying both sides with P_1 from the left, we find $B_{12}MP_2 = 0$. By using the primeness of $\mathfrak{A}$, we obtain $P_2MP_2 = 0$. Thus, $M_{22} = 0$. Similarly, we can find $M_{11} = 0$. Hence, $M = 0$. □

Lemma 3.4. *For any $S_{ij}, T_{ij} \in \mathfrak{A}_{ij}, (1 \leq i \neq j \leq 2)$, we have*

$$\delta(S_{ij} + T_{ij}) = \delta(S_{ij}) + \delta(T_{ij}).$$

Proof. Since, we have

$$[-\frac{i}{2}I, i(S_{ij} + P_i)]_* \bullet (T_{ij} + P_j) = (S_{ij} + T_{ij}) + S_{ij}^* + T_{ij}S_{ij}^*.$$

It follows from Lemma 3.3, that

$$\begin{aligned}\delta(S_{ij} + T_{ij}) + \delta(S_{ij}^*) + \delta(T_{ij}S_{ij}^*) &= \delta\Big((S_{ij} + T_{ij}) + S_{ij}^* + T_{ij}S_{ij}^*\Big) \\ &= \delta([-\frac{i}{2}I, i(S_{ij} + P_i)]_* \bullet (T_{ij} + P_j)) \\ &= [\delta(-\frac{i}{2}I), i(S_{ij} + P_i)]_* \bullet (T_{ij} + P_j) \\ &\quad + [-\frac{i}{2}I, \delta(i(S_{ij} + P_i))]_* \bullet (T_{ij} + P_j) \\ &\quad + [-\frac{i}{2}I, i(S_{ij} + P_i)]_* \bullet \delta(T_{ij} + P_j)\end{aligned}$$

$$\begin{aligned} &= \delta([-\frac{i}{2}I, iS_{ij}]_* \bullet T_{ij}) + \delta([-\frac{i}{2}I, iS_{ij}]_* \bullet P_j) \\ &\quad +\delta([-\frac{i}{2}I, iP_i]_* \bullet T_{ij}) + \delta([-\frac{i}{2}I, iP_i]_* \bullet P_j) \\ &= \delta(T_{ij}S_{ij}^*) + \delta(S_{ij} + S_{ij}^*) + \delta(T_{ij}) \\ &= \delta(S_{ij}) + \delta(S_{ij}^*) + \delta(T_{ij}S_{ij}^*) + \delta(T_{ij}). \end{aligned}$$

Hence, $\delta(S_{ij} + T_{ij}) = \delta(S_{ij}) + \delta(T_{ij})$. □

Lemma 3.5. *For any $S_{ii}, T_{ii} \in \mathfrak{A}_{ii}$, $(1 \le i \le 2)$, we have*

$$\delta(S_{ii} + T_{ii}) = \delta(S_{ii}) + \delta(T_{ii}).$$

Proof. For $i = 1$, write $M = \delta(S_{11} + T_{11}) - \delta(S_{11}) - \delta(T_{11})$. We have

$$\begin{aligned} \delta([P_1, S_{11} + T_{11}]_* \bullet P_2) &= [\delta(P_1), S_{11} + T_{11}]_* \bullet P_2 + [P_1, \delta(S_{11} + T_{11})]_* \bullet P_2 \\ &\quad +[P_1, S_{11} + T_{11}]_* \bullet \delta(P_2). \end{aligned}$$

On the other side, by using Lemma 3.2, we have

$$\begin{aligned} \delta([P_1, S_{11} + T_{11}]_* \bullet P_2) &= \delta([P_1, S_{11}]_* \bullet P_2) + \delta([P_1, T_{11}]_* \bullet P_2) \\ &= [\delta(P_1), S_{11} + T_{11}]_* \bullet P_2 + [P_1, \delta(S_{11}) + \delta(T_{11})]_* \bullet P_2 \\ &\quad +[P_1, S_{11} + T_{11}]_* \bullet \delta(P_2). \end{aligned}$$

By comparing the last two equations, we get $[P_1, M]_* \bullet P_2 = 0$. That means $P_1MP_2 + P_2M^*P_1 = 0$. Multiplying both sides from left by P_1, we get $P_1MP_2 = 0$. Similarly, we can show $P_2MP_1 = 0$.

For any $B_{ij} \in \mathfrak{A}_{ij}$, we have

$$\begin{aligned} \delta([B_{12}, S_{11} + T_{11}]_* \bullet P_1) &= [\delta(B_{12}), S_{11} + T_{11}]_* \bullet P_1 + [B_{12}, \delta(S_{11} + T_{11})]_* \bullet P_1 \\ &\quad +[B_{12}, S_{11} + T_{11}]_* \bullet \delta(P_1). \end{aligned}$$

On the other side, by Lemma 3.2, we have

$$\begin{aligned} \delta([B_{12}, S_{11} + T_{11}]_* \bullet P_1) &= \delta([B_{12}, S_{11}]_* \bullet P_1) + \delta([B_{12}, T_{11}]_* \bullet P_1) \\ &= [\delta(B_{12}), S_{11} + T_{11}]_* \bullet P_1 + [B_{12}, \delta(S_{11}) + \delta(T_{11})]_* \bullet P_1 \\ &\quad +[B_{12}, S_{11} + T_{11}]_* \bullet \delta(P_1). \end{aligned}$$

By comparing the above two equations and then multiplying both sides from right by P_2, we obtain $B_{12}MP_2 = 0$. By using the primeness of $\mathfrak{A}$, we get $M_{22} = 0$. Hence, $M = M_{11}$. Now, again on the one hand, we have

$$\begin{aligned} \delta([S_{11} + T_{11}, B_{12}]_* \bullet P_2) &= [\delta(S_{11} + T_{11}), B_{12}]_* \bullet P_2 + [S_{11} + T_{11}, \delta(B_{12})]_* \bullet P_2 \\ &\quad +[S_{11} + T_{11}, B_{12}]_* \bullet \delta(P_2). \end{aligned}$$

On the other hand, from Lemma 3.3 and Lemma 3.4 that for any $B_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned} \delta([S_{11} + T_{11}, B_{12}]_* \bullet P_2) &= \delta(S_{11}B_{12}) + \delta(T_{11}B_{12}) + \delta(B_{12}^*S_{11}^*) + \delta(B_{12}^*T_{11}^*) \\ &= \delta([S_{11}, B_{12}]_* \bullet P_2) + \delta([T_{11}, B_{12}]_* \bullet P_2) \\ &= [\delta(S_{11}) + \delta(T_{11}), B_{12}]_* \bullet P_2 + [S_{11} + T_{11}, \delta(B_{12})]_* \bullet P_2 \\ &\quad +[S_{11} + T_{11}, B_{12}]_* \bullet \delta(P_2). \end{aligned}$$

By comparing the last two expressions, we get $[M_{11}, B_{12}]_* \bullet P_2 = 0$. By using the primeness of $\mathfrak{A}$, we obtain $M_{11} = 0$. Hence, the proof is complete. Similarly, we can show the case for $i = 2$. □

Lemma 3.6. *δ is additive.*

Proof. Let $S, T \in \mathfrak{A}$ and write $S = \sum_{i,j=1}^{2} S_{ij},\ T = \sum_{i,j=1}^{2} T_{ij}$. Then by using Lemma 3.3, Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned}\delta(S+T) &= \delta(\sum_{i,j=1}^{2} S_{ij} + \sum_{i,j=1}^{2} T_{ij}) \\ &= \delta(\sum_{i,j=1}^{2}(S_{ij}+T_{ij})) \\ &= \sum_{i,j=1}^{2} \delta(S_{ij}+T_{ij}) \\ &= \sum_{i,j=1}^{2} \delta(S_{ij}) + \delta(T_{ij}) \\ &= \delta(\sum_{i,j=1}^{2} S_{ij}) + \delta(\sum_{i,j=1}^{2} T_{ij}) \\ &= \delta(S) + \delta(T).\end{aligned}$$

□

Lemma 3.7. *δ has the following properties:*

1. $\delta(iI)^* = \delta(iI)$.
2. *For any* $\lambda \in \mathbb{R}, \delta(\lambda I) \in \mathbb{R}I$.
3. *For all* $S \in \mathfrak{A}$ *with* $S = S^*, \delta(S) = \delta(S)^*$.
4. *For any* $\lambda \in \mathbb{C}, \delta(\lambda I) \in \mathbb{C}I$.

Proof. (1) We have,

$$\delta([iI, iI]_* \bullet (iI)) = -4\delta(iI).$$

On the other hand, we have

$$\begin{aligned}\delta([iI, iI]_* \bullet (iI)) &= [\delta(iI), iI]_* \bullet (iI) + [iI, \delta(iI)]_* \bullet (iI) \\ &\quad +[iI, iI]_* \bullet \delta(iI) \\ &= -8\delta(iI) + 4\delta^*(iI).\end{aligned}$$

By comparing the above two equations, we get, $\delta(iI)^* = \delta(iI)$.
(2) For any $\lambda \in \mathbb{R}$, we have

$$0 = \delta([\lambda I, S]_* \bullet I) = [\delta(\lambda I), S]_* \bullet I = \delta(\lambda I)(S - S^*) - (S - S^*)\delta(\lambda I)^*.$$

Thus, $\delta(\lambda I)(S - S^*) = (S - S^*)\delta(\lambda I)^*$ holds for all $S \in \mathfrak{A}$ and hence $\delta(\lambda I)S = S\delta(\lambda I)^*$ for all $S = -S^* \in \mathfrak{A}$. Since every S is of the form of $S = S_1 + iS_2$, where $S_1 = \frac{S+S^*}{2}$ and $S_2 = \frac{S-S^*}{2i}$, it follows that $\delta(\lambda I)S = S\delta(\lambda I)^*$ for all $S \in \mathfrak{A}$. By Lemma 2.1, we have $\delta(\lambda I) \in \mathbb{R}I$.
(3) By using Lemma 3.7 (2), we have for $S = S^*$

$$\begin{aligned}0 = \delta([S, I]_* \bullet B) &= [\delta(S), I]_* \bullet B + [S, \delta(I)]_* \bullet B + [S, I]_* \bullet \delta(B) \\ &= [\delta(S), I]_* \bullet B \\ &= (\delta(S) - \delta(S)^*) \bullet B \\ &= (\delta(S) - \delta(S)^*)B - B(\delta(S) - \delta(S)^*)\end{aligned}$$

for all $B \in \mathfrak{A}$. That means, $\delta(S) - \delta(S)^* = [\delta(S), I]_* \in \mathbb{F}I$. In particular, $\delta(S) - \delta(S)^* = \lambda I$ for some $\lambda \in \mathbb{C}$. Also, we have

$$\begin{aligned} 0 &= \delta([S,S]_* \bullet B) \\ &= [\delta(S), S]_* \bullet B + [S, \delta(S)]_* \bullet B \\ &= (S(\delta(S) - \delta(S)^*)) \bullet B \\ &= \lambda(SB - BS) \end{aligned}$$

for all $B \in \mathfrak{A}$. Suppose that $\lambda \neq 0$, then $S \in \mathbb{F}I$, which is a contradiction. Thus, $\lambda = 0$. Hence, $\psi(S) = \psi(S)^*$.

(4) For any $\lambda \in \mathbb{C}$ and $S \in \mathfrak{A}$ with $S = S^*$. Using Lemma 3.7 (3), we see that

$$0 = \delta([S, \lambda I]_* \bullet T) = [\delta(S), \lambda I]_* \bullet T + [S, \delta(\lambda I)]_* \bullet T + [S, \lambda I]_* \bullet \delta(T) = [S, \delta(\lambda I)]_* \bullet T$$

for all $T \in \mathfrak{A}$. That means $[S, \lambda I]_* = [S, \lambda I] \in \mathbb{F}I$. Now, by using Lemma 2.2, we get $[S, \lambda I] = 0$. Thus, $\delta(\lambda I)S = S\delta(\lambda I)$ for all $S = S^*$. Since every S is of the form of $S = S_1 + iS_2$, where $S_1 = \frac{S+S^*}{2}$ and $S_2 = \frac{S-S^*}{2i}$. It follows that

$$\delta(\lambda I)S = S\delta(\lambda I)$$

for all $S \in \mathfrak{A}$. Hence, $\delta(\lambda I) \in \mathbb{C}I$.

Lemma 3.8. 1. $P_1\delta(P_1)P_2 = -P_1\delta(P_2)P_2, \quad P_2\delta(P_1)P_1 = -P_2\delta(P_2)P_1$.

2. $P_1\delta(P_2)P_1 = P_2\delta(P_1)P_2 = 0$.

Proof. (1). Let $1 \leq i \neq j \leq 2$. It follows from Lemma 3.7 that

$$\begin{aligned} 0 = \delta([P_1, P_2]_* \bullet P_1) &= [\delta(P_1), P_2]_* \bullet P_1 + [P_1, \delta(P_2)]_* \bullet P_1 + [P_1, P_2]_* \bullet \delta(P_1) \\ &= -P_2\delta(P_1)P_1 - P_1\delta(P_1)P_2 + 2P_1\delta(P_2)P_1 - \delta(P_2)P_1 - P_1\delta(P_2). \end{aligned}$$

Multiplying both sides by P_1 from left and by P_2 from the right, we get

$$P_1\delta(P_1)P_2 = -P_1\delta(P_2)P_2.$$

Similarly, we can show that $P_2\delta(P_1)P_1 = -P_2\delta(P_2)P_1$.

(2). On the other hand, we get

$$\begin{aligned} \delta([iI, iP_1]_* \bullet P_2) &= [\delta(iI), iP_1]_* \bullet P_2 + [iI, \delta(iP_1)]_* \bullet P_2 + [iI, iP_1]_* \bullet \delta(P_2) \\ &= -iP_1\delta(iI)P_2 + iP_2\delta(iI)P_1 + 2i\delta(iP_1)P_2 - 2iP_2\delta(iP_1) - 2P_1\delta(P_2) \\ &\quad -2\delta(P_2)P_1. \end{aligned}$$

Multiplying both sides of the above equation by P_1 from left and right, we obtain that $P_1\delta(P_2)P_1 = 0$. Similarly, $P_2\delta(P_1)P_2 = 0$. □

Let $M = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$. Then $M = -M^*$. We define a map $\psi : \mathfrak{A} \to \mathfrak{A}$ by

$$\psi(S) = \delta(S) - (SM - MS)$$

for all $S \in \mathfrak{A}$. It is easy to verify that ψ also satifies $\psi([S,T]_* \bullet U) = [\psi(S), T]_* \bullet U + [S, \psi(T)]_* \bullet U + [S,T]_* \bullet \psi(U)$ and has following properties.

Remark 3.9. 1. $\psi(P_i) = P_i\, \delta(P_i)\, P_i \in \mathfrak{A}_{ii}, i = 1, 2$.

2. $\psi(iI)^* = \psi(iI)$.
3. $\psi(S) = \psi(S)^*$ *for all* $S = S^* \in \mathfrak{A}$.
4. ψ *is additive.*
5. ψ *is a* $*$*-derivation if and only if* δ *is a* $*$*-derivation.*

Lemma 3.10. $\psi(P_i) = 0$ *and* $\psi(\mathfrak{A}_{ij}) \subseteq \mathfrak{A}_{ij}$.

Proof. For any $S_{12} \in \mathfrak{A}_{12}$. By the properties of ψ, we have

$$\begin{aligned} \psi(iS_{12}) &= \psi([\frac{i}{2}I, P_1]_* \bullet S_{12}) \\ &= [\frac{i}{2}I, \psi(P_1)]_* \bullet S_{12} + [\frac{i}{2}I, P_1]_* \bullet \psi(S_{12}) \\ &= i(\psi(P_1)S_{12} - S_{12}\psi(P_1)^* + P_1\psi(S_{12}) - \psi(S_{12})P_1) \\ &= i(\psi(P_1)S_{12} + P_1\psi(S_{12}) - \psi(S_{12})P_1). \end{aligned}$$

Multiplying both sides of the above equation by P_1 and P_2 from the left and right respectively, we get

$$P_1\psi(iS_{12})P_1 = P_2\psi(iS_{12})P_2 = 0.$$

Hence, $\psi(iS_{12}) = P_1\psi(iS_{12})P_2 + P_2\psi(iS_{12})P_1$. On the other hand, for all $B \in \mathfrak{A}$, we have

$$0 = \psi([iS_{12}, P_1]_* \bullet B) = [\psi(iS_{12}), P_1]_* \bullet B.$$

Thus, $\psi(iS_{12})P_1 - P_1\psi(iS_{12})^* \in \mathbb{R}I$. Multiplying both sides by P_2 from the left and P_1 from the right, we get $P_2\psi(iS_{12})P_1 = 0$. Thus, $\psi(iS_{12}) \subseteq \mathfrak{A}_{12}$. Since, S_{12} is arbitary. Hence, $\psi(\mathfrak{A}_{12}) \subseteq \mathfrak{A}_{12}$. Similarly, we can show that $\psi(\mathfrak{A}_{21}) \subseteq \mathfrak{A}_{21}$.
Now, by using the additivity of ψ and for any $S_{12} \in \mathfrak{A}_{12}$, we have

$$\psi([S_{12}, P_2]_* \bullet P_2) = \psi(S_{12} + S_{12}^*) = \psi(S_{12}) + \psi(S_{12}^*).$$

On the other hand, we have

$$\begin{aligned} \psi([S_{12}, P_2]_* \bullet P_2) &= [\psi(S_{12}), P_2]_* \bullet P_2 + [S_{12}, \psi(P_2)]_* \bullet P_2 + [S_{12}, P_2]_* \bullet \psi(P_2) \\ &= \psi(S_{12}) + \psi(S_{12})^* + 2S_{12}\psi(P_2) + \psi(S_{12})^*S_{12}^* + \psi(P_2)S_{12}^*. \end{aligned}$$

By comparing the above two equations, we get

$$\psi(S_{12}^*) = \psi(S_{12})^* + 2S_{12}\psi(P_2) + \psi(S_{12})^*S_{12}^* + \psi(P_2)S_{12}^*.$$

Multiplying both sides of the above equation by P_1 from the left and by P_2 from the right, we have $S_{12}\psi(P_2)P_2 = 0$ for all $S_{12} \in \mathfrak{A}_{12}$. By using primeness of $\mathfrak{A}$, we get $P_2\psi(P_2)P_2 = 0$. Now, by using Remark 3.9 (1), we get $P_2\delta(P_2)P_2 = 0$. Hence, $\psi(P_2) = 0$. Similarly, we can show that $\psi(P_1) = 0$.
For every $S_{11} \in \mathfrak{A}_{11}$, we have

$$0 = \psi([P_1, S_{11}]_* \bullet P_2) = [P_1, \psi(S_{11})]_* \bullet P_2 = P_1\psi(S_{11})P_2 + P_2\psi(S_{11})^*P_1 \tag{1}$$

and

$$0 = \psi([P_2, S_{11}]_* \bullet P_1) = [P_2, \psi(S_{11})]_* \bullet P_1 = P_2\psi(S_{11})P_1 + P_1\psi(S_{11})^*P_2. \tag{2}$$

Multiplying both sides from the left by P_1 to equation (1) and by P_2 from left to equation (2), we have $P_1\psi(S_{11})P_2 = P_2\psi(S_{11})P_1 = 0$.
On the other hand, for any $M_{12} \in \mathfrak{A}_{12}$, we have

$$0 = \psi([M_{12}, S_{11}]_* \bullet P_2) = [M_{12}, \psi(S_{11})]_* \bullet P_2 = M_{12}\psi(S_{11})P_2 + P_2\psi(S_{11})^*M_{12}^*.$$

Multiplying both sides with P_2 from the right, we have $M_{12}\psi(S_{11})P_2 = 0$. By using the primeness of $\mathfrak{A}$, we get $P_2\psi(S_{11})P_2 = 0$. Hence, $\psi(\mathfrak{A}_{11}) \subseteq \mathfrak{A}_{11}$. Similarly, we can show that $\psi(\mathfrak{A}_{22}) \subseteq \mathfrak{A}_{22}$. □

Lemma 3.11. *For every $S_{ii}, T_{ii} \in \mathfrak{A}_{ii}, S_{ij}, T_{ij} \in \mathfrak{A}_{ij}, T_{ji} \in \mathfrak{A}_{ji}, T_{jj} \in \mathfrak{A}_{jj} (1 \leq i \neq j \leq 2)$, we have*

1. $\psi(S_{ij}T_{ji}) = \psi(S_{ij})T_{ji} + S_{ij}\psi(T_{ji})$.
2. $\psi(S_{ii}T_{ij}) = \psi(S_{ii})T_{ij} + S_{ii}\psi(T_{ij})$.
3. $\psi(S_{ij}T_{jj}) = \psi(S_{ij})T_{jj} + S_{ij}\psi(T_{jj})$.
4. $\psi(S_{ii}T_{ii}) = \psi(S_{ii})T_{ii} + S_{ii}\psi(T_{ii})$.

Proof. (1) It follows from Lemma 3.10 that

$$\begin{aligned}\psi(S_{ij}T_{ji}) = \psi([P_i, S_{ij}]_* \bullet T_{ji}) &= [P_i, \psi(S_{ij})]_* \bullet T_{ji} + [P_i, S_{ij}]_* \bullet \psi(T_{ji}) \\ &= \psi(S_{ij})T_{ij} + S_{ij}\psi(T_{ji}).\end{aligned}$$

(2) For every $X_{ji} \in \mathfrak{A}_{ji}, (1 \le i \ne j \le 2)$, we have from (1) that

$$\psi([S_{ii}, T_{ij}]_* \bullet X_{ji}) = \psi(S_{ii}T_{ij}X_{ji}) = \psi(S_{ii}T_{ij})X_{ji} + S_{ii}T_{ij}\psi(X_{ji}).$$

On the other hand, we have

$$\begin{aligned}\psi([S_{ii}, T_{ij}]_* \bullet X_{ji}) &= [\psi(S_{ii}), T_{ij}]_* \bullet X_{ji} + [S_{ii}, \psi(T_{ij})]_* \bullet X_{ji} + [S_{ii}, T_{ij}]_* \bullet \psi(X_{ji}) \\ &= \psi(S_{ii})T_{ij}X_{ji} + S_{ii}\psi(T_{ij})X_{ji} + S_{ii}T_{ij}\psi(X_{ji}).\end{aligned}$$

By comparing the above two equations, we have $(\psi(S_{ii}T_{ij}) - \psi(S_{ii})T_{ij} - S_{ii}\psi(T_{ij}))X_{ji} = 0$ for all $X_{ji} \in \mathfrak{A}_{ji}$. By using the primeness of $\mathfrak{A}$, we have

$$\psi(S_{ii}T_{ij}) = \psi(S_{ii})T_{ij} + S_{ii}\psi(T_{ij}).$$

(3) For every $X_{ji} \in \mathfrak{A}_{ji}, (1 \le i \ne j \le 2)$, using Lemma 3.11, (1) and (2), we get

$$\begin{aligned}\psi(S_{ij}T_{jj})X_{ji} + S_{ij}T_{jj}\psi(X_{ji}) &= \psi(S_{ij}T_{jj}X_{ji}) \\ &= \psi(S_{ij})T_{jj}X_{ji} + S_{ij}\psi(T_{jj}X_{ji}) \\ &= \psi(S_{ij})T_{jj}X_{ji} + S_{ij}\psi(T_{jj})X_{ji} + S_{ij}T_{jj}\psi(X_{ji}).\end{aligned}$$

Hence, $(\psi(S_{ij}T_{jj}) - (\psi(S_{ij})T_{jj} + S_{ij}\psi(T_{jj})))X_{ji} = 0$ for all $X_{ji} \in \mathfrak{A}_{ji}$. Then, by using the primeness of $\mathfrak{A}$, we have

$$\psi(S_{ij}T_{jj}) = \psi(S_{ij})T_{jj} + S_{ij}\psi(T_{jj}).$$

(4) For every $X_{ij} \in \mathfrak{A}_{ij}, (1 \le i \ne j \le 2)$, we have from (2) that

$$\begin{aligned}\psi(S_{ii}T_{ii})X_{ij} + S_{ii}T_{ii}\psi(X_{ij}) &= \psi(S_{ii}T_{ii}X_{ij}) \\ &= \psi(S_{ii})T_{ii}X_{ij} + S_{ii}\psi(T_{ii}X_{ij}) \\ &= \psi(S_{ii})T_{ii}X_{ij} + S_{ii}\psi(T_{ii})X_{ij} + S_{ii}T_{ii}\psi(X_{ij}).\end{aligned}$$

Hence, $(\psi(S_{ii}T_{ii}) - (\psi(S_{ii})T_{ii} + S_{ii}\psi(T_{ii})))X_{ij} = 0$ for all $X_{ij} \in \mathfrak{A}_{ij}$. Then, by using the primeness of $\mathfrak{A}$, we have

$$\psi(S_{ii}T_{ii}) = \psi(S_{ii})T_{ii} + S_{ii}\psi(T_{ii}).$$

Now, by using (1), (2), (3), (4) and the additivity of ψ, we get $\psi(ST) = \psi(S)T + S\psi(T)$.
□

Lemma 3.12. $\psi(S^*) = \psi(S)^*$ *for all* $S \in \mathfrak{A}$.

Proof. We have $\psi(P_1) = 0$ and $\psi(P_2) = 0$. Then

$$0 = \psi(I) = -\psi((iI)(iI)) = \psi(iI)iI + iI\psi(iI) = 2i\psi(iI).$$

Thus, $\psi(iI) = 0$. Hence, $\psi(iS) = \psi(iI(S)) = i\psi(S)$. For any $S \in \mathfrak{A}$, applying Remark 3.9 (3), we have

$$\begin{aligned}\psi(S^*) &= \psi(\Re S - i\Im S) = \psi(\Re S) - \psi(i\Im S) \\ &= \psi(\Re S) - i\psi(\Im S) = \psi(\Re S)^* - i\psi(\Im S)^* \\ &= \psi(\Re S)^* + (i\psi(\Im S))^* = \psi(\Re S)^* + \psi(i\Im S)^* \\ &= \psi(\Re S + i\Im S)^* = \psi(S)^*.\end{aligned}$$

□

Proof of Theorem 3.1 By using Lemma 3.6, Lemma 3.11, Lemma 3.12 and the Remark 3.9, we get δ is an additive $*$-derivation. □

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