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Generalized matrix functions, permutation matrices and symmetric matrices

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Abstract. The purpose of this paper is to study generalized matrix functions only using the permutation matrices and symmetric matrices. Firstly the zeroness of a generalized matrix function and then the equality of two generalized matrix functions on the permutation matrices and symmetric matrices will be examined. Secondly generalized matrix functions preserving commutativity of the permutation matrices or commutativity of the symmetric matrices will be characterized. Thirdly generalized matrix functions which preserve product of the permutation matrices or product of the symmetric matrices will be investigated. Finally the Cayley-Hamilton Theorem for generalized characteristic polynomials using the permutation matrices and symmetric matrices will be studied.

1. Introduction

Throughout the paper denote by $M_n(\mathbb{C})$ the set of all n-by-n matrices over $\mathbb{C}$ and let $\mathbb{S}_n$ and $\mathbb{A}_n$ be the symmetric group and the alternating group of degree n, respectively. Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be an arbitrary function. The *generalized matrix function* associated with G and χ is the function $d_\chi^G : M_n(\mathbb{C}) \to \mathbb{C}$ given by

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $A = (a_{ij}) \in M_n(\mathbb{C})$. The determinant and the permanent are two famous generalized matrix functions. In fact, if $G = \mathbb{S}_n$ and $\chi = \varepsilon$ is the alternating character of G, then $d_\chi^G = \det$ is the determinant and if $G = \mathbb{S}_n$ and $\chi = 1_G$ is the principal character of G, then $d_\chi^G = \mathrm{per}$ is the permanent. Clearly if $\chi, \varphi : G \to \mathbb{C}$ are two functions and $\lambda \in \mathbb{C}$, then $d_{\chi+\lambda\varphi}^G = d_\chi^G + \lambda d_\varphi^G$, and if $\hat{\chi}$ is the unique extension of χ to $\mathbb{S}_n$ which vanishes outside of G, then $d_\chi^G = d_{\hat{\chi}}^{\mathbb{S}_n}$. We refer the reader to [6] and [7] for some deep information about generalized matrix functions. The reader may also consult the papers [3], [4], [5], [8], and the references therein.

Let us introduce some notations and preliminaries which will be used throughout. For each $\sigma \in \mathbb{S}_n$, let

$$\mathrm{Fix}(\sigma) = \{i : 1 \le i \le n, \sigma(i) = i\}$$

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be the set of fixed points of σ and $l(\sigma) = n - |\mathrm{Fix}(\sigma)|$ be the length of σ. Obviously $\sigma = 1$ if and only if $l(\sigma) = 0$, and also $l(\sigma) \neq 1$ for all $\sigma \in \mathbb{S}_n$. It should be noted that $l(\sigma) \leq 2$ if and only if $\sigma = 1$ or σ is a transposition, and $l(\sigma) \leq 3$ if and only if $\sigma = 1$ or σ is a transposition or σ is a 3-cycle. It is important to note that the composition of permutations in $\mathbb{S}_n$ means left-to-right, that is, $(\sigma\tau)(i) = \tau(\sigma(i))$, for any $\sigma, \tau \in \mathbb{S}_n$. Also we know that each $1 \neq \sigma \in \mathbb{S}_n$ can be uniquely written as a product of (nontrivial) disjoint cycles. The number of (nontrivial) disjoint cycles in the decomposition of σ is denoted by $c(\sigma)$.

Let $E_{rs} = (\delta_{ir}\delta_{sj}) \in M_n(\mathbb{C})$ be the standard matrix units, i.e., the matrix which has 1 in the (r,s)-th entry and zeros elsewhere. Also for each $\sigma \in \mathbb{S}_n$, let $A_\sigma = (\delta_{\sigma(i)j}) \in M_n(\mathbb{C})$ be the permutation matrix induced by σ. It can be easily verified that for any $\sigma, \tau \in \mathbb{S}_n$:
(1) $A_\sigma = I_n$ if and only if $\sigma = 1$;
(2) $A_{\sigma\tau} = A_\sigma A_\tau$;
(3) $\det A_\sigma = \mathrm{sgn}(\sigma)$;
(4) A_σ is diagonalizable;
(5) if σ has order m, then each eigenvalue of A_σ is an m-th root of unity;
(6) $A_\sigma^{-1} = A_{\sigma^{-1}} = A_\sigma^t$;
(7) A_σ is a symmetric matrix if and only if $\sigma^2 = 1$;
(8) $E_{rs}A_\sigma = E_{r\sigma(s)}$ and $A_\sigma E_{rs} = E_{\sigma^{-1}(r)s}$.

The set of involutions of $\mathbb{S}_n$ plays a crucial role in the paper and so we denote it by

$$\mathbb{T}_n = \{\sigma \in \mathbb{S}_n : \ \sigma^2 = 1\}.$$

Finally for each $\sigma \in \mathbb{S}_n$, let $S_\sigma = (a_{ij}) \in M_n(\mathbb{C})$ in which a_{ij} is given by

$$a_{ij} = \begin{cases} 1 & \text{if } \ \sigma(i) = j \ \text{ or } \ \sigma(j) = i \\ 0 & \text{otherwise.} \end{cases}$$

These matrices were originally introduced in [8]. It is clear that for any $\sigma \in \mathbb{S}_n$:
(1) $S_\sigma = I_n$ if and only if $\sigma = 1$;
(2) S_σ is a symmetric matrix;
(3) $S_\sigma = S_{\sigma^{-1}}$.

It should be noted that the matrices S_σ may be singular or nonsingular, for example, $S_{(1234)}$ is singular while $S_{(123)}$ is not. It is not difficult to see that for any $\sigma \in \mathbb{S}_n$ the following are equivalent:
(1) $\sigma \in \mathbb{T}_n$;
(2) σ is a product of disjoint transpositions;
(3) A_σ is a symmetric matrix;
(4) $S_\sigma = A_\sigma$;
(5) 2 is not an eigenvalue of S_σ.

The main objective of this paper is the study of generalized matrix functions using the permutation matrices and symmetric matrices. The results presented here can be viewed as deep generalizations or analogs of the results presented in [3], [4], and [8]. The most results obtained here, regarding the generalized matrix function d_χ^G, are proven first in the case χ is a character of G and then for an arbitrary function χ. This is because one can obtain more results and the proofs also are easier when χ is a character of G compared with the case χ is not a character of G.

The paper is organized as follows. In section 2, first we examine the zeroness of a generalized matrix function on the permutation matrices or on the symmetric matrices, and then, as a corollary, some equivalent conditions will be given for the equality of two generalized matrix functions on the symmetric matrices. In section 3, generalized matrix functions preserving commutativity of the permutation matrices or commutativity of the symmetric matrices will be characterized. In section 4, we concentrate on those generalized matrix functions which preserve product of the permutation matrices or product of the symmetric matrices. Finally, in section 5, our study is focused on the so-called generalized characteristic polynomials and the Cayley-Hamilton Theorem.

2. Zeroness and equality

We begin this section with a rather simple theorem.

Theorem 2.1. *Let $G \leqslant \mathbb{S}_n$ and χ be an irreducible character of G. Then either $d_\chi^G = \det$ or $d_\chi^G = \mathrm{per}$ if and only if $d_\chi^G(A_\sigma) \neq 0$ for all $\sigma \in \mathbb{S}_n$.*

Proof. It is trivial that if $\sigma \in \mathbb{S}_n$, then

$$\det(A_\sigma) = \varepsilon(\sigma) = \pm 1 \neq 0,$$

$$\mathrm{per}(A_\sigma) = 1_{\mathbb{S}_n}(\sigma) = 1 \neq 0.$$

Conversely, if $d_\chi^G(A_\sigma) \neq 0$ for all $\sigma \in \mathbb{S}_n$, then $\hat{\chi}(\sigma) \neq 0$ for all $\sigma \in \mathbb{S}_n$ and so $G = \mathbb{S}_n$ and χ is an irreducible character of $\mathbb{S}_n$ which vanishes nowhere. By Burnside's Theorem, see Theorem 3.15 in [2], every nonlinear irreducible character vanishes somewhere, hence we deduce that χ is a linear character of $\mathbb{S}_n$. Since $1_{\mathbb{S}_n}$ and ε are the only linear characters of $\mathbb{S}_n$, one has either $\chi = \varepsilon$ or $\chi = 1_{\mathbb{S}_n}$ and the proof is complete. □

Irreducibility of the character χ in Theorem 2.1 is essential, as the following example shows.

Example 2.2. *It is obvious that $\chi = 1_{\mathbb{S}_2} + 2\varepsilon$ is a character of $\mathbb{S}_2$ which is not irreducible. Also $d_\chi^{\mathbb{S}_2}(A_\sigma) = \chi(\sigma) = 1 + 2\varepsilon(\sigma) \neq 0$ for all $\sigma \in \mathbb{S}_2$ and $d_\chi^{\mathbb{S}_2} \neq \det$ and $d_\chi^{\mathbb{S}_2} \neq \mathrm{per}$.*

For any character χ of a group G, one has $|\chi(g)| \leq \chi(1)$ for any $g \in G$. It is proved that the set

$$Z(\chi) = \{g \in G : |\chi(g)| = \chi(1)\}$$

is a normal subgroup of G, see Chapter 2 of [2]. Every nonlinear irreducible character vanishes somewhere by Burnside's Theorem. Hence one can deduce that an irreducible character χ of G is linear if and only if $G = Z(\chi)$. It should also be remarked that if a character χ of G is the sum of two characters φ and ψ of G, then $Z(\chi) \subseteq Z(\varphi) \cap Z(\psi)$. The zeroness of d_χ^G on the symmetric matrices, when χ is a character of G, is examined in the next theorem.

Theorem 2.3. *Let $G \leqslant \mathbb{S}_n$ and χ be a character of G. Then the following are equivalent:*
(i) $d_\chi^G = \chi(1)\det$;
(ii) $d_\chi^G(A) = \chi(1)\det(A)$ *for all symmetric matrices* $A \in M_n(\mathbb{C})$;
(iii) $d_\chi^G(S_\sigma) = \chi(1)\det(S_\sigma)$ *for all* $\sigma \in \mathbb{S}_n$;
(iv) $d_\chi^G(A_{(k\,k+1)}) = -\chi(1)$ *for all* $1 \leq k \leq n-1$;
(v) $d_\chi^G(A) = 0$ *for all singular symmetric matrices* $A \in M_n(\mathbb{C})$;
(vi) $d_\chi^G(A) \neq 0$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$;
(vii) $d_\chi^G(I_n + A_{(k\,k+1)}) = 0$ *for all* $1 \leq k \leq n-1$.

Proof. First note that $S_{(k\,k+1)} = A_{(k\,k+1)}$ and the matrix $I_n + A_{(k\,k+1)}$ is a singular symmetric matrix for any $1 \leq k \leq n-1$. So (i) ⇒ (ii) ⇒ (iii) ⇒ (iv), (ii) ⇒ (vi), and (ii) ⇒ (v) ⇒ (vii) are obvious. Hence it suffices to prove (iv) ⇒ (i), (vi) ⇒ (i), and (vii) ⇒ (i).

In the remaining cases, we first claim that $\hat{\chi}((k\ k+1)) = -\chi(1)$ for all $1 \le k \le n-1$, where $n \ge 2$. The claim is obvious if hypothesis (iv) holds. For any $1 \le k \le n-1$, using the matrix

$$I_n + A_{(k\,k+1)} = \left[\begin{array}{c|cc|c} 2I_{k-1} & \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \\ \hline \mathbf{0} & 1 & 1 & \mathbf{0} \\ & 1 & 1 & \\ \hline \mathbf{0} & \multicolumn{2}{c|}{\mathbf{0}} & 2I_{n-k-1} \end{array}\right]$$

one has by hypothesis (vii) that

$$0 = d_\chi^G(I_n + A_{(k\,k+1)}) = 2^{n-2}(\chi(1) + \hat{\chi}((k\ k+1))),$$

implying that $\hat{\chi}((k\ k+1)) = -\chi(1)$. Now assuming hypothesis (vi), suppose by way of contradiction that $\hat{\chi}((k\ k+1)) \neq -\chi(1)$ for some $1 \le k \le n-1$. Since χ is a character of G, $\chi(1)$ is nonzero and so the matrix

$$A = \left[\begin{array}{c|cc|c} I_{k-1} & \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \\ \hline \mathbf{0} & -\frac{\hat{\chi}((k\,k+1))}{\chi(1)} & 1 & \mathbf{0} \\ & 1 & 1 & \\ \hline \mathbf{0} & \multicolumn{2}{c|}{\mathbf{0}} & I_{n-k-1} \end{array}\right]$$

is a nonsingular symmetric matrix. But this contradicts hypothesis (vi) because $d_\chi^G(A) = 0$. Hence the claim is proved.

Now by the claim $\hat{\chi}((k\ k+1)) = -\chi(1) \neq 0$, for any $1 \le k \le n-1$, and so G contains all transpositions $(k\ k+1)$ and hence $\chi((k\ k+1)) = -\chi(1)$, where $1 \le k \le n-1$. In particular, $|\chi((k\ k+1))| = \chi(1)$, for any $1 \le k \le n-1$, which implies that the subgroup $Z(\chi)$ also contains all transpositions $(k\ k+1)$, where $1 \le k \le n-1$. But the set $\{(k\ k+1) \mid 1 \le k \le n-1\}$ generates $\mathbb{S}_n$ and thus $G = Z(\chi) = \mathbb{S}_n$ and $\hat{\chi} = \chi$. Therefore if φ is an irreducible constituent of χ, then $\mathbb{S}_n = Z(\varphi)$ by the remark mentioned before the theorem. This means that all irreducible constituents of χ are linear. But $1_{\mathbb{S}_n}$ and ε are the only linear characters of $\mathbb{S}_n$ and hence $\chi = r1_{\mathbb{S}_n} + s\varepsilon$ for some nonnegative integers r, s. Thus

$$-(r+s) = -\chi(1) = \chi((12)) = r - s,$$

which implies that $r = 0$ and so $\chi = \chi(1)\varepsilon$. This completes the proof. □

The next example shows that Theorem 2.3 does not hold if χ is not a character of G. It is also not true even if χ is a class function of G.

Example 2.4. (i) *Let $\chi : \mathbb{S}_3 \to \mathbb{C}$ be a function given by*

$$\chi(\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ -1 & \text{if } l(\sigma) = 2 \\ 2 & \text{if } \sigma = (123) \\ 0 & \text{if } \sigma = (132) \end{cases}$$

One can see that $d_\chi^{\mathbb{S}_3}(A) = \det(A)$ for all symmetric matrices $A \in M_3(\mathbb{C})$.

(ii) *Let χ be a class function of $\mathbb{A}_3$ given by*

$$\chi(\sigma) = \begin{cases} 0 & \text{if } \sigma = 1 \\ 1 & \text{if } \sigma = (123) \\ -1 & \text{if } \sigma = (132) \end{cases}$$

One can easily verify that $d_\chi^{\mathbb{A}_3}(A) = 0$ for all symmetric matrices $A \in M_3(\mathbb{C})$.

To state our next results, we need some preliminaries. We define a binary relation on $\mathbb{S}_n$ as follows. For any $\sigma, \tau \in \mathbb{S}_n$, we say that

$$\sigma \sim \tau \text{ if and only if } \prod_{i=1}^{n} a_{i\sigma(i)} = \prod_{i=1}^{n} a_{i\tau(i)},$$

for all symmetric matrices $A = (a_{ij}) \in M_n(\mathbb{C})$. It is clear that $\sim$ is an equivalence relation on $\mathbb{S}_n$. The equivalence class of $\sigma \in \mathbb{S}_n$ is denoted by $[\sigma]$. The next lemma gives us some information about $\sim$.

Lemma 2.5. *Let $\sigma, \tau \in \mathbb{S}_n - \{1\}$. Then*
(i) $[1] = \{1\}$;
(ii) $\{\sigma, \sigma^{-1}\} \subseteq [\sigma]$;
(iii) *if $\sigma = \sigma_1 \ldots \sigma_s$ and $\tau = \tau_1 \ldots \tau_s$ are the decompositions of σ and τ into disjoint cycles, where the cycles σ_i, τ_i have the same moving points for any $1 \le i \le s$, then*

$$\sigma \sim \tau \text{ if and only if } \sigma_i \sim \tau_i, \text{ for any } 1 \le i \le s;$$

(iv) *if $\sigma = \sigma_1 \ldots \sigma_s$ is the decomposition of σ into disjoint cycles, then*

$$\{\sigma_1^{n_1} \ldots \sigma_s^{n_s} : \ n_1, \ldots, n_s \in \{-1, 1\}\} \subseteq [\sigma];$$

(v) *if $\tau(k) \in \{\sigma(k), \sigma^{-1}(k)\}$ for any $1 \le k \le n$, then* $\mathrm{Fix}(\tau) = \mathrm{Fix}(\sigma)$ *and $c(\tau) \ge c(\sigma)$;*
(vi) *if σ, τ are two cycles and $\tau(k) \in \{\sigma(k), \sigma^{-1}(k)\}$ for any $1 \le k \le n$, then $\tau \in \{\sigma, \sigma^{-1}\}$;*
(vii) *if $\sigma = \sigma_1 \ldots \sigma_s$ is the decomposition of σ into disjoint cycles, $\tau(k) \in \{\sigma(k), \sigma^{-1}(k)\}$ for any $1 \le k \le n$, and $c(\tau) \le c(\sigma)$, then*

$$\tau \in \{\sigma_1^{n_1} \ldots \sigma_s^{n_s} : \ n_1, \ldots, n_s \in \{-1, 1\}\};$$

(viii) *if $\sigma = \sigma_1 \ldots \sigma_s$ is the decomposition of σ into disjoint cycles, then*

$$[\sigma] = \{\sigma_1^{n_1} \ldots \sigma_s^{n_s} : \ n_1, \ldots, n_s \in \{-1, 1\}\};$$

(ix) $|[\sigma]| = 2^m$ *for some nonnegative integer m;*
(x) $\sigma \in \mathbb{T}_n$ *if and only if* $|[\sigma]| = 1$.

Proof. (i): Taking the symmetric matrix $A = (a_{ij})$ to be I_n gives $[1] = \{1\}$.
(ii): It is enough to show that $\sigma^{-1} \in [\sigma]$. For any symmetric matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ we have

$$\begin{aligned} \prod_{i=1}^{n} a_{i\sigma^{-1}(i)} &= \prod_{j=1}^{n} a_{\sigma(j)\sigma^{-1}(\sigma(j))} \\ &= \prod_{j=1}^{n} a_{\sigma(j)j} \\ &= \prod_{j=1}^{n} a_{j\sigma(j)}, \end{aligned}$$

which means that $\sigma \sim \sigma^{-1}$.
(iii): By hypothesis the cycles σ_i, τ_i have the same set Ω_i of moving points for any $1 \le i \le s$ and so σ, τ have

the same set Ω_0 of fixed points. Now we have

$$\begin{aligned}
\sigma \sim \tau \quad &\Leftrightarrow \quad \prod_{i=1}^{n} a_{i\sigma(i)} = \prod_{i=1}^{n} a_{i\tau(i)}, \text{ for any symmetric } (a_{ij})\\
&\Leftrightarrow \quad \prod_{i\notin\Omega_0} a_{i\sigma(i)} = \prod_{i\notin\Omega_0} a_{i\tau(i)}, \text{ for any symmetric } (a_{ij})\\
&\Leftrightarrow \quad \prod_{i=1}^{s}\Big(\prod_{j\in\Omega_i} a_{j\sigma_i(j)}\Big) = \prod_{i=1}^{s}\Big(\prod_{j\in\Omega_i} a_{j\tau_i(j)}\Big), \text{ for any symmetric } (a_{ij})\\
&\Leftrightarrow \quad \prod_{j\in\Omega_i} a_{j\sigma_i(j)} = \prod_{j\in\Omega_i} a_{j\tau_i(j)}, \ \forall 1 \le i \le s, \text{ for any symmetric } (a_{ij})\\
&\Leftrightarrow \quad \prod_{j=1}^{n} a_{j\sigma_i(j)} = \prod_{j=1}^{n} a_{j\tau_i(j)}, \ \forall 1 \le i \le s, \text{ for any symmetric } (a_{ij})\\
&\Leftrightarrow \quad \sigma_i \sim \tau_i, \ \forall 1 \le i \le s.
\end{aligned}$$

(iv): It comes from (ii) and (iii).
(v): Since $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\sigma^{-1})$, using hypothesis we obtain $\mathrm{Fix}(\tau) = \mathrm{Fix}(\sigma)$. Now assume that $\sigma = \sigma_1 \dots \sigma_s$ is the decomposition of σ into disjoint cycles and Ω_i is the set of moving points of σ_i for any $1 \le i \le s$. Thus Ω_i is the set of moving points of σ_i^{-1} and by hypothesis $\tau(k) \in \Omega_i$ for any $k \in \Omega_i$, which means that τ maps each Ω_i into itself. Hence τ_i, the restriction of τ to Ω_i, lies in $\mathbb{S}_{\Omega_i}$ and therefore $\tau = \tau_1 \dots \tau_s$, that is, $c(\tau) \ge c(\sigma)$.
(vi): The two cycles σ, τ have the same fixed points by (v) and so have the same moving points. It then follows that

$$\sigma = (a\ \sigma(a) \dots \sigma^{m-1}(a)), \quad \tau = (a\ \tau(a) \dots \tau^{m-1}(a)),$$

for some $m \ge 2$ and $1 \le a \le n$. By hypothesis we have either $\tau(a) = \sigma(a)$ or $\tau(a) = \sigma^{-1}(a)$ and will show that either $\tau = \sigma$ or $\tau = \sigma^{-1}$, respectively. Suppose that $\tau(a) = \sigma(a)$ and $2 \le k \le m-1$ is the least number so that $\tau^k(a) \ne \sigma^k(a)$. So by hypothesis

$$\tau^k(a) = \tau(\tau^{k-1}(a)) = \tau(\sigma^{k-1}(a)) = \sigma^{-1}(\sigma^{k-1}(a)) = \sigma^{k-2}(a) = \tau^{k-2}(a),$$

which implies that $\tau^2(a) = a$, that is, $m = 2$, a contradiction. Similarly, if $\tau(a) = \sigma^{-1}(a)$, then $\tau = \sigma^{-1}$.
(vii): We have $c(\tau) = c(\sigma)$ using hypothesis and (v). Now the proof of (v) tells us that $\tau = \tau_1 \dots \tau_s$ is the decomposition of τ into disjoint cycles and Ω_i is the set of moving points of the cycles σ_i, τ_i for any $1 \le i \le s$. Since $\tau_i(k) \in \{\sigma_i(k), \sigma_i^{-1}(k)\}$ for any $1 \le k \le n$, one deduces by (vi) that $\tau_i \in \{\sigma_i, \sigma_i^{-1}\}$ for any $1 \le i \le s$ and the proof is complete.
(viii): One part is trivial by (iv). Suppose now that $\sigma \sim \tau$ and so we have for the symmetric matrix $S_\sigma = (a_{ij})$ that

$$\prod_{i=1}^{n} a_{i\tau(i)} = \prod_{i=1}^{n} a_{i\sigma(i)} = 1,$$

implying that $a_{i\tau(i)} = 1$ for any $1 \le i \le n$. It follows by definition of S_σ that $\tau(i) \in \{\sigma(i), \sigma^{-1}(i)\}$ for any $1 \le i \le n$. Hence we deduce that $c(\tau) \ge c(\sigma)$ by (v). In a similar manner, using the symmetric matrix S_τ, one obtains $c(\sigma) \ge c(\tau)$. Therefore $c(\sigma) = c(\tau)$ and the result follows by (vii).
(ix) and (x): It is clear from (ii) and (viii). □

It should be noted that for any $\sigma \in \mathbb{S}_n$ the set $[\sigma]$ is contained in $\sigma^{\mathbb{S}_n}$, the conjugacy class of σ in $\mathbb{S}_n$, and hence every class function of $\mathbb{S}_n$ is constant on $[\sigma]$.

Taking a closer look at Example 2.4 would lead us to the following generalization of Theorem 2.3. Notice that the following can be compared with Theorem 2.1 of [3] or Theorem 2.1 of [4]. The zeroness of $d_\chi^{\mathbb{S}_n}$ on the symmetric matrices is examined in the next theorem when χ is an arbitrary function on $\mathbb{S}_n$.

Theorem 2.6. *Let $\chi : \mathbb{S}_n \to \mathbb{C}$ be a function. Then the following are equivalent:*
(i) $\sum_{\tau\in[\sigma]} \chi(\tau) = |[\sigma]|\chi(1)\varepsilon(\sigma)$ *for all* $\sigma \in \mathbb{S}_n$*;*
(ii) $d_\chi^{\mathbb{S}_n}(A) = \chi(1)\det(A)$ *for all symmetric matrices* $A \in M_n(\mathbb{C})$*;*
(iii) $d_\chi^{\mathbb{S}_n}(S_\sigma) = \chi(1)\det(S_\sigma)$ *for all* $\sigma \in \mathbb{S}_n$*;*
(iv) $d_\chi^{\mathbb{S}_n}(A) = 0$ *for all singular symmetric matrices* $A \in M_n(\mathbb{C})$*;*
(v) $d_\chi^{\mathbb{S}_n}(A) = \chi(1)\det(A)$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$*.*
Also, any one of the conditions (i) − (v) *together with the condition* $\chi(1) \neq 0$ *is equivalent to:*
(vi) $d_\chi^{\mathbb{S}_n}(A) \neq 0$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$*.*

Proof. (ii) ⇒ (iii), (ii) ⇒ (iv), (ii) ⇒ (v), ((ii) &$\chi(1) \neq 0$) ⇒ (vi), and ((v) &$\chi(1) \neq 0$) ⇒ (vi) are obvious. So it suffices to show that (i) ⇒ (ii), (iii) ⇒ (i), (iv) ⇒ (i), and (vi) ⇒ ((i) &$\chi(1) \neq 0$).
(i) ⇒ (ii): Let Ω be the set of representatives for the equivalence classes of $\sim$ on $\mathbb{S}_n$. Using hypothesis and Lemma 2.5, for any symmetric matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ we have

$$\begin{aligned}
d_\chi^{\mathbb{S}_n}(A) &= \sum_{\sigma\in\Omega}\sum_{\tau\in[\sigma]} \chi(\tau) \prod_{i=1}^n a_{i\tau(i)} \\
&= \sum_{\sigma\in\Omega}\Big(\sum_{\tau\in[\sigma]} \chi(\tau)\Big) \prod_{i=1}^n a_{i\sigma(i)} \\
&= \sum_{\sigma\in\Omega} |[\sigma]|\chi(1)\varepsilon(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \\
&= \chi(1)\sum_{\sigma\in\Omega}\Big(\sum_{\tau\in[\sigma]} \varepsilon(\tau)\Big) \prod_{i=1}^n a_{i\sigma(i)} \\
&= \chi(1)\sum_{\sigma\in\Omega}\sum_{\tau\in[\sigma]} \varepsilon(\tau) \prod_{i=1}^n a_{i\tau(i)} \\
&= \chi(1)\det(A),
\end{aligned}$$

which completes the proof.
(iii) ⇒ (i), (iv) ⇒ (i), and (vi) ⇒ ((i) &$\chi(1) \neq 0$): If $\varphi : \mathbb{S}_n \to \mathbb{C}$ is a function given by

$$\varphi(\sigma) = \chi(\sigma) - \chi(1)\varepsilon(\sigma),$$

then

$$\sum_{\tau\in[\sigma]} \varphi(\tau) = \sum_{\tau\in[\sigma]} \chi(\tau) - \sum_{\tau\in[\sigma]} \chi(1)\varepsilon(\tau) = \sum_{\tau\in[\sigma]} \chi(\tau) - |[\sigma]|\chi(1)\varepsilon(\sigma),$$

and so we have to show that

$$\sum_{\tau\in[\sigma]} \varphi(\tau) = 0,$$

for any $\sigma \in \mathbb{S}_n$. Obviously

$$\sum_{\tau\in[1]} \varphi(\tau) = \varphi(1) = 0.$$

By way of contradiction, choose $1 \neq \sigma \in \mathbb{S}_n$ so that $l(\sigma)$ is minimal and $c(\sigma)$ is maximal and

$$\sum_{\tau\in[\sigma]} \varphi(\tau) \neq 0.$$

Thus if $\tau \in \mathbb{S}_n$ and either $l(\tau) < l(\sigma)$ or $l(\tau) = l(\sigma)$ and $c(\tau) > c(\sigma)$, then

$$\sum_{\alpha\in[\tau]} \varphi(\alpha) = 0.$$

Now let $A = (a_{ij})$ be the symmetric matrix $xI_n + S_\sigma$, where $x \in \mathbb{C}$. By definition of $\sim$, the set

$$\Omega = \{\tau \in \mathbb{S}_n : \prod_{i=1}^{n} a_{i\tau(i)} \neq 0\}$$

is a union of equivalence classes of $\sim$. Clearly

$$\prod_{i=1}^{n} a_{i\sigma(i)} = (x+1)^{n-l(\sigma)}$$

and so $\sigma \in \Omega$ if $x \neq -1$. Let Γ, containing σ if $x \neq -1$, be the set of representatives for the equivalence classes of $\sim$ contained in Ω. If $\alpha \in \Gamma$, then by definition of A we have $\alpha(i) \in \{i, \sigma(i), \sigma^{-1}(i)\}$ for any $1 \le i \le n$ and so $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\alpha)$. If $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\alpha)$, then $l(\alpha) < l(\sigma)$ and so by the choice of σ we have

$$\sum_{\tau\in[\alpha]} \varphi(\tau) = 0.$$

But if $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\alpha)$, then $l(\sigma) = l(\alpha)$ and $\alpha(i) \in \{\sigma(i), \sigma^{-1}(i)\}$ for any $1 \le i \le n$. Hence by (v) of Lemma 2.5, $c(\alpha) \ge c(\sigma)$. If $c(\alpha) > c(\sigma)$, then again by the choice of σ we obtain

$$\sum_{\tau\in[\alpha]} \varphi(\tau) = 0.$$

Now if $c(\alpha) = c(\sigma)$, then by (vii) of Lemma 2.5, $\alpha \in [\sigma]$ and so $\alpha = \sigma$, because $\alpha, \sigma \in \Gamma$.

Therefore

$$\begin{aligned}
d_\chi^{\mathbb{S}_n}(A) - \chi(1)\det(A) &= d_\varphi^{\mathbb{S}_n}(A) \\
&= \sum_{\alpha\in\Gamma}\sum_{\tau\in[\alpha]} \varphi(\tau) \prod_{i=1}^{n} a_{i\tau(i)} \\
&= \sum_{\alpha\in\Gamma}\sum_{\tau\in[\alpha]} \varphi(\tau) \prod_{i=1}^{n} a_{i\alpha(i)} \\
&= \sum_{\alpha\in\Gamma}\Big(\sum_{\tau\in[\alpha]} \varphi(\tau)\Big) \prod_{i=1}^{n} a_{i\alpha(i)} \\
&= \Big(\sum_{\tau\in[\sigma]} \varphi(\tau)\Big) \prod_{i=1}^{n} a_{i\sigma(i)} \\
&= (x+1)^{n-l(\sigma)} \sum_{\tau\in[\sigma]} \varphi(\tau). \qquad (*)
\end{aligned}$$

Putting $x = 0$ in relation $(*)$ will result in

$$d_\chi^{\mathbb{S}_n}(S_\sigma) - \chi(1)\det(S_\sigma) = \sum_{\tau\in[\sigma]} \varphi(\tau) \neq 0,$$

which contradicts hypothesis (iii). This completes the proof of (iii) $\Rightarrow$ (i).

Since $\mathrm{tr}(S_\sigma) = n - l(\sigma) \neq n$, hence S_σ has an eigenvalue $\lambda \neq 1$. Thus the matrix $A = -\lambda I_n + S_\sigma$ is singular and so by putting $x = -\lambda$ in relation (∗) we obtain

$$d_\chi^{\mathbb{S}_n}(A) = (1-\lambda)^{n-l(\sigma)} \sum_{\tau\in[\sigma]} \varphi(\tau) \neq 0,$$

contradicting hypothesis (iv). This completes the proof of (iv) ⇒ (i).

Assuming hypothesis (vi), we have $\chi(1) = d_\chi^{\mathbb{S}_n}(I_n) \neq 0$. Hence it can be easily seen that $p(x) = \chi(1)\det(xI_n + S_\sigma)$ and

$$q(x) = p(x) + (x+1)^{n-l(\sigma)} \sum_{\tau\in[\sigma]} \varphi(\tau)$$

are both polynomials in x having degree n and leading coefficient $\chi(1)$.

Suppose by way of contradiction that each root of $q(x)$ is also a root of $p(x)$. So each root of $q(x)$ is a root of $q(x) - p(x)$, too. We get a contradiction if $l(\sigma) = n$. So if $l(\sigma) < n$, then it follows that -1 is the only root of $q(x)$. Thus $q(x) = \chi(1)(x+1)^n$ and

$$p(x) = \chi(1)(x+1)^n - (x+1)^{n-l(\sigma)} \sum_{\tau\in[\sigma]} \varphi(\tau).$$

One can see using $l(\sigma) \geq 2$ that the coefficient of x^{n-1} in $p(x)$ is $\chi(1)n$. But this implies that the coefficient of x^{n-1} in $\det(xI_n - S_\sigma)$ is $-n$, that is, $\mathrm{tr}(S_\sigma) = n$, a contradiction. Hence one can choose some root λ of $q(x)$ such that λ is not a root of $p(x)$. Therefore the matrix $A = \lambda I_n + S_\sigma$ is nonsingular and so by putting $x = \lambda$ in relation (∗) we have

$$d_\chi^{\mathbb{S}_n}(A) = q(\lambda) = 0,$$

which contradicts hypothesis (vi). Therefore the proof of (vi) ⇒ ((i) &$\chi(1) \neq 0$) and so the proof of the theorem is completed. □

As a consequence of Theorem 2.6 we obtain:

Corollary 2.7. *Let $G \leqslant \mathbb{S}_n$ and χ be a class function of G with $\chi(1) \neq 0$. Then the following are equivalent:*
(i) $d_\chi^G = \chi(1)\det$;
(ii) $d_\chi^G(A) = \chi(1)\det(A)$ *for all symmetric matrices* $A \in M_n(\mathbb{C})$;
(iii) $d_\chi^G(S_\sigma) = \chi(1)\det(S_\sigma)$ *for all* $\sigma \in \mathbb{S}_n$;
(iv) $d_\chi^G(A) = 0$ *for all singular symmetric matrices* $A \in M_n(\mathbb{C})$;
(v) $d_\chi^G(A) = \chi(1)\det(A)$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$;
(vi) $d_\chi^G(A) \neq 0$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$.

Proof. First note that $d_\chi^G = d_{\hat{\chi}}^{\mathbb{S}_n}$. Now, using (x) of Lemma 2.5 and $\chi(1) \neq 0$, the condition (i) in Theorem 2.6 reduces to $\hat{\chi}(\sigma) = \chi(1)\varepsilon(\sigma) \neq 0$ for any transposition $\sigma \in \mathbb{S}_n$, which means that $G = \mathbb{S}_n$. Hence χ is a class function of $\mathbb{S}_n$ and the condition (i) in Theorem 2.6 reduces to $\chi = \chi(1)\varepsilon$ and hence $d_\chi^G = \chi(1)\det$. □

Another interesting corollary of Theorem 2.6 is the following which is a generalization of Theorem 3.9 and Corollary 3.12 of [8]. It can also be compared with Theorem 2.2 of [3].

Corollary 2.8. *Let $\chi, \varphi : \mathbb{S}_n \to \mathbb{C}$ be two functions. Then the following are equivalent:*
(i) $\sum_{\tau\in[\sigma]} \chi(\tau) = \sum_{\tau\in[\sigma]} \varphi(\tau)$ *for all* $\sigma \in \mathbb{S}_n$;
(ii) $d_\chi^{\mathbb{S}_n}(A) = d_\varphi^{\mathbb{S}_n}(A)$ *for all symmetric matrices* $A \in M_n(\mathbb{C})$;
(iii) $d_\chi^{\mathbb{S}_n}(S_\sigma) = d_\varphi^{\mathbb{S}_n}(S_\sigma)$ *for all* $\sigma \in \mathbb{S}_n$;
(iv) $d_\chi^{\mathbb{S}_n}(A) = d_\varphi^{\mathbb{S}_n}(A)$ *for all singular symmetric matrices* $A \in M_n(\mathbb{C})$ *and* $\chi(1) = \varphi(1)$;
(v) $d_\chi^{\mathbb{S}_n}(A) = d_\varphi^{\mathbb{S}_n}(A)$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$.

Proof. First note that if $\sigma = 1$, then $S_\sigma = I_n$ and so we have the equality

$$\chi(1) = d_\chi^{\mathbb{S}_n}(I_n) = d_\varphi^{\mathbb{S}_n}(I_n) = \varphi(1),$$

by any one of the conditions (i)-(v). Now consider the function $\theta : \mathbb{S}_n \to \mathbb{C}$ given by

$$\theta(\sigma) = \varepsilon(\sigma) + \chi(\sigma) - \varphi(\sigma).$$

Hence $\theta(1) = 1$ if each of the conditions (i)-(v) is satisfied. Since

$$\sum_{\tau\in[\sigma]} \theta(\tau) = |[\sigma]|\varepsilon(\sigma) + \sum_{\tau\in[\sigma]} \chi(\tau) - \sum_{\tau\in[\sigma]} \varphi(\tau),$$

for all $\sigma \in \mathbb{S}_n$ and since for any matrix $A \in M_n(\mathbb{C})$

$$d_\theta^{\mathbb{S}_n}(A) = \det(A) + d_\chi^{\mathbb{S}_n}(A) - d_\varphi^{\mathbb{S}_n}(A),$$

hence by applying Theorem 2.6 for θ the result follows. □

It should be remarked that if χ, φ are two class functions of $\mathbb{S}_n$, then the condition (i) in the above corollary is equivalent to "$\chi = \varphi$". Also, the condition "$\chi(1) = \varphi(1)$" is essential in part (iv) of Corollary 2.8, as part (ii) of Example 2.4 shows.

Applying Corollary 2.8 will immediately give the next corollary which generalizes Corollary 3.13 of [8].

Corollary 2.9. *Let χ, φ be two class functions of $\mathbb{S}_n$. Then the following are equivalent:*
(i) $\chi = \varphi$;
(ii) $d_\chi^{\mathbb{S}_n}(A) = d_\varphi^{\mathbb{S}_n}(A)$ *for all symmetric matrices* $A \in M_n(\mathbb{C})$;
(iii) $d_\chi^{\mathbb{S}_n}(S_\sigma) = d_\varphi^{\mathbb{S}_n}(S_\sigma)$ *for all* $\sigma \in \mathbb{S}_n$;
(iv) $d_\chi^{\mathbb{S}_n}(A) = d_\varphi^{\mathbb{S}_n}(A)$ *for all singular symmetric matrices* $A \in M_n(\mathbb{C})$ *and* $\chi(1) = \varphi(1)$;
(v) $d_\chi^{\mathbb{S}_n}(A) = d_\varphi^{\mathbb{S}_n}(A)$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$.

Notice that a result similar to the above corollary does not hold if χ, φ are two class functions of a proper subgroup of $\mathbb{S}_n$.

Example 2.10. *Let χ be the irreducible character of $\mathbb{A}_3$ given by*

$$\chi(\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ \omega & \text{if } \sigma = (123) \\ \omega^2 & \text{if } \sigma = (132) \end{cases}$$

where ω is a primitive cube root of unity. It can be easily verified that if $\varphi = \bar{\chi}$, the conjugate of χ, then $d_\chi^{\mathbb{A}_3}(A) = d_\varphi^{\mathbb{A}_3}(A)$ for all symmetric matrices $A \in M_3(\mathbb{C})$.

3. Preserving commutativity

The first result of this section gives a criterion for a map to be a class function.

Theorem 3.1. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a function. Then χ is a class function of G if and only if $d_\chi^G(A_\alpha A_\beta) = d_\chi^G(A_\beta A_\alpha)$ for all $\alpha, \beta \in G$.*

Proof. Assume that χ is a class function of G and $\alpha, \beta \in G$ are arbitrary. Then

$$\begin{aligned} d_\chi^G(A_\alpha A_\beta) &= d_\chi^G(A_{\alpha\beta}) \\ &= \chi(\alpha\beta) \\ &= \chi(\alpha(\beta\alpha)\alpha^{-1}) \\ &= \chi(\beta\alpha) \\ &= d_\chi^G(A_{\beta\alpha}) \\ &= d_\chi^G(A_\beta A_\alpha). \end{aligned}$$

Conversely, if $\alpha, \beta \in G$ are arbitrary, then by hypothesis

$$\begin{aligned} \chi(\alpha) &= d_\chi^G(A_\alpha) \\ &= d_\chi^G(A_{(\alpha\beta)\beta^{-1}}) \\ &= d_\chi^G(A_{\alpha\beta}A_{\beta^{-1}}) \\ &= d_\chi^G(A_{\beta^{-1}}A_{\alpha\beta}) \\ &= d_\chi^G(A_{\beta^{-1}\alpha\beta}) \\ &= \chi(\beta^{-1}\alpha\beta), \end{aligned}$$

which means that χ is a class function of G. □

As a corollary we obtain:

Corollary 3.2. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a function such that $\chi(\sigma) \neq 0$ for some $1 \neq \sigma \in G$. If $n \neq 4$ and $d_\chi^G(A_\alpha A_\beta) = d_\chi^G(A_\beta A_\alpha)$ for all $\alpha, \beta \in \mathbb{S}_n$, then either $G = \mathbb{A}_n$ or $G = \mathbb{S}_n$.*

Proof. Since $d_{\hat{\chi}}^{\mathbb{S}_n} = d_\chi^G$, it follows by Theorem 3.1 that $\hat{\chi}$ is a class function of $\mathbb{S}_n$. By hypothesis, $\hat{\chi}$ is nonzero on $\sigma^{\mathbb{S}_n}$, the conjugacy class of σ in $\mathbb{S}_n$, and so $\sigma^{\mathbb{S}_n} \subseteq G$. If N is the subgroup of $\mathbb{S}_n$ generated by $\sigma^{\mathbb{S}_n}$, then N is a nontrivial normal subgroup of $\mathbb{S}_n$ contained in G. But $\mathbb{S}_n$ has at most three normal subgroups for $n \neq 4$. Therefore, either $G = \mathbb{A}_n$ or $G = \mathbb{S}_n$ and the proof is completed. □

We need a lemma, whose easy proof is omitted, which plays a key role in the sequel.

Lemma 3.3. (i) *Let $\sigma = (a_1a_2 \ldots a_m) \in \mathbb{S}_n$, where $m \geq 2$. Then $\sigma = \alpha\beta$, where $\alpha, \beta \in \mathbb{T}_n$ are defined as follows:*

$$\alpha = (a_1a_m)(a_2a_{m-1}) \cdots (a_{l-1}a_{l+2})(a_la_{l+1}),$$

$$\beta = (a_ma_2)(a_{m-1}a_3) \cdots (a_{l+3}a_{l-1})(a_{l+2}a_l),$$

if $m = 2l$ is even, and

$$\alpha = (a_1a_m)(a_2a_{m-1}) \cdots (a_{l-1}a_{l+3})(a_la_{l+2}),$$

$$\beta = (a_ma_2)(a_{m-1}a_3) \cdots (a_{l+3}a_l)(a_{l+2}a_{l+1}),$$

if $m = 2l + 1$ is odd;
(ii) *For each $\sigma \in \mathbb{S}_n$ there exist $\alpha, \beta \in \mathbb{T}_n$ such that $\sigma = \alpha\beta$ and* $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$.

It is a well-known problem that if $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ is a function, then $\chi(\sigma) = \chi(\sigma^{-1})$ for all $\sigma \in G$ if and only if $d_\chi^G(A) = d_\chi^G(A^t)$ for all $A \in M_n(\mathbb{C})$, see either Exercise 7 in Section 2.4 of [6] or Exercise 3 in Chapter 7 of [7]. The authors in Theorem 3.14 of [8] added another equivalent condition to that problem, that is,

$$d_\chi^G(AB) = d_\chi^G(BA) \text{ for all symmetric matrices } A, B \in M_n(\mathbb{C}).$$

Their proof uses a theorem due to Frobenius saying that every square complex matrix is a product of two symmetric complex matrices, see [1]. The next theorem not only generalizes Theorem 3.14 of [8] but also is comparable with Theorem 2.2 of [4]. Of course, our proof is different than theirs.

Theorem 3.4. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a function. Then the following are equivalent:*
(i) *$\chi(\sigma) = \chi(\sigma^{-1})$ for all $\sigma \in G$;*
(ii) *$d_\chi^G(A_\sigma) = d_\chi^G(A_{\sigma^{-1}})$ for all $\sigma \in G$;*
(iii) *$d_\chi^G(A) = d_\chi^G(A^t)$ for all $A \in M_n(\mathbb{C})$;*
(iv) *$d_\chi^G(AB) = d_\chi^G(BA)$ for all symmetric matrices $A, B \in M_n(\mathbb{C})$;*
(v) *$d_\chi^G(AB) = d_\chi^G(BA)$ for all nonsingular symmetric matrices $A, B \in M_n(\mathbb{C})$;*
(vi) *$d_\chi^G(AB) = d_\chi^G(BA)$ for all nonsingular symmetric matrix $A \in M_n(\mathbb{C})$ and singular symmetric matrix $B \in M_n(\mathbb{C})$;*
(vii) *$d_\chi^G(AB) = d_\chi^G(BA)$ for all singular symmetric matrices $A, B \in M_n(\mathbb{C})$;*
(viii) *$d_\chi^G(S_\alpha S_\beta) = d_\chi^G(S_\beta S_\alpha)$ for all $\alpha, \beta \in \mathbb{S}_n$;*
(ix) *$d_\chi^G(A_\alpha A_\beta) = d_\chi^G(A_\beta A_\alpha)$ for all $\alpha, \beta \in \mathbb{T}_n$.*

Proof. (i) $\Leftrightarrow$ (ii), (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ix), (iv) $\Rightarrow$ (vi), (iv) $\Rightarrow$ (vii), (iv) $\Rightarrow$ (viii) $\Rightarrow$ (ix) are obvious. We have to prove the following remaining cases.
(i) $\Rightarrow$ (iii): Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and $A^t = (b_{ij})$. Then by hypothesis

$$\begin{aligned} d_\chi^G(A) &= \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \\ &= \sum_{\sigma \in G} \chi(\sigma^{-1}) \prod_{i=1}^n b_{\sigma(i)i} \\ &= \sum_{\sigma \in G} \chi(\sigma^{-1}) \prod_{j=1}^n b_{j\sigma^{-1}(j)} \\ &= \sum_{\tau \in G} \chi(\tau) \prod_{j=1}^n b_{j\tau(j)} \\ &= d_\chi^G(A^t). \end{aligned}$$

(vi) $\Rightarrow$ (vii): Let $A, B \in M_n(\mathbb{C})$ be singular symmetric matrices. Hence there exists some $\epsilon > 0$ such that for all $0 < x < \epsilon$ the matrix $xI_n + A$ is nonsingular and so by hypothesis

$$d_\chi^G((xI_n + A)B) = d_\chi^G(B(xI_n + A)).$$

But both sides of the above equality are polynomials in x and therefore their constant coefficients are equal, that is,

$$d_\chi^G(AB) = d_\chi^G(BA),$$

as required.
(vii) $\Rightarrow$ (i): The assertion is true for the elements of $\mathbb{T}_n$. Suppose by way of contradiction that $\sigma \in \mathbb{S}_n - \mathbb{T}_n$ has been chosen so that $l(\sigma)$ is minimal and $c(\sigma)$ is maximal and $\chi(\sigma) \neq \chi(\sigma^{-1})$. Thus if $\tau \in \mathbb{S}_n$ and either $l(\tau) < l(\sigma)$ or $l(\tau) = l(\sigma)$ and $c(\tau) > c(\sigma)$, then $\chi(\tau) = \chi(\tau^{-1})$.

Let $\sigma = \sigma_1 \dots \sigma_s$ be the decomposition of σ into disjoint cycles, where $\sigma_1 = (a_1 a_2 \dots a_k)$ is a k-cycle with $k \geq 3$. By Lemma 3.3 there exist disjoint permutations $\alpha_1, \dots, \alpha_s \in \mathbb{T}_n$ and disjoint permutations $\beta_1, \dots, \beta_s \in \mathbb{T}_n$ so that

$$\alpha = \alpha_1 \dots \alpha_s, \;\; \beta = \beta_1 \dots \beta_s, \;\; \sigma = \alpha\beta,$$

where

$$\alpha_1 = (a_1 a_k)(a_2 a_{k-1}) \cdots (a_{l-1} a_{l+2})(a_l a_{l+1}),$$

$$\beta_1 = (a_k a_2)(a_{k-1} a_3) \cdots (a_{l+3} a_{l-1})(a_{l+2} a_l),$$

if $k = 2l$ is even, and

$$\alpha_1 = (a_1a_k)(a_2a_{k-1})\cdots(a_{l-1}a_{l+3})(a_la_{l+2}),$$

$$\beta_1 = (a_ka_2)(a_{k-1}a_3)\cdots(a_{l+3}a_l)(a_{l+2}a_{l+1}),$$

if $k = 2l + 1$ is odd.

Now the symmetric matrices

$$A = A_\alpha + E_{a_1a_1} + E_{a_ka_k}, \quad B = A_\beta + E_{a_2a_2} + E_{a_ka_k}$$

are clearly singular and

$$\begin{aligned} AB &= A_\sigma + E_{\alpha(a_2)a_2} + E_{\alpha(a_k)a_k} + E_{a_1\beta(a_1)} + E_{a_k\beta(a_k)} + E_{a_ka_k} \\ &= A_\sigma + E_{a_{k-1}a_2} + E_{a_1a_k} + E_{a_1a_1} + E_{a_ka_2} + E_{a_ka_k}, \end{aligned}$$

$$BA = (AB)^t = A_{\sigma^{-1}} + E_{a_2a_{k-1}} + E_{a_ka_1} + E_{a_1a_1} + E_{a_2a_k} + E_{a_ka_k}.$$

Now consider the sets

$$\Omega = \{\tau \in \mathbb{S}_n : \prod_{i=1}^{n} u_{i\tau(i)} \neq 0\},$$

$$\Gamma = \{\tau \in \mathbb{S}_n : \prod_{i=1}^{n} v_{i\tau(i)} \neq 0\},$$

where $AB = (u_{ij})$ and $BA = (v_{ij})$. Obviously, $u_{ij} = v_{ji} \in \{0, 1\}$ and so

$$\tau^{-1} \in \Gamma \Longleftrightarrow \tau \in \Omega \Longleftrightarrow u_{i\tau(i)} = 1, \ \forall 1 \le i \le n.$$

It is not also difficult to see that if $\tau \in \Omega$, then

$$\tau(a_1) \in \{a_1, a_2, a_k\}, \quad \tau(a_{k-1}) \in \{a_2, a_k\}, \quad \tau(a_k) \in \{a_1, a_2, a_k\},$$

$$\tau(b) = \sigma(b), \ \forall b \notin \{a_1, a_{k-1}, a_k\}.$$

Three cases can occur for $\tau \in \Omega$:
If $\tau(a_1) = a_1$, then either $\tau(a_{k-1}) = a_2$ and $\tau(a_k) = a_k$ or $\tau(a_{k-1}) = a_k$ and $\tau(a_k) = a_2$. In this case $\tau \in \{\tau_1, \tau_2\}$, where

$$\tau_1 = (a_2 \dots a_{k-1})\sigma_2 \dots \sigma_s, \quad \tau_2 = (a_2 \dots a_k)\sigma_2 \dots \sigma_s.$$

If $\tau(a_1) = a_2$, then $\tau(a_{k-1}) = a_k$ and so $\tau(a_k) = a_1$. In this case $\tau = \sigma$.
If $\tau(a_1) = a_k$, then $\tau(a_{k-1}) = a_2$ and so $\tau(a_k) = a_1$. In this case

$$\tau = \tau_3 = (a_1a_k)(a_2 \dots a_{k-1})\sigma_2 \dots \sigma_s.$$

Since $l(\tau_1) + 1 = l(\tau_2) < l(\sigma)$, $l(\tau_3) = l(\sigma)$ and $c(\tau_3) > c(\sigma)$, hence by the choice of σ we have

$$\chi(\tau_i) = \chi(\tau_i^{-1}), \ 1 \le i \le 3.$$

Therefore by hypothesis

$$\begin{aligned}
\chi(\sigma)+\chi(\tau_1)+\chi(\tau_2)+\chi(\tau_3) &= \sum_{\tau\in\Omega}\chi(\tau)\\
&= d_\chi^G(AB)\\
&= d_\chi^G(BA)\\
&= \sum_{\tau\in\Gamma}\chi(\tau)\\
&= \sum_{\tau\in\Omega}\chi(\tau^{-1})\\
&= \chi(\sigma^{-1})+\chi(\tau_1^{-1})+\chi(\tau_2^{-1})+\chi(\tau_3^{-1})\\
&= \chi(\sigma^{-1})+\chi(\tau_1)+\chi(\tau_2)+\chi(\tau_3),
\end{aligned}$$

which means that $\chi(\sigma)=\chi(\sigma^{-1})$, a contradiction.
(ix) ⇒ (i): Let $\sigma\in G$ be arbitrary. By Lemma 3.3, there exist $\alpha,\beta\in\mathbb{T}_n$ such that $\sigma=\alpha\beta$. Hence by hypothesis

$$\begin{aligned}
\chi(\sigma) &= d_\chi^G(A_\sigma)\\
&= d_\chi^G(A_{\alpha\beta})\\
&= d_\chi^G(A_\alpha A_\beta)\\
&= d_\chi^G(A_\beta A_\alpha)\\
&= d_\chi^G(A_{\beta\alpha})\\
&= d_\chi^G(A_{\sigma^{-1}})\\
&= \chi(\sigma^{-1}).
\end{aligned}$$

□

4. Preserving product

We begin this section with a useful theorem characterizing generalized matrix functions which preserve the product of permutation matrices.

Theorem 4.1. *Let $G\leqslant\mathbb{S}_n$ and χ be a character of G. Then the following are equivalent:*
(i) *either* $d_\chi^G=\det$ *or* $d_\chi^G=\mathrm{per}$;
(ii) $d_\chi^G(A_\alpha A_\beta)=d_\chi^G(A_\alpha)d_\chi^G(A_\beta)$ *for all* $\alpha,\beta\in\mathbb{S}_n$;
(iii) $d_\chi^G(A_\alpha A_\beta)=d_\chi^G(A_\alpha)d_\chi^G(A_\beta)$ *for all* $\alpha,\beta\in\mathbb{T}_n$;
(iv) $d_\chi^G(A_\alpha A_\beta)=d_\chi^G(A_\alpha)d_\chi^G(A_\beta)$ *for all* $\alpha,\beta\in\mathbb{S}_n$ *with* $l(\alpha),l(\beta)\leq 2$;
(v) $d_\chi^G(A_{\alpha^2})=d_\chi^G(A_\alpha)^2$ *for all* $\alpha\in\mathbb{S}_n$;
(vi) $d_\chi^G(I_n)=d_\chi^G(A_\alpha)d_\chi^G(A_{\alpha^{-1}})$ *for all* $\alpha\in\mathbb{S}_n$;
(vii) $d_\chi^G(I_n)=d_\chi^G(A_\alpha)^2$ *for all* $\alpha\in\mathbb{S}_n$ *with* $l(\alpha)\leq 2$.

Proof. (ii) ⇒ (iii) ⇒ (iv) ⇒ (vii), (ii) ⇒ (v) ⇒ (vii), and (ii) ⇒ (vi) ⇒ (vii) are obvious. We have to prove the following remaining cases.
(i) ⇒ (ii): The assertion is true if $d_\chi^G=\det$. So assume that $d_\chi^G=\mathrm{per}$ and $\alpha,\beta\in\mathbb{S}_n$ are arbitrary. Then

$$\mathrm{per}(A_\alpha A_\beta)=\mathrm{per}(A_{\alpha\beta})=1_{\mathbb{S}_n}(\alpha\beta)=1_{\mathbb{S}_n}(\alpha)1_{\mathbb{S}_n}(\beta)=\mathrm{per}(A_\alpha)\mathrm{per}(A_\beta).$$

(vii) ⇒ (i): For any $\alpha\in\mathbb{S}_n$ with $l(\alpha)\leq 2$ one has

$$\chi(1)=d_\chi^G(I_n)=d_\chi^G(A_\alpha)^2=\hat{\chi}(\alpha)^2.$$

Hence $\chi(1) = \chi(1)^2$ and so $\chi(1) = 1$, for χ is a character. Also if $\alpha \in \mathbb{S}_n$ is a transposition, then $\hat{\chi}(\alpha)^2 = \chi(1) = 1$ and so G contains all transpositions of $\mathbb{S}_n$. Therefore $G = \mathbb{S}_n$ and χ is a linear character of $\mathbb{S}_n$. It follows that $\chi = 1_{\mathbb{S}_n}$ or $\chi = \varepsilon$ which concludes the proof. □

Some parts of Theorem 4.1 may not be true if χ is not a character of G, as the following example shows.

Example 4.2. *Let $\chi : \mathbb{S}_4 \to \mathbb{C}$ be a class function given by*

$$\chi(\sigma) = \begin{cases} 1 & \text{if } l(\sigma) \neq 2 \\ -1 & \text{if } l(\sigma) = 2 \end{cases}$$

It is easy to see that parts (iv), (v), (vi), *and* (vii) *of Theorem* 4.1 *hold but* $d_\chi^{\mathbb{S}_4} \neq \det$ *and* $d_\chi^{\mathbb{S}_4} \neq \operatorname{per}$.

Our next result is a generalization of Theorem 3.19 of [8]. There are of course some errors in Theorem 3.19 of [8] and its proof, because matrices other than the matrices S_α have been employed by the authors when $n = 2$. In fact Theorem 3.19 of [8] is not true for $n = 2$, as part (iv) of Theorem 4.1 shows.

Corollary 4.3. *Let $G \leqslant \mathbb{S}_n$ and χ be a character of G, where $n \neq 2$. Then the following are equivalent:*
(i) $d_\chi^G = \det$;
(ii) $d_\chi^G(S_\alpha S_\beta) = d_\chi^G(S_\alpha) d_\chi^G(S_\beta)$ *for all $\alpha, \beta \in \mathbb{S}_n$ with $l(\alpha), l(\beta) \leq 3$;*
(iii) $d_\chi^G(S_\alpha^2) = d_\chi^G(S_\alpha)^2$ *for all $\alpha \in \mathbb{S}_n$ with $l(\alpha) \leq 3$.*

Proof. It suffices to prove (iii) ⇒ (i). Since $S_\alpha = A_\alpha$ for any $\alpha \in \mathbb{S}_n$ with $l(\alpha) \leq 2$, hence, by hypothesis and Theorem 4.1, $d_\chi^G = \det$ or $d_\chi^G = \operatorname{per}$. We may assume by way of contradiction that $n \geq 3$ and $d_\chi^G = \operatorname{per}$. So if $\alpha = (123)$ then

$$S_\alpha = \left[\begin{array}{ccc|c} 0 & 1 & 1 & \\ 1 & 0 & 1 & \mathbf{0} \\ 1 & 1 & 0 & \\ \hline & \mathbf{0} & & I_{n-3} \end{array}\right]$$

and

$$S_\alpha^2 = \left[\begin{array}{ccc|c} 2 & 1 & 1 & \\ 1 & 2 & 1 & \mathbf{0} \\ 1 & 1 & 2 & \\ \hline & \mathbf{0} & & I_{n-3} \end{array}\right]$$

and so

$$\operatorname{per}(S_\alpha^2) = 16 \neq 4 = \operatorname{per}(S_\alpha)^2,$$

a contradiction, completing the proof. □

We show by means of an example that Corollary 4.3 may not be true if χ is not a character of G. In fact, Corollary 4.3 need not be true even if χ is a class function of G.

Example 4.4. *Let $\chi : \mathbb{S}_7 \to \mathbb{C}$ be a class function given by*

$$\chi(\sigma) = \begin{cases} 1 & \text{if } l(\sigma) = 7 \\ 0 & \text{if } l(\sigma) < 7 \end{cases}$$

For each $\sigma \in \mathbb{S}_7$ consider the set

$$\Omega_\sigma = \{\tau \in \mathbb{S}_7 : \prod_{i=1}^{7} a_{i\tau(i)} \neq 0\},$$

where $S_\sigma = (a_{ij})$. Obviously, $\tau \in \Omega_\sigma$ if and only if $a_{i\tau(i)} = 1$ for all $1 \leq i \leq 7$. Hence if $\tau \in \Omega_\sigma$, then $\mathrm{Fix}(\tau) = \mathrm{Fix}(\sigma)$ *and so $l(\tau) = l(\sigma)$. Now we have*

$$d_\chi^{\mathbb{S}_7}(S_\sigma) = \sum_{\tau\in\mathbb{S}_7} \chi(\tau) \prod_{i=1}^{7} a_{i\tau(i)} = \sum_{\tau\in\Omega_\sigma} \chi(\tau) \prod_{i=1}^{7} a_{i\tau(i)} = \sum_{\tau\in\Omega_\sigma} \chi(\tau).$$

Therefore if $\alpha, \beta \in \mathbb{S}_7$ with $l(\alpha), l(\beta) \leq 3$, then $d_\chi^{\mathbb{S}_7}(S_\alpha) = d_\chi^{\mathbb{S}_7}(S_\beta) = 0$. It suffices to show that $d_\chi^{\mathbb{S}_7}(S_\alpha S_\beta) = 0$. To this end, let $k \in \mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$. Now if $S_\alpha = (a_{ij})$ and $S_\beta = (b_{ij})$, then for any $1 \leq i \leq 7$

$$a_{ik} = a_{ki} = \delta_{ik},$$

$$b_{ik} = b_{ki} = \delta_{ik}.$$

Hence if $S_\alpha S_\beta = (c_{ij})$, then it can be easily seen that $c_{ik} = c_{ki} = \delta_{ik}$ for any $1 \leq i \leq 7$. Now

$$\begin{aligned}
d_\chi^{\mathbb{S}_7}(S_\alpha S_\beta) &= \sum_{\tau\in\mathbb{S}_7} \chi(\tau) \prod_{i=1}^{7} c_{i\tau(i)} \\
&= \sum_{\tau\in\mathbb{S}_7} \chi(\tau) c_{k\tau(k)} \prod_{\substack{i=1\\ i\neq k}}^{7} c_{i\tau(i)} \\
&= \sum_{\tau\in\mathbb{S}_7} \chi(\tau) \delta_{k\tau(k)} \prod_{\substack{i=1\\ i\neq k}}^{7} c_{i\tau(i)} \\
&= \sum_{\substack{\tau\in\mathbb{S}_7\\ \tau(k)=k}} \chi(\tau) \prod_{\substack{i=1\\ i\neq k}}^{7} c_{i\tau(i)} \\
&= 0,
\end{aligned}$$

the last equality holds because $l(\tau) < 7$ and so $\chi(\tau) = 0$.

As another consequence of Theorem 4.1 one can achieve the following which can be compared with Corollary 2.4 of [4].

Corollary 4.5. *Let $G \leqslant \mathbb{S}_n$ and χ be a character of G. Then the following are equivalent:*
(i) $d_\chi^G = \det$;
(ii) $d_\chi^G(AB) = d_\chi^G(A)d_\chi^G(B)$ *for all symmetric matrices $A, B \in M_n(\mathbb{C})$;*
(iii) $d_\chi^G(AB) = d_\chi^G(A)d_\chi^G(B)$ *for all nonsingular symmetric matrices $A, B \in M_n(\mathbb{C})$;*
(iv) $d_\chi^G(A^2) = d_\chi^G(A)^2$ *for all symmetric matrices $A \in M_n(\mathbb{C})$;*
(v) $d_\chi^G(A^2) = d_\chi^G(A)^2$ *for all nonsingular symmetric matrices $A \in M_n(\mathbb{C})$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious. So it is sufficient to show that (v) $\Rightarrow$ (i). If $\sigma \in \mathbb{S}_n$ with $l(\sigma) \leq 2$, then A_σ is a nonsingular symmetric matrix and so by hypothesis

$$d_\chi^G(A_\sigma)^2 = d_\chi^G(A_\sigma^2) = d_\chi^G(A_{\sigma^2}) = d_\chi^G(I_n),$$

hence $d_\chi^G = \det$ or $d_\chi^G = \mathrm{per}$, by Theorem 4.1. We may assume by way of contradiction that $n \geq 2$ and $d_\chi^G = \mathrm{per}$. So if

$$A = \left[\begin{array}{cc|c} 2 & 1 & \multirow{2}{*}{$\mathbf{0}$} \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & I_{n-2} \end{array}\right]$$

then

$$A^2 = \left[\begin{array}{cc|c} 5 & 3 & \multirow{2}{*}{$\mathbf{0}$} \\ 3 & 2 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & I_{n-2} \end{array}\right]$$

and one gets

$$\mathrm{per}(A^2) = 19 \neq 9 = \mathrm{per}(A)^2,$$

a contradiction, which completes the proof. □

The analog of Corollary 4.5 for singular symmetric matrices is the following which can be compared with Corollary 2.3 of [4].

Theorem 4.6. *Let $G \leqslant \mathbb{S}_n$ and χ be a character of G. Then the following are equivalent:*
(i) $d_\chi^G = \chi(1)\det$;
(ii) $d_\chi^G(AB) = d_\chi^G(A)d_\chi^G(B)$ *for all nonsingular symmetric matrix $A \in M_n(\mathbb{C})$ and singular symmetric matrix $B \in M_n(\mathbb{C})$;*
(iii) $d_\chi^G(AB) = d_\chi^G(A)d_\chi^G(B)$ *for all singular symmetric matrices $A, B \in M_n(\mathbb{C})$;*
(iv) $d_\chi^G(A^2) = d_\chi^G(A)^2$ *for all singular symmetric matrices $A \in M_n(\mathbb{C})$.*

Proof. It suffices to show that (ii) ⇒ (iii) and (iv) ⇒ (i).
(ii) ⇒ (iii): Let $A, B \in M_n(\mathbb{C})$ be singular symmetric matrices. Hence there exists some $\epsilon > 0$ such that for all $0 < x < \epsilon$ the matrix $xI_n + A$ is nonsingular and so by hypothesis

$$d_\chi^G((xI_n + A)B) = d_\chi^G(xI_n + A)d_\chi^G(B).$$

But both sides of the above equality are polynomials in x and therefore their constant coefficients are equal, that is,

$$d_\chi^G(AB) = d_\chi^G(A)d_\chi^G(B),$$

as desired.
(iv) ⇒ (i): For any $1 \leq k \leq n-1$, the matrices

$$A = \left[\begin{array}{c|c|c} I_{k-1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_{n-k-1} \end{array}\right], \quad B = \left[\begin{array}{c|c|c} I_{k-1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_{n-k-1} \end{array}\right]$$

are singular symmetric matrices and so

$$A^2 = \left[\begin{array}{c|c|c} I_{k-1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_{n-k-1} \end{array}\right], \quad B^2 = \left[\begin{array}{c|c|c} I_{k-1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} 5 & 10 \\ 10 & 20 \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_{n-k-1} \end{array}\right].$$

Hence by hypothesis

$$4(\chi(1) + \hat{\chi}((k\ k+1))) = d_\chi^G(A^2) = d_\chi^G(A)^2 = (\chi(1) + \hat{\chi}((k\ k+1)))^2,$$

$$100(\chi(1) + \hat{\chi}((k\ k+1))) = d_\chi^G(B^2) = d_\chi^G(B)^2 = 16(\chi(1) + \hat{\chi}((k\ k+1)))^2,$$

which imply that

$$d_\chi^G(I_n + A_{(k\ k+1)}) = \chi(1) + \hat{\chi}((k\ k+1)) = 0.$$

Now the result follows by Theorem 2.3. □

It should be remarked that Example 2.4 shows that if χ is not a character of G, then parts (iv) and (v) of Corollary 4.5 are not equivalent to part (i) of Corollary 4.5, and part (iv) of Theorem 4.6 is not equivalent to part (i) of Theorem 4.6.

In the sequel we want to prove results analogous to the above for an arbitrary function χ on G. The next generalizes Theorem 4.1.

Theorem 4.7. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a nonzero function. Then the following are equivalent:*
(i) *either $d_\chi^G = \det$ or $d_\chi^G = \mathrm{per}$;*
(ii) $d_\chi^G(A_\alpha A_\beta) = d_\chi^G(A_\alpha) d_\chi^G(A_\beta)$ *for all $\alpha, \beta \in \mathbb{S}_n$;*
(iii) $d_\chi^G(A_\alpha A_\beta) = d_\chi^G(A_\alpha) d_\chi^G(A_\beta)$ *for all $\alpha, \beta \in \mathbb{T}_n$.*

Proof. We need only to prove that (iii) ⇒ (i). For any $\alpha, \beta \in \mathbb{T}_n$ one by hypothesis has

$$\hat{\chi}(\alpha\beta) = \hat{\chi}(\alpha)\hat{\chi}(\beta), \qquad (1)$$

and in particular,

$$\chi(1) = \chi(1)^2, \quad \chi(1) = \hat{\chi}(\alpha)^2. \qquad (2)$$

Also if $\sigma \in \mathbb{S}_n$, then by Lemma 3.3 there exist $\alpha, \beta \in \mathbb{T}_n$ such that $\sigma = \alpha\beta$ and so by (1) one obtains

$$\hat{\chi}(\sigma) = \hat{\chi}(\alpha)\hat{\chi}(\beta). \qquad (3)$$

If $\chi(1) = 0$, then (2) and (3) imply that $\hat{\chi} = 0$, which is a contradiction. Thus $\chi(1) = 1$ and $\hat{\chi}$ vanishes nowhere. This implies that $G = \mathbb{S}_n$, $\hat{\chi} = \chi$ and $\chi(\alpha) = \pm 1$ for any transposition α.

Since $\alpha, \beta \in \mathbb{T}_n$, there exist disjoint transpositions $\alpha_1, \dots, \alpha_s$ and disjoint transpositions $\beta_1, \dots, \beta_r$ such that

$$\alpha = \alpha_1 \dots \alpha_s, \quad \beta = \beta_1 \dots \beta_r.$$

It can be deduced using induction from (1) that

$$\chi(\alpha_1 \dots \alpha_s) = \chi(\alpha_1) \dots \chi(\alpha_s),$$

$$\chi(\beta_1 \dots \beta_r) = \chi(\beta_1) \dots \chi(\beta_r),$$

and hence one has by (3) that

$$\chi(\sigma) = \chi(\alpha_1) \dots \chi(\alpha_s)\chi(\beta_1) \dots \chi(\beta_r). \qquad (4)$$

We now claim that χ has the same value on all transpositions. We may assume that $n \geq 3$ and $(a_1a_2a_3)$ is a 3-cycle. From

$$(a_1a_2a_3) = (a_1a_2)(a_1a_3) = (a_2a_3)(a_1a_2) = (a_1a_3)(a_2a_3)$$

and (1) we get

$$\chi((a_1a_2a_3)) = \chi((a_1a_2))\chi((a_1a_3)) = \chi((a_2a_3))\chi((a_1a_2)) = \chi((a_1a_3))\chi((a_2a_3)),$$

implying that

$$\chi((a_1a_2)) = \chi((a_2a_3)) = \chi((a_1a_3)).$$

Also if $n \geq 4$ and $(a_2a_3a_4)$ is a 3-cycle, then

$$\chi((a_2a_3)) = \chi((a_2a_4)) = \chi((a_3a_4)).$$

Now the claim becomes clear from the two latter relations.

If χ has value 1 on all transpositions, then (4) shows that $\chi = 1_{\mathbb{S}_n}$ and so $d_\chi^G = \mathrm{per}$.

If χ has value -1 on all transpositions, then (4) shows that $\chi = \varepsilon$ and so $d_\chi^G = \det$. This completes the proof. □

The first consequence of Theorem 4.7 is the following which generalizes Corollary 2.4 of [4] and Corollary 4.5.

Corollary 4.8. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a nonzero function. Then the following are equivalent:*
(i) $d_\chi^G = \det$;
(ii) $d_\chi^G(AB) = d_\chi^G(A)d_\chi^G(B)$ *for all symmetric matrices* $A, B \in M_n(\mathbb{C})$;
(iii) $d_\chi^G(AB) = d_\chi^G(A)d_\chi^G(B)$ *for all nonsingular symmetric matrices* $A, B \in M_n(\mathbb{C})$.

Proof. It suffices to show that (iii) ⇒ (i). Since $\sigma \in \mathbb{T}_n$ if and only if A_σ is a nonsingular symmetric matrix, hence $d_\chi^G = \det$ or $d_\chi^G = \text{per}$, by Theorem 4.7. We may assume by way of contradiction that $n \geq 2$ and $d_\chi^G = \text{per}$. Now similar to the proof of Corollary 4.5 if

$$A = \left[\begin{array}{cc|c} 2 & 1 & \multirow{2}{*}{$\mathbf{0}$} \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & I_{n-2} \end{array}\right]$$

then one gets

$$\text{per}(A^2) = 19 \neq 9 = \text{per}(A)^2,$$

a contradiction, which completes the proof. □

The second consequence of Theorem 4.7 is the following which can be compared with Corollary 4.3.

Corollary 4.9. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a nonzero function, where $n \neq 2$. Then the following are equivalent:*
(i) $d_\chi^G = \det$;
(ii) $d_\chi^G(S_\alpha S_\beta) = d_\chi^G(S_\alpha)d_\chi^G(S_\beta)$ *for all* $\alpha, \beta \in \mathbb{S}_n$;
(iii) $d_\chi^G(S_\alpha S_\beta) = d_\chi^G(S_\alpha)d_\chi^G(S_\beta)$ *for all nonsingular* S_α, S_β.

Proof. It is sufficient to show that (iii) ⇒ (i). Since $S_\alpha = A_\alpha, S_\beta = A_\beta$ for any $\alpha, \beta \in \mathbb{T}_n$, hence, by hypothesis and Theorem 4.7, $d_\chi^G = \det$ or $d_\chi^G = \text{per}$. We may assume by way of contradiction that $n \geq 3$ and $d_\chi^G = \text{per}$. Now similar to the proof of Corollary 4.3 if $\alpha = (123)$ then

$$\text{per}(S_\alpha^2) = 16 \neq 4 = \text{per}(S_\alpha)^2,$$

a contradiction, completing the proof. □

The next example shows that the condition

$$d_\chi^G(S_\alpha^2) = d_\chi^G(S_\alpha)^2 \text{ for all } \alpha \in \mathbb{S}_n$$

cannot be inserted as an equivalent condition in Corollary 4.9.

Example 4.10. *Let $\chi : \mathbb{S}_3 \to \mathbb{C}$ be a class function given by*

$$\chi(\sigma) = \begin{cases} \frac{1}{2} & \text{if } l(\sigma) = 3 \\ 0 & \text{if } l(\sigma) < 3 \end{cases}$$

It is easy to see that

$$d_\chi^{\mathbb{S}_3}(S_\alpha^2) = d_\chi^{\mathbb{S}_3}(S_\alpha) = 0, \text{ if } \alpha \in \mathbb{S}_3 \text{ with } l(\alpha) \leq 2$$

and

$$d_\chi^{\mathbb{S}_3}(S_\alpha^2) = d_\chi^{\mathbb{S}_3}(S_\alpha) = 1, \text{ if } \alpha \in \{(123), (132)\}.$$

As a generalization of Corollary 2.3 of [4] and Theorem 4.6 we obtain:

Theorem 4.11. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a function. Then the following are equivalent:*
(i) $d^G_\chi = \chi(1)\det$;
(ii) $d^G_\chi(AB) = d^G_\chi(A)d^G_\chi(B)$ *for all nonsingular symmetric matrix* $A \in M_n(\mathbb{C})$ *and singular symmetric matrix* $B \in M_n(\mathbb{C})$;
(iii) $d^G_\chi(AB) = d^G_\chi(A)d^G_\chi(B)$ *for all singular symmetric matrices* $A, B \in M_n(\mathbb{C})$.

Proof. Obviously (i) ⇒ (ii) and the proof of (ii) ⇒ (iii) is the same as that of Theorem 4.6. So it suffices to show that (iii) ⇒ (i).
First we claim that if $\varphi : \mathbb{S}_n \to \mathbb{C}$ is a function such that $\varphi(1) = 0$ and

$$d^{\mathbb{S}_n}_\varphi(AB) = d^{\mathbb{S}_n}_\varphi(A)d^{\mathbb{S}_n}_\varphi(B)$$

for all singular symmetric matrices $A, B \in M_n(\mathbb{C})$, then $\varphi = 0$.

First we show that the assertion is true for the elements of $\mathbb{T}_n$. Let $1 \neq \tau \in \mathbb{T}_n$ and $\tau = \tau_1 \dots \tau_s$ be the decomposition of τ into disjoint transpositions with $\tau_1 = (a_1a_2)$. Then the symmetric matrices

$$A = A_\tau + E_{a_1a_1} + E_{a_2a_2}, \quad B = A_\tau - E_{a_1a_1} - E_{a_2a_2}$$

are clearly singular and

$$AB = I_n - E_{a_1a_1} - E_{a_2a_2}.$$

Hence by hypothesis

$$0 = d^{\mathbb{S}_n}_\varphi(AB) = d^{\mathbb{S}_n}_\varphi(A)d^{\mathbb{S}_n}_\varphi(B) = (\varphi(\tau) + \varphi(\tau_2 \dots \tau_s))^2.$$

Now by induction on the number of transpositions in the decomposition of τ one obtains $\varphi(\tau) = 0$.

To complete the proof of the claim, by way of contradiction choose $\sigma \in \mathbb{S}_n$ so that $l(\sigma)$ is minimal and $c(\sigma)$ is maximal and $\varphi(\sigma) \neq 0$. Thus $\sigma \notin \mathbb{T}_n$ and if $\tau \in \mathbb{S}_n$ and either $l(\tau) < l(\sigma)$ or $l(\tau) = l(\sigma)$ and $c(\tau) > c(\sigma)$, then $\varphi(\tau) = 0$.

Let $\sigma = \sigma_1 \dots \sigma_s$ be the decomposition of σ into disjoint cycles, where $\sigma_1 = (a_1a_2 \dots a_k)$ is a k-cycle with $k \geq 3$. Let also $\alpha, \beta \in \mathbb{T}_n$ be the permutations defined in the proof of Theorem 3.4, (vii) ⇒ (i). Thus, in a similar manner, the symmetric matrices

$$A = A_\alpha + E_{a_1a_1} + E_{a_ka_k}, \quad B = A_\beta + E_{a_2a_2} + E_{a_ka_k}$$

are singular,

$$AB = A_\sigma + E_{a_{k-1}a_2} + E_{a_1a_k} + E_{a_1a_1} + E_{a_ka_2} + E_{a_ka_k},$$

and

$$d^{\mathbb{S}_n}_\varphi(AB) = \varphi(\sigma) + \varphi(\tau_1) + \varphi(\tau_2) + \varphi(\tau_3),$$

where

$$\tau_1 = (a_2 \dots a_{k-1})\sigma_2 \dots \sigma_s,$$

$$\tau_2 = (a_2 \dots a_k)\sigma_2 \dots \sigma_s,$$

$$\tau_3 = (a_1a_k)(a_2 \dots a_{k-1})\sigma_2 \dots \sigma_s.$$

Since $l(\tau_1) + 1 = l(\tau_2) < l(\sigma)$, $l(\tau_3) = l(\sigma)$ and $c(\tau_3) > c(\sigma)$, so by the choice of σ we have

$$\varphi(\tau_1) = \varphi(\tau_2) = \varphi(\tau_3) = 0.$$

Hence $d_{\varphi}^{\mathbb{S}_n}(AB) = \varphi(\sigma)$.

Since $\alpha, \beta, (a_1a_k)\alpha, (a_2a_k)\beta \in \mathbb{T}_n$ and φ is zero on the elements of $\mathbb{T}_n$,

$$d_{\varphi}^{\mathbb{S}_n}(A) = \varphi(\alpha) + \varphi((a_1a_k)\alpha) = 0,$$

$$d_{\varphi}^{\mathbb{S}_n}(B) = \varphi(\beta) + \varphi((a_2a_k)\beta) = 0.$$

Therefore by hypothesis

$$\varphi(\sigma) = d_{\varphi}^{\mathbb{S}_n}(AB) = d_{\varphi}^{\mathbb{S}_n}(A)d_{\varphi}^{\mathbb{S}_n}(B) = 0,$$

which is a contradiction and the proof of the claim is completed.

To complete the proof, suppose that $\varphi = \hat{\chi} - \chi(1)\varepsilon$ and so $\varphi(1) = 0$. Thus for all singular symmetric matrices $A, B \in M_n(\mathbb{C})$

$$\begin{aligned} d_{\varphi}^{\mathbb{S}_n}(AB) &= d_{\hat{\chi}}^{\mathbb{S}_n}(AB) - \chi(1)\det(AB) \\ &= d_{\chi}^{G}(AB) \\ &= d_{\chi}^{G}(A)d_{\chi}^{G}(B) \\ &= d_{\hat{\chi}}^{\mathbb{S}_n}(A)d_{\hat{\chi}}^{\mathbb{S}_n}(B) \\ &= (d_{\hat{\chi}}^{\mathbb{S}_n}(A) - \chi(1)\det(A))(d_{\hat{\chi}}^{\mathbb{S}_n}(B) - \chi(1)\det(B)) \\ &= d_{\varphi}^{\mathbb{S}_n}(A)d_{\varphi}^{\mathbb{S}_n}(B). \end{aligned}$$

Therefore $\varphi = 0$ by the claim and so $\hat{\chi} = \chi(1)\varepsilon$. This completes the proof of the theorem. □

5. Generalized characteristic polynomial

In this final section we would like to look at generalized matrix functions from a different angle. Let us recall a definition from [7]. Suppose that $G \leqslant \mathbb{S}_n$ and χ is a complex valued function defined on G. For any $A \in M_n(\mathbb{C})$, the *generalized characteristic polynomial* of A associated with G and χ is denoted by $C_{\chi}^{G}(x, A)$ and defined by $C_{\chi}^{G}(x, A) = d_{\chi}^{G}(xI_n - A)$.

It is easy to see that $C_{\chi}^{G}(x, A)$ is a polynomial of degree at most n over $\mathbb{C}$. In fact, if $A = (x_{ij})$ and

$$C_{\chi}^{G}(x, A) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

then the coefficients a_i are polynomials in n^2 variables $x_{11}, x_{12}, \ldots, x_{nn}$ with coefficients in $\mathbb{C}$. For instance, it can be easily verified that

$$a_n = \chi(1), \quad a_{n-1} = -\chi(1)\mathrm{tr}A, \quad a_0 = (-1)^n d_{\chi}^{G}(A).$$

Notice that if $G = \mathbb{S}_n$ and χ is an irreducible character of S_n, then $C_{\chi}^{G}(x, A)$ is called the χ-th immanantal polynomial of A. It is clear that the ε-th immanantal polynomial of A is the ordinary characteristic polynomial of A, where ε is the alternating character of S_n.

By the Cayley-Hamilton Theorem, we know that every square matrix satisfies its own ordinary characteristic polynomial. The authors in Theorem 2.7 of [4] showed for a given generalized matrix function that if every nonsingular (respectively singular) square matrix satisfies its own generalized characteristic polynomial, then the generalized matrix function is a multiple of the determinant.

The following results will indeed refine Theorem 2.7 of [4]. Similar to the previous sections, we verify the problem first for a character χ of G and then for an arbitrary function χ on G. Recall that if a square matrix A satisfies a polynomial $p(x)$, then the minimal polynomial of A divides $p(x)$. In particular, all eigenvalues of A are roots of $p(x)$.

Theorem 5.1. *Let $G \leqslant \mathbb{S}_n$ and χ be a character of G. Then the following are equivalent:*
(i) $d_\chi^G = \chi(1)\det$;
(ii) $C_\chi^G(A_\sigma, A_\sigma) = \mathbf{0}$ *for all* $\sigma \in \mathbb{S}_n$;
(iii) $C_\chi^G(A_\sigma, A_\sigma) = \mathbf{0}$ *for all* $\sigma \in \mathbb{T}_n$;
(iv) $C_\chi^G(A, A) = \mathbf{0}$ *for all symmetric matrices* $A \in M_n(\mathbb{C})$;
(v) $C_\chi^G(A, A) = \mathbf{0}$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$;
(vi) $C_\chi^G(A, A) = \mathbf{0}$ *for all singular symmetric matrices* $A \in M_n(\mathbb{C})$;
(vii) $C_\chi^G(S_\sigma, S_\sigma) = \mathbf{0}$ *for all* $\sigma \in \mathbb{S}_n$;
(viii) $C_\chi^G(A_\sigma, A_\sigma) = \mathbf{0}$ *for all* $\sigma \in \mathbb{S}_n$ *with* $l(\sigma) \leq 2$;
(ix) $C_\chi^G(I_n + A_\sigma, I_n + A_\sigma) = \mathbf{0}$ *for all* $\sigma \in \mathbb{S}_n$ *with* $l(\sigma) \leq 2$.

Proof. If $d_\chi^G = \chi(1)\det$, then $C_\chi^G(x, A) = \chi(1)\det(xI_n - A)$ and so we obtain (i) ⇒ (ii) and (i) ⇒ (iv) by the Cayley-Hamilton Theorem. Now (ii) ⇒ (iii) ⇒ (viii), (iv) ⇒ (v) ⇒ (viii), (iv) ⇒ (vi) ⇒ (ix), and (iv) ⇒ (vii) ⇒ (viii) are obvious. So it is sufficient to show that (viii) ⇒ (i) and (ix) ⇒ (i).

If $C_\chi^G(A, A) = \mathbf{0}$ for a matrix A, then each eigenvalue of A is a root of $C_\chi^G(x, A)$. We know that ± 1 are the only eigenvalues of $A_{(k\,k+1)}$ and $0, 2$ are the only eigenvalues of $I_n + A_{(k\,k+1)}$ for any $1 \leq k \leq n-1$. Hence by hypothesis (viii)

$$0 = C_\chi^G(-1, A_{(k\,k+1)}) = d_\chi^G(-I_n - A_{(k\,k+1)}),$$

and by hypothesis (ix)

$$0 = C_\chi^G(0, I_n + A_{(k\,k+1)}) = d_\chi^G(-I_n - A_{(k\,k+1)}).$$

In both cases one has

$$(-1)^n d_\chi^G(I_n + A_{(k\,k+1)}) = 0.$$

The result now follows by Theorem 2.3. □

The following example shows that Theorem 5.1 may not be true if χ is not a character of G.

Example 5.2. *Let χ be a class function of $\mathbb{A}_3$ given by*

$$\chi(\sigma) = \begin{cases} 0 & \text{if } \sigma = 1 \\ 1 & \text{if } \sigma = (123) \\ -1 & \text{if } \sigma = (132) \end{cases}$$

It can be easily seen that for all symmetric matrices $A \in M_3(\mathbb{C})$ one has $d_\chi^{\mathbb{A}_3}(A) = 0$ and so $C_\chi^{\mathbb{A}_3}(x, A)$ is the zero polynomial. Hence parts (iii) – (ix) *of Theorem* 5.1 *hold but* $d_\chi^{\mathbb{A}_3} \neq \chi(1)\det$.

The next theorem tells us that if all permutation matrices satisfy their generalized characteristic polynomials associated with G and χ, then d_χ^G is a multiple of the determinant.

Theorem 5.3. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a function. Then* $d_\chi^G = \chi(1)\det$ *if and only if* $C_\chi^G(A_\sigma, A_\sigma) = \mathbf{0}$ *for all* $\sigma \in \mathbb{S}_n$.

Proof. Assume that $C_\chi^G(A_\sigma, A_\sigma) = \mathbf{0}$ for any $\sigma \in \mathbb{S}_n$. It then follows that each eigenvalue of A_σ is a root of $C_\chi^G(x, A_\sigma)$. If $\varphi : \mathbb{S}_n \to \mathbb{C}$ is a function given by

$$\varphi(\sigma) = \hat{\chi}(\sigma) - \chi(1)\varepsilon(\sigma),$$

then we have to show by induction on $l(\sigma)$ that $\varphi = 0$. Obviously $\varphi(1) = 0$ and so let $l(\sigma) \geq 2$ and $\sigma = \sigma_1 \ldots \sigma_s$ be the decomposition of σ into disjoint cycles. Since $(A_{\sigma_1})^{l(\sigma_1)} = I_n$ and $A_{\sigma_1} \neq I_n$, hence A_{σ_1} is diagonalizable and so it has an eigenvalue $\lambda \neq 1$ such that $\lambda^{l(\sigma_1)} = 1$. Let $X = (x_1, \ldots, x_n)^t$ be an eigenvector of A_{σ_1} associated with λ. Obviously if $j \in \mathrm{Fix}(\sigma_1)$ then $x_j = 0$, for $\lambda \neq 1$. This implies that $A_{\sigma_i}X = X$, for any $2 \leq i \leq s$. Since $A_\sigma = A_{\sigma_1} \ldots A_{\sigma_s}$, we deduce that $A_\sigma X = \lambda X$, which means that λ is an eigenvalue of A_σ.

Now suppose $\lambda I_n - A_\sigma = (a_{ij})$ and suppose $\tau \in \mathbb{S}_n$ is chosen such that $\prod_{i=1}^n a_{i\tau(i)} \neq 0$. Then $\tau(i) \in \{i, \sigma(i)\}$ for any $1 \leq i \leq n$ and so $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\tau)$. If $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\tau)$, then $l(\tau) < l(\sigma)$ and so by induction we have $\varphi(\tau) = 0$. But if $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\tau)$, then clearly $\tau = \sigma$. Therefore,

$$\begin{aligned}
0 &= C_\chi^G(\lambda, A_\sigma) \\
&= d_\chi^G(\lambda I_n - A_\sigma) \\
&= d_{\hat{\chi}}^{\mathbb{S}_n}(\lambda I_n - A_\sigma) \\
&= d_\varphi^{\mathbb{S}_n}(\lambda I_n - A_\sigma) + \chi(1)\det(\lambda I_n - A_\sigma) \\
&= d_\varphi^{\mathbb{S}_n}(\lambda I_n - A_\sigma) \\
&= \sum_{\tau \in \mathbb{S}_n} \varphi(\tau) \prod_{i=1}^n a_{i\tau(i)} \\
&= \varphi(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \\
&= (\lambda - 1)^{n-l(\sigma)}(-1)^{l(\sigma)}\varphi(\sigma),
\end{aligned}$$

which completes the proof. □

We close the paper with a theorem which says that if all symmetric matrices satisfy their generalized characteristic polynomials associated with G and χ, then d_χ^G is a multiple of the determinant on symmetric matrices.

Theorem 5.4. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a function. Then the following are equivalent:*
(i) $\sum_{\tau \in [\sigma]} \hat{\chi}(\tau) = |[\sigma]|\chi(1)\varepsilon(\sigma)$ *for all* $\sigma \in \mathbb{S}_n$*;*
(ii) $C_\chi^G(A, A) = \mathbf{0}$ *for all symmetric matrices* $A \in M_n(\mathbb{C})$*;*
(iii) $C_\chi^G(A, A) = \mathbf{0}$ *for all nonsingular symmetric matrices* $A \in M_n(\mathbb{C})$*;*
(iv) $C_\chi^G(A, A) = \mathbf{0}$ *for all singular symmetric matrices* $A \in M_n(\mathbb{C})$*;*
(v) $C_\chi^G(S_\sigma, S_\sigma) = \mathbf{0}$ *for all* $\sigma \in \mathbb{S}_n$*.*

Proof. (ii) ⇒ (iii), (ii) ⇒ (iv), and (ii) ⇒ (v) are clear. It suffices to prove the following cases.
(i) ⇒ (ii): For any symmetric matrix $A \in M_n(\mathbb{C})$ we have by Theorem 2.6 that $d_\chi^G(A) = \chi(1)\det(A)$ and so $C_\chi^G(x, A) = \chi(1)\det(xI_n - A)$. Now the result follows by the Cayley-Hamilton Theorem.
(iii) ⇒ (i), (iv) ⇒ (i), and (v) ⇒ (i): Let $\varphi : \mathbb{S}_n \to \mathbb{C}$ be a function given by

$$\varphi(\sigma) = \hat{\chi}(\sigma) - \chi(1)\varepsilon(\sigma).$$

We show that

$$\sum_{\tau \in [\sigma]} \varphi(\tau) = 0,$$

for any $\sigma \in \mathbb{S}_n$, which is clearly true for $\sigma = 1$. By way of contradiction, choose $1 \neq \sigma \in \mathbb{S}_n$ so that $l(\sigma)$ is minimal and $c(\sigma)$ is maximal and

$$\sum_{\tau \in [\sigma]} \varphi(\tau) \neq 0.$$

Thus if $\tau \in \mathbb{S}_n$ and either $l(\tau) < l(\sigma)$ or $l(\tau) = l(\sigma)$ and $c(\tau) > c(\sigma)$, then

$$\sum_{\alpha \in [\tau]} \varphi(\alpha) = 0.$$

Similar to the proof of Theorem 2.6 one can deduce for the symmetric matrix $\lambda I_n + S_\sigma$ that

$$d_\chi^G(\lambda I_n + S_\sigma) - \chi(1)\det(\lambda I_n + S_\sigma) = (\lambda + 1)^{n-l(\sigma)} \sum_{\tau \in [\sigma]} \varphi(\tau),$$

where $\lambda \in \mathbb{C}$. In particular,

$$C_\chi^G(x, \lambda I_n + S_\sigma) - \chi(1)\det(xI_n - (\lambda I_n + S_\sigma)) = (-1)^n(\lambda - x + 1)^{n-l(\sigma)} \sum_{\tau \in [\sigma]} \varphi(\tau). \ (\star)$$

Notice that if $C_\chi^G(A, A) = \mathbf{0}$ for a matrix A, then each eigenvalue of A is a root of $C_\chi^G(x, A)$. Now if $\sigma \in \mathbb{T}_n - \{1\}$, then S_σ has -1 as an eigenvalue and if $\sigma \in \mathbb{S}_n - \mathbb{T}_n$, then S_σ has 2 as an eigenvalue. Hence in either case S_σ has an eigenvalue $\lambda_0 \neq 1$.

Assuming (v) and putting $x = \lambda_0$ and $\lambda = 0$ in relation $(\star)$ will result in

$$0 = C_\chi^G(\lambda_0, S_\sigma) - \chi(1)\det(\lambda_0 I_n - S_\sigma) = (-1)^n(1 - \lambda_0)^{n-l(\sigma)} \sum_{\tau \in [\sigma]} \varphi(\tau),$$

a contradiction. This completes the proof of (v) $\Rightarrow$ (i).

Assuming (iv) and notting that $\lambda_0 I_n - S_\sigma$ is singular, one can get by putting $x = 0$ and $\lambda = -\lambda_0$ in relation $(\star)$ that

$$0 = C_\chi^G(0, -\lambda_0 I_n + S_\sigma) - \chi(1)\det(\lambda_0 I_n - S_\sigma) = (-1)^n(1 - \lambda_0)^{n-l(\sigma)} \sum_{\tau \in [\sigma]} \varphi(\tau),$$

again a contradiction. This completes the proof of (iv) $\Rightarrow$ (i).

Finally assuming (iii) and choosing $\lambda_1 \in \mathbb{C}$ so that $\lambda_1 I_n + S_\sigma$ is nonsingular, one can see that $\lambda_0 + \lambda_1$ is an eigenvalue of $\lambda_1 I_n + S_\sigma$. Now putting $x = \lambda_0 + \lambda_1$ and $\lambda = \lambda_1$ in relation $(\star)$ will result in

$$\begin{aligned} 0 &= C_\chi^G(\lambda_0 + \lambda_1, \lambda_1 I_n + S_\sigma) - \chi(1)\det((\lambda_0 + \lambda_1)I_n - (\lambda_1 I_n + S_\sigma)) \\ &= (-1)^n(1 - \lambda_0)^{n-l(\sigma)} \sum_{\tau \in [\sigma]} \varphi(\tau), \end{aligned}$$

again a contradiction. This completes the proof of (iii) $\Rightarrow$ (i) and therefore the proof of the theorem is completed. □

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