Filomat 37:10 (2023), 3105–3117
https://doi.org/10.2298/FIL2310105Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Regularity theory for quasilinear elliptic equations of p-Schrödinger type with certain potentials

Fengping Yao[a]

[a]*Department of Mathematics, Shanghai University, Shanghai 200444, China*

Abstract. In this paper we study the regularity theory in Orlicz spaces for the following divergence quasilinear elliptic equations of p-Schrödinger type with certain potentials in the whole space $\mathbb{R}^n$ under some proper conditions

$$-\text{div}\,\left(|\nabla u|^{p-2}\nabla u\right)+V(x)\,|u|^{p-2}\,u=-\text{div}\left(|\mathbf{f}|^{p-2}\,\mathbf{f}\right).$$

Especially when $p=2$, the above equation can be reduced to the classical linear divergence elliptic Schrödinger equation

$$-\Delta u+V(x)u=-\text{div}\,\mathbf{f}.$$

Moreover, we would like to remark that the results in this work generalize the results of our previous paper [50].

1. Introduction

The Calderón-Zygmund type estimates (L^p-type estimates), whose main purpose is to obtain L^p bounds in Sobolev spaces for a variety of operators and solutions of equations, have been proved to be a power tool in many aspects of harmonic analysis and partial differential equations. As we know, every differentiable function in the classical function spaces is required to have derivative at any point of the domain. However, this condition is very harsh in partial differential equations. Sobolev spaces, which consist of some kinds of functions with weak derivatives are commonly used in many fields of mathematics, have turned out to be one of the most powerful tools in analysis created in the 20th century. The definitions and properties of Sobolev spaces can be found in numerous monographs and textbooks (see [2]). Subsequently, Orlicz spaces [2, 13, 30] which were introduced by Orlicz [38] have been studied as the most natural generalization of Sobolev spaces as a result of the need in various practical problems to use wider spaces of functions than Sobolev spaces. The theory of Orlicz spaces plays a fairly important role in a wide variety of fields of mathematics including partial differential equations, Fourier analysis, geometry, probability theory, stochastic analysis, and insurance and financial mathematics (see [40]).

The aim of this paper is to study the regularity in Orlicz spaces for the following divergence quasilinear elliptic equations of p-Schrödinger type

$$-\text{div}\,\left(|\nabla u|^{p-2}\nabla u\right)+V(x)\,|u|^{p-2}\,u=-\text{div}\left(|\mathbf{f}|^{p-2}\,\mathbf{f}\right)\quad\text{ in }\ \mathbb{R}^n,\qquad n\geq 2,\tag{1}$$

2020 *Mathematics Subject Classification*. Primary 35J10, 35J15
Keywords. p-Schrödinger, regularity, Orlicz, quasilinear, elliptic, divergence, whole space.
Received: 24 May 2022; Revised: 07 August 2022; Accepted: 27 October 2022
Communicated by Marko Nedeljkov
Email address: yfp@shu.edu.cn (Fengping Yao)

where $p > 1$, $x = (x^1, ..., x^n) \in \mathbb{R}^n$, the nonnegative potential $V > 0$ a.e., $V \in L^{\infty}_{loc}(\mathbb{R}^n)$ and

$$\sup_{B_r} |V(x)| \le C \fint_{B_r} V(x)\, dx \quad \text{for any ball} \quad B_r \subset \mathbb{R}^n, \tag{2}$$

where

$$\fint_{B_r} V(x)\, dx = \frac{1}{|B_r|} \int_{B_r} V(x)\, dx.$$

In other words, this means that $V \in A_\infty$ (see [43], Pages 194-197). Generally speaking, we say that V belongs to the class of the reverse Hölder weights A_q for some $1 < q < \infty$ if $V \in L^q_{loc}(\mathbb{R}^n)$, $V > 0$ a.e. and

$$\left(\fint_{B_r} V^q(x)\, dx \right)^{1/q} \le C \fint_{B_r} V(x)\, dx. \tag{3}$$

If $q = \infty$, then the left hand side (3) is replaced by the essential supremum in (2). It is easy to check that $V \in A_q$ for every $q > 1$ if $V \in A_\infty$.

Especially when $p = 2$, (1) is reduced to the classical linear divergence elliptic Schrödinger equation

$$-\Delta u + V(x) u = -\mathrm{div}\, \mathbf{f}. \tag{4}$$

As is well known, the Schrödinger equation is one of the most fundamental equations of physics for describing the spatial and temporal behavior of quantum-mechanical systems, which was proposed by Austrian physicist Erwin Schrödinger in 1926. The Schrödinger equation is widely used within physics and takes several different forms depending on the physical situation. In certain physical situations it may be appropriate to incorporate some kind of randomness into the equation. There are the time dependent equation used for describing progressive waves and the time independent form of this equation used for describing standing waves. The Schrödinger time-independent equation can be solved analytically for a few simple systems. Solutions to the Schrödinger equation describe not only subatomic particles, atoms, and molecules, but also macroscopic systems, possibly even the whole universe. Due to its deeper modern physical and mathematical significance, L^p-type estimates for (4) have been paid considerable attention in recent decades. We refer the readers to Refs. [39, 41, 42, 45, 49, 53] and the references therein for Calderón-Zygmund type estimates of the elliptic Schrödinger operator $L = -\Delta + V(x)$, where $V(x)$ is a nonnegative potential. Moreover, there are also many investigations [16, 26, 44] on the L^p estimates for the parabolic Schrödinger equation $u_t - \Delta u + Vu = f$.

In the last years, the nonlinear Schrödinger equations (NLSEs) have been widely investigated by several authors, which are models for different physical phenomena: the propagation in birefringent optical fibers, Kerr-like photorefractive media in optics and Bose-Einstein condensates. There are many famous NLSEs such as derivative NLSE, Kundu Mukherjee Naskar model, Fokas-Lenells equation, Biswas-Arshed equation and so on. When $V(x) \equiv 0$, the elliptic PDE of p-Schrödinger type (1) can be simplified to the classical elliptic PDE of p-Laplacian type

$$-\mathrm{div}\, \left(|\nabla u|^{p-2} \nabla u \right) = -\mathrm{div} \left(|\mathbf{f}|^{p-2}\, \mathbf{f} \right). \tag{5}$$

There have been a wide research activities (see [4, 8, 12, 17, 27, 29, 36]) on the study on L^p-type regularity for weak solutions to (5) and the general case with the different coefficients and domains. Meanwhile, some authors [5, 14, 15, 34, 35, 52] also investigated regularity estimates in Orlicz and Lorentz spaces for weak solutions of the elliptic PDEs of p-Laplacian type. Moreover, many authors [7, 9–11, 18, 19, 23, 25] also studied various kinds of regularity estimates for weak solutions to (5) and the general case. For the better part of a decade, some scholars [22, 31, 46] began to study the corresponding regularity estimates including Lipschitz regularity and L^p-type estimates for weak solutions of the elliptic equations with p-growth of Schrödinger type. Just recently, some authors [3, 48] considered the Schrödinger-Kirchhoff type operator involving the fractional p-Laplacian $L = (-\Delta)^s_p u + V(x)|u|^{p-2}u$. On the other hand, the authors have investigated the global regularity estimates in Sobolev and Orlicz spaces for weak solutions of elliptic PDE of p-Laplacian type (5) in the whole space $\mathbb{R}^n$ (see [20, 51]) and the general $p(x)$-Laplacian case in the whole space (see [21]).

Actually, we [47] have obtained the regularity estimates in Orlicz spaces for the Poisson equation $-\Delta u = f$ in the whole space $\mathbb{R}^n$, in which we proved that

$$\int_{\mathbb{R}^n} \phi\left(\left|D^2u\right|\right) dx \leq C \int_{\mathbb{R}^n} \phi\left(\left|f\right|\right) dx, \tag{6}$$

where C is a positive constant independent of u and f. Indeed, if $\phi(t) = t^p$, the above estimate is reduced to the classical L^p estimate. Meanwhile, we also verify that the global $\Delta_2 \cap \nabla_2$ condition on the Orlicz function ϕ is optimal. Here we want to mention that $\phi(t) = t^p \ln(1+t)$ for $p > 1$ satisfies the global $\Delta_2 \cap \nabla_2$ condition. Furthermore, we [50] also proved the above regularity estimates in Orlicz spaces

$$\int_{\mathbb{R}^n} \phi\left(\left|D^2u\right|\right) dx + \int_{\mathbb{R}^n} \phi\left(|Vu|\right) dx \leq C \int_{\mathbb{R}^n} \phi\left(\left|f\right|\right) dx \tag{7}$$

for the Schrödinger equation (1) in $\mathbb{R}^n$ when V satisfies the condition (2) and $\phi \in \Delta_2 \cap \nabla_2$. In this work we will extend the result in [50] in the context of the nonlinear elliptic p-Schrödinger equation (1) in $\mathbb{R}^n$. Similarly to the previous paper [50], we assume that $V \in A_\infty$ due to the technical difficulties. We remark that the major difficulties are the nonlinearity of the equation itself, the inhomogeneity of Orlicz function $\phi(t)$ and the unboundedness of the domain.

As usual, the solutions of (1) are taken in a weak sense. For the sake of contrast and better understanding, we first recall the definition of the weak solution of the classical elliptic PDE of p-Laplacian type (5) in $\mathbb{R}^n$ (see [20, 27]). More precisely, we say that $u \in D^p(\mathbb{R}^n)$ is a weak solution of (5) in $\mathbb{R}^n$ with $\mathbf{f} \in L^p(\mathbb{R}^n)$ if for each $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = \int_{\mathbb{R}^n} |\mathbf{f}|^{p-2}\, \mathbf{f} \cdot \nabla\varphi \, dx,$$

where $D^p(\mathbb{R}^n) := \left\{ u \in L^p_{loc}(\mathbb{R}^n) \,\middle|\, \nabla u \in L^p(\mathbb{R}^n) \right\}$. We now state the definition of weak solutions in this paper.

Definition 1.1. *Assume that $\mathbf{f} \in L^p(\mathbb{R}^n)$. A function $u \in D^p_V(\mathbb{R}^n)$ is a weak solution of (1) if for each $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + V(x)\,|u|^{p-2}\, u \cdot \varphi \, dx = \int_{\mathbb{R}^n} |\mathbf{f}|^{p-2}\, \mathbf{f} \cdot \nabla\varphi \, dx,$$

where $D^p_V(\mathbb{R}^n) := \left\{ u \in L^p_{loc}(\mathbb{R}^n) \,\middle|\, \nabla u \in L^p(\mathbb{R}^n) \text{ and } V|u|^p \in L^1(\mathbb{R}^n) \right\}$.

Now let us state the main result of this work.

Theorem 1.2. *Assume that $\phi \in \Delta_2 \cap \nabla_2$ (see Definition 2.1) and $V \in A_\infty$. If u is the weak solution of the nonlinear elliptic p-Schrödinger type equation (1), then we have*

$$\int_{\mathbb{R}^n} \phi\left([\Phi(u, V)]^p\right) dx \leq C \int_{\mathbb{R}^n} \phi\left(|\mathbf{f}|^p\right) dx, \tag{8}$$

where

$$\Phi(u, V) := |\nabla u| + V^{\frac{1}{p}} |u|. \tag{9}$$

Here we notice that when $\phi(t) = t^q$, the result in the above theorem can be simplified to be the classical L^p-type estimates. We would like to stress that our approach is very much influenced by [1, 37]. In [1] Acerbi and Mingione found a new covering/iteration approach (see [37] for its origin) involving the large-M-inequality principle (here $M = \frac{1}{\delta}$ in (11)) to overcome the difficulty in the scaling-invariant problem for the parabolic p-Laplacian equations. Remarkably enough, the above method takes advantage of a stopping time argument and Vitali's covering lemma without using the Calderón-Zygmund decomposition and maximal functions, which were usually used to get the L^p-type regularity estimates for all kinds of linear/nonlinear elliptic problems. As a matter of fact, this approach has been widely used in L^p-type regularity theory for various kinds of nonlinear elliptic and parabolic equations.

2. Proof of the main result

This section is devoted to the proof of the main result stated in Theorem 1.2. We shall give some definitions and lemmas, which will be used later. For convenience of the readers, we first recall some definitions and conclusions on the properties of the general Orlicz spaces.

Definition 2.1. *A convex function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ is said to be a Young function if*

$$\lim_{t\to 0+} \frac{\phi(t)}{t} = \lim_{t\to+\infty} \frac{t}{\phi(t)} = 0 \quad \text{and} \quad \phi(0) = 0.$$

A Young function ϕ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if

$$\phi(2t) \le K\phi(t) \quad \text{for every } t > 0 \text{ and some constant } K > 0.$$

Moreover, a Young function ϕ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if

$$\phi(t) \le \frac{\phi(at)}{2a} \quad \text{for every } t > 0 \text{ and some constant } a > 1.$$

Remark 2.2. *The two simple examples for functions satisfying the $\Delta_2 \cap \nabla_2$ condition are*

$$\phi_1(t) = t^p \quad \text{and} \quad \phi_2(t) = t^p \ln(1 + t) \quad \text{for } p > 1.$$

Actually, another interesting example is related to (p, q)-growth condition given by appropriate gluing of the monomials in Page 600 of [6] and Page 314 of [32]. We remark that

$$\phi_3(t) = t \ln(1 + t) \notin \nabla_2 \quad \text{and} \quad \phi_4(t) = e^t \notin \Delta_2.$$

Actually, $\phi \in \Delta_2 \cap \nabla_2$ if and only if there exist constants $A_2 \ge A_1 > 0$ and $\alpha_1 \ge \alpha_2 > 1$ such that

$$A_1 \left(\frac{s}{t}\right)^{\alpha_2} \le \frac{\phi(s)}{\phi(t)} \le A_2 \left(\frac{s}{t}\right)^{\alpha_1} \quad \text{for any } 0 < t \le s. \tag{10}$$

Definition 2.3. *Let ϕ be a Young function. Then the Orlicz class $K^\phi(\Omega)$ is the set of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying*

$$\int_\Omega \phi(|g|)\, dx < \infty.$$

The Orlicz space $L^\phi(\Omega)$ is the linear hull of $K^\phi(\Omega)$.

Lemma 2.4. *Let ϕ be a Young function with $\phi \in \Delta_2 \cap \nabla_2$ and $g \in L^\phi(\Omega)$. Then we have*

1. *$K^\phi(\Omega) = L^\phi(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $L^\phi(\Omega)$.*
2. $\displaystyle\int_\Omega \phi(|g|)\, dx = \int_0^\infty |\{x \in \Omega : |g| > \mu\}|\, d\left[\phi(\mu)\right].$

Proof. The result (1) can follow from [2]. Moreover, we see that

$$\int_\Omega \phi(|g|)\, dx = \int_\Omega \int_0^{|g|} d\left[\phi(\mu)\right] dx = \int_\Omega \int_0^\infty \chi_{\{x\in\Omega:|g|>\mu\}}\, d\left[\phi(\mu)\right] dx,$$

where χ_Q is the characteristic function of the set Q in $\mathbb{R}^n$. Then by Fubini's theorem, we can easily obtain the result. □

Moreover, we have the following integral inequality.

Lemma 2.5. *If $\phi \in \triangle_2 \cap \nabla_2$ and $g \in L^{\phi}(\mathbb{R}^n)$, then for any $b_1, b_2 > 0$ we have*

$$\int_0^\infty \frac{1}{\mu} \left\{ \int_{\{x \in \mathbb{R}^n : |g| > b_1 \mu\}} |g| dx \right\} d\left[\phi(b_2\mu)\right] \le C(b_1, b_2, \phi) \int_{\mathbb{R}^n} \phi(|g|)\, dx.$$

Proof. We interchange the order of integration and use integration by parts to deduce that

$$\begin{aligned} I &:= \int_0^\infty \frac{1}{\mu} \left\{ \int_{\{x \in \mathbb{R}^n : |g| > b_1 \mu\}} |g| dx \right\} d\left[\phi(b_2\mu)\right] \\ &= \int_{\mathbb{R}^n} |g| \left\{ \int_0^{|g|/b_1} \frac{d\left[\phi(b_2\mu)\right]}{\mu} \right\} dx \\ &\le \int_{\mathbb{R}^n} |g| \left\{ \frac{\phi(b_2|g|/b_1)}{(|g|/b_1)} + \int_0^{|g|/b_1} \frac{\phi(b_2\mu)}{\mu^2} d\mu \right\} dx. \end{aligned}$$

Thus, it follows from (10) that

$$\begin{aligned} I &\le C \int_{\mathbb{R}^n} \phi(|g|)\, dx + C \int_{\mathbb{R}^n} \phi(b_2|g|/b_1)|g|^{1-\alpha_2} \left\{ \int_0^{|g|/b_1} \frac{1}{\mu^{2-\alpha_2}}\, d\mu \right\} dx \\ &\le C \int_{\mathbb{R}^n} \phi(|g|)\, dx, \end{aligned}$$

which finishes our proof. □

Now we give the important iteration-covering procedure, which was first introduced by [1, 37]. Now we define

$$\lambda_0^p := \int_{\mathbb{R}^n} \left[\Phi\left(u, V\right)\right]^p\, dx + \frac{1}{\delta} \int_{\mathbb{R}^n} |\mathbf{f}|^p\, dx, \tag{11}$$

where $\Phi\left(u, V\right)$ is defined in (9) and $\delta \in (0, 1)$ is a small enough constant which will be determined later. Set

$$u_\lambda := \frac{u}{\lambda_0 \lambda} \quad \text{and} \quad \mathbf{f}_\lambda := \frac{\mathbf{f}}{\lambda_0 \lambda} \quad \text{for any } \lambda > 0. \tag{12}$$

Then u_λ is still the solution of (1) with $\mathbf{f}_\lambda$ replacing $\mathbf{f}$. Moreover, we write

$$J\left[u_\lambda, \mathbf{f}_\lambda, V, B\right] := \fint_B \left[\Phi\left(u_\lambda, V\right)\right]^p\, dx + \frac{1}{\delta} \fint_B |\mathbf{f}_\lambda|^p\, dx \quad \text{for any domain } B \subset \mathbb{R}^n$$

and the level set

$$E(u_\lambda, V, 1) := \left\{ x \in \mathbb{R}^n :\ \Phi\left(u_\lambda, V\right) > 1 \right\}.$$

Next, we will decompose the level set $E(u_\lambda, V, 1)$ into a family of disjoint small domains.

Lemma 2.6. *For any $\lambda > 0$, there exists a family of disjoint balls $\left\{B_{\rho_i}(x_i)\right\}_{i \ge 1}$ with $x_i \in E(u_\lambda, V, 1)$ and $\rho_i = \rho(x_i, \lambda) > 0$ such that*

$$J\left[u_\lambda, \mathbf{f}_\lambda, V, B_{\rho_i}(x_i)\right] = 1, \tag{13}$$

$$J\left[u_\lambda, \mathbf{f}_\lambda, V, B_{\rho}(x_i)\right] < 1 \qquad \textit{for any } \rho > \rho_i \tag{14}$$

and

$$E(u_\lambda, V, 1) \subset \bigcup_{i \ge 1} B_{5\rho_i}(x_i) \cup \textit{negligible set}. \tag{15}$$

Proof. Fix any $x \in \mathbb{R}^n$ and $\rho \geq \rho_0 = \rho_0(\lambda) > 0$ with $\lambda^p |B_{\rho_0}| = 1$. Then it follows from (11) and (12) that

$$\begin{aligned} J\left[u_\lambda, \mathbf{f}_\lambda, V, B_\rho(x)\right] &= \fint_{B_\rho(x)} [\Phi(u_\lambda, V)]^p \, dx + \frac{1}{\delta} \fint_{B_\rho(x)} |\mathbf{f}_\lambda|^p \, dx \\ &\leq \frac{1}{|B_\rho(x)|} \left[\int_{\mathbb{R}^n} [\Phi(u_\lambda, V)]^p \, dx + \frac{1}{\delta} \int_{\mathbb{R}^n} |\mathbf{f}_\lambda|^p \, dx \right] \\ &\leq \frac{1}{|B_\rho(x)| \lambda^p} \leq 1, \end{aligned}$$

which implies that

$$\sup_{x \in \mathbb{R}^n} \sup_{\rho \geq \rho_0} J\left[u_\lambda, \mathbf{f}_\lambda, V, B_\rho(x)\right] \leq 1. \tag{16}$$

Thus, by using Lebesgue's differentiation theorem, for a.e. $x \in E(u_\lambda, V, 1)$ we know that

$$\lim_{\rho \to 0} J\left[u_\lambda, \mathbf{f}_\lambda, V, B_\rho(x)\right] > 1,$$

which implies that there exists some $\rho > 0$ such that

$$J\left[u_\lambda, \mathbf{f}_\lambda, V, B_\rho(x)\right] > 1.$$

So, from (16) one can select a radius $\rho_x \in (0, \rho_0]$ such that

$$\rho_x = \max\left\{\rho \in (0, \rho_0] : J\left[u_\lambda, \mathbf{f}_\lambda, V, B_\rho(x)\right] = 1\right\},$$

$$J\left[u_\lambda, \mathbf{f}_\lambda, V, B_{\rho_x}(x)\right] = 1 \quad \text{and} \quad J\left[u_\lambda, \mathbf{f}_\lambda, V, B_\rho(x)\right] < 1 \ \text{ for any } \ \rho > \rho_x.$$

Therefore, we apply Vitali's covering lemma to find a family of disjoint balls $\left\{B_{\rho_i}(x_i)\right\}_{i \geq 1}$ such that (13)-(15) hold. □

Lemma 2.7. *Under the same assumed conditions as those in the above lemma, we have*

$$|B_{\rho_i}(x_i)| \quad \leq \quad 3 \int_{\left\{x \in B_{\rho_i}(x_i) :\ [\Phi(u_\lambda, V)]^p > \frac{1}{3}\right\}} [\Phi(u_\lambda, V)]^p \, dx + \frac{3}{\delta} \int_{\left\{x \in B_{\rho_i}(x_i) :\ |\mathbf{f}_\lambda|^p > \frac{\delta}{3}\right\}} |\mathbf{f}_\lambda|^p \, dx. \tag{17}$$

Proof. From (13) in the lemma above we see

$$|B_{\rho_i}(x_i)| = \int_{B_{\rho_i}(x_i)} [\Phi(u_\lambda, V)]^p \, dx + \frac{1}{\delta} \int_{B_{\rho_i}(x_i)} |\mathbf{f}_\lambda|^p \, dx.$$

Therefore, by splitting the two integrals above as follows we have

$$|B_{\rho_i}(x_i)| \leq \int_{\left\{x \in B_{\rho_i}(x_i) : [\Phi(u_\lambda, V)]^p > \frac{1}{3}\right\}} [\Phi(u_\lambda, V)]^p \, dx + \frac{1}{3} |B_{\rho_i}(x_i)| + \frac{1}{\delta} \int_{\left\{x \in B_{\rho_i}(x_i) :\ |\mathbf{f}_\lambda|^p > \frac{\delta}{3}\right\}} |\mathbf{f}_\lambda|^p \, dx + \frac{1}{3} |B_{\rho_i}(x_i)|.$$

Thus we can finish the proof. □

Since $V \in A_\infty$, we know that $V \in A_t$ for every $t > 1$. And then we'll recall the following type of reverse-Hölder inequality.

Lemma 2.8 ([43], Page 195). *Assume that $V \in A_\infty$. Then we have*

$$\fint_B g\,dx \leq \left(\frac{C}{V(B)}\int_B V g^t\,dx\right)^{\frac{1}{t}}$$

for every $t > 1$, any nonnegative integrable function g and all balls B, where

$$V(B) = \int_B V(x)\,dx.$$

Subsequently, we recall the local bounded property for weak solutions of the reference equation

$$-\mathrm{div}\,\left(|\nabla h|^{p-2}\nabla h\right) + V(x)\,|h|^{p-2}\,h = 0, \tag{18}$$

whose proof is totally similar to the proof of Lemma 3.5 in [31]. Here we used Theorem 7.3 in [28] for any $\theta > 0$.

Lemma 2.9. *Assume that $V \in A_\infty$. If $h(x)$ is a weak solution of (18), then for every $\theta > 0$ there exists a positive constant $C = C(\theta)$ such that*

$$\sup_{B_r}|h| \leq C\left(\fint_{B_{2r}}|h|^\theta\,dx\right)^{\frac{1}{\theta}}. \tag{19}$$

Moreover, we shall prove the following local Lipschitz regularity of weak solutions to the reference equation (18).

Lemma 2.10. *Assume that $V \in A_\infty$. If $h(x)$ is a weak solution of (18), then there exists a positive constant C, depending on n, such that*

$$\sup_{B_r}|h|^p \leq \frac{C}{V(B_{2r})}\int_{B_{2r}} V|h|^p\,dx.$$

Proof. Using Lemma 2.8 and Lemma 2.9 with $\theta = \frac{p}{t}$, we find that

$$\sup_{B_r}|h|^p \leq C\left(\fint_{B_{2r}}[|h|^p]^{\frac{1}{t}}\,dx\right)^t \leq \frac{C}{V(B_{2r})}\int_{B_{2r}} V|h|^p\,dx.$$

Therefore, this completes our proof. □

Next, we shall derive comparison results between h and the weak solution u of (1).

Lemma 2.11. *Let $1 < p < +\infty$. For any $\epsilon > 0$, there exists a small positive constant $\delta = \delta(\epsilon, n, p)$ such that if u is the weak solution of (1),*

$$\left(\fint_{B_{2r}}[\Phi(u_\lambda, V)]^p\,dx\right)^{\frac{1}{p}} < 1 \quad \text{and} \quad \left(\fint_{B_{2r}}|\mathbf{f}_\lambda|^p\,dx\right)^{\frac{1}{p}} < \delta, \tag{20}$$

then we have

$$\int_{B_{2r}}[\Phi(h, V)]^p\,dx \leq \int_{B_{2r}}[\Phi(u_\lambda, V)]^p\,dx \tag{21}$$

and

$$\left(\fint_{B_{2r}}[\Phi(u_\lambda - h, V)]^p\,dx\right)^{\frac{1}{p}} < \epsilon, \tag{22}$$

where h is the weak solution of (18) in B_{2r} with $h = u_\lambda$ on ∂B_{2r}.

Proof. If u and h are the weak solutions of (1) in Ω and (18) in B_{2r} with $h = u_\lambda$ on ∂B_{2r} respectively, then by selecting the test function $\varphi = u_\lambda - h$ in Definition 1.1 we find that

$$\int_{B_{2r}} |\nabla h|^{p-2}\nabla h \cdot \nabla(u_\lambda - h) + V(x)\,|h|^{p-2}\,h \cdot (u_\lambda - h)\,dx = 0 \tag{23}$$

and

$$\int_{B_{2r}} |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla(u_\lambda - h) + V(x)\,|u_\lambda|^{p-2}\,u_\lambda \cdot (u_\lambda - h)\,dx = \int_{B_{2r}} |\mathbf{f}_\lambda|^{p-2}\,\mathbf{f}_\lambda \cdot \nabla(u_\lambda - h)\,dx. \tag{24}$$

Using (23) and Young's inequality, we deduce that

$$\begin{aligned}\int_{B_{2r}} [\Phi(h,V)]^p\ dx &= \int_{B_{2r}} |\nabla h|^p + V(x)\,|h|^p\ dx \\ &\le C\int_{B_{2r}} |\nabla u_\lambda|^p + V(x)\,|u_\lambda|^p\ dx = \int_{B_{2r}} [\Phi(u_\lambda, V)]^p\ dx,\end{aligned} \tag{25}$$

which implies that (21) is true. Moreover, after a direct calculation we use (23)-(24) to show the resulting expression as

$$I_1 + I_2 = I_3, \tag{26}$$

where

$$\begin{aligned} I_1 &:= \int_{B_{2r}} \left(|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla h|^{p-2}\nabla h\right)\cdot \nabla\left(u_\lambda - h\right)\ dx, \\ I_2 &:= \int_{B_{2r}} V(x)\left(|u_\lambda|^{p-2}u_\lambda - |h|^{p-2}h\right)\cdot\left(u_\lambda - h\right)\ dx, \\ I_3 &:= \int_{B_{2r}} |\mathbf{f}_\lambda|^{p-2}\,\mathbf{f}_\lambda \cdot \nabla(u_\lambda - h)\,dx. \end{aligned}$$

Estimate of I_1. We divide it into two cases:
Case 1. $p \ge 2$. Using the elementary inequality

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi - \eta) \ge C|\xi - \eta|^p$$

for every $\xi, \eta \in \mathbb{R}^n$ with $C = C(p)$, we have

$$I_1 \ge C\int_{B_{2r}} |\nabla(u_\lambda - h)|^p\,dx.$$

Case 2. $1 < p < 2$. From the following elementary inequality

$$C|\xi - \eta|^p \le (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi - \eta) + \tau|\eta|^p$$

for every $\xi, \eta \in \mathbb{R}^n$ and any small constant $\tau > 0$ with $C = C(p, \tau)$, we have

$$I_1 + \tau\int_{B_{2r}} |\nabla u_\lambda|^p\,dx \ge C\int_{B_{2r}} |\nabla(u_\lambda - h)|^p\,dx.$$

Estimate of I_2. Similarly to the proofs of I_1, for any $p > 1$ we have

$$I_2 + \tau\int_{B_{2r}} V(x)|u_\lambda|^p\,dx \ge C\int_{B_{2r}} V(x)\,|u_\lambda - h|^p\ dx.$$

Estimate of I_3. Now we apply Young's inequality to conclude that

$$I_3 \le \tau\int_{B_{2r}} |\nabla(u_\lambda - h)|^p\ dx + C(\tau)\int_{B_{2r}} |\mathbf{f}_\lambda|^p\ dx.$$

Finally, we combine all the estimates of I_i $(1 \le i \le 3)$ and choose $\tau > 0$ small enough to see that

$$\begin{aligned}
\fint_{B_{2r}} [\Phi(u_\lambda - h, V)]^p \, dx &= \fint_{B_{2r}} |\nabla(u_\lambda - h)|^p + V(x)\,|u_\lambda - h|^p \, dx \\
&\le C\fint_{B_{2r}} |\mathbf{f}_\lambda|^p \, dx + \tau \fint_{B_{2r}} |\nabla u_\lambda|^p + V(x)|u_\lambda|^p \, dx \\
&= C\fint_{B_{2r}} |\mathbf{f}_\lambda|^p \, dx + \tau \fint_{B_{2r}} [\Phi(u_\lambda, V)]^p \, dx \\
&\le C\delta^p + C\tau \le \epsilon^p,
\end{aligned}$$

where we have used (20) and (25), and selected δ, τ small enough satisfying the last inequality. Therefore, we finish the proof of this lemma. □

Now we shall finish the proof of the main result: Theorem 1.2.

Proof. Fix $i \ge 1$. In view of Lemma 2.6, we obtain

$$\fint_{B_{10\rho_i}(x_i)} [\Phi(u_\lambda, V)]^p \, dx \le 1 \quad \text{and} \quad \fint_{B_{10\rho_i}(x_i)} |\mathbf{f}_\lambda|^p \, dx \le \delta^p. \tag{27}$$

Let h be the weak solution of (18) in $B_{10\rho_i}(x_i)$ with $h = u_\lambda$ on $\partial B_{10\rho_i}(x_i)$. Then it follows from (27) and Lemma 2.11 that

$$\begin{aligned}
\fint_{B_{10\rho_i}(x_i)} V|h|^p \, dx &\le C\fint_{B_{10\rho_i}(x_i)} V\,|u_\lambda - h|^p \, dx + C\fint_{B_{10\rho_i}(x_i)} V\,|u_\lambda|^p \, dx \\
&\le C\fint_{B_{10\rho_i}(x_i)} [\Phi(u_\lambda - h, V)]^p \, dx + C\fint_{B_{10\rho_i}(x_i)} [\Phi(u_\lambda, V)]^p \, dx \\
&\le C.
\end{aligned}$$

Then from the above inequality and Lemma 2.10 we find that

$$\begin{aligned}
\sup_{B_{6\rho_i}(x_i)} V|h|^p &\le C \sup_{B_{6\rho_i}(x_i)} V \left[V(B_{10\rho_i}(x_i))\right]^{-1} \int_{B_{10\rho_i}(x_i)} V|h|^p \, dx \\
&\le C \sup_{B_{10\rho_i}(x_i)} V \left(\fint_{B_{10\rho_i}(x_i)} V \, dx\right)^{-1},
\end{aligned}$$

which follows from the fact that $V \in A_\infty$ that

$$\sup_{B_{6\rho_i}(x_i)} V|h|^p \le N_1 \quad \text{for some constant} \quad N_1 > 1. \tag{28}$$

Using the above inequality (28) and then recalling [24, 33], we get the following local Lipschitz regularity

$$\sup_{B_{5\rho_i}(x_i)} |\nabla h|^p \le C \left\{\fint_{B_{6\rho_i}(x_i)} |\nabla h|^p \, dx\right\}^{\frac{1}{p}},$$

which implies that

$$\begin{aligned}\sup_{B_{5\rho_i}(x_i)} |\nabla h|^p &\leq C\left\{\fint_{B_{10\rho_i}(x_i)} |\nabla h|^p \, dx\right\}^{\frac{1}{p}} \\ &\leq C\left\{\fint_{B_{10\rho_i}(x_i)} [\Phi(u_\lambda, V)]^p \, dx\right\}^{\frac{1}{p}} \\ &\leq N_2 \quad \text{for some constant} \quad N_2 > 1, \end{aligned} \tag{29}$$

where we have used (21) and (27). Thus, (28) and (29) imply that

$$\sup_{B_{5\rho_i}(x_i)} [\Phi(h, V)]^p \leq N_1 + N_2 =: N_3. \tag{30}$$

Choosing $\mu = [\lambda\lambda_0]^p$, where λ_0 is defined in (11), we deduce from (12) and (30) that

$$\begin{aligned}&\left|\left\{x \in B_{5\rho_i}(x_i) : [\Phi(u, V)]^p > 2^p N_3 \mu\right\}\right| \\ &= \left|\left\{x \in B_{5\rho_i}(x_i) : [\Phi(u_\lambda, V)]^p > 2^p N_3\right\}\right| \\ &\leq \left|\left\{x \in B_{5\rho_i}(x_i) : [\Phi(u_\lambda - h, V)]^p > N_3\right\}\right| + \left|\left\{x \in B_{5\rho_i}(x_i) : [\Phi(h, V)]^p > N_3\right\}\right| \\ &= \left|\left\{x \in B_{5\rho_i}(x_i) : [\Phi(u_\lambda - h, V)]^p > N_3\right\}\right| \\ &\leq \frac{1}{N_3} \int_{B_{5\rho_i}(x_i)} [\Phi(u_\lambda - h, V)]^p \, dx,\end{aligned}$$

where we have used the inequality

$$(a + b)^p \leq 2^{p-1} (a^p + b^p) \quad \text{for any} \quad a, b > 0.$$

So, we use (27) and Lemma 2.11 to prove that

$$\left|\left\{x \in B_{5\rho_i}(x_i) : [\Phi(u, V)]^p > 2^p N_3 \mu\right\}\right| \leq C\epsilon^p |B_{\rho_i}(x_i)|.$$

Therefore, from Lemma 2.7 we see that

$$\begin{aligned}&\left|\left\{x \in B_{5\rho_i}(x_i) : [\Phi(u, V)]^p > 2^p N_3 \mu\right\}\right| \\ &\leq \frac{C\epsilon^p}{\mu}\left(\int_{\left\{x \in B_{\rho_i}(x_i):\, [\Phi(u,V)]^p > \frac{\mu}{3}\right\}} [\Phi(u, V)]^p \, dx + \frac{1}{\delta}\int_{\left\{x \in B_{\rho_i}(x_i):\, |\mathbf{f}|^p > \frac{\delta\mu}{3}\right\}} |\mathbf{f}|^p \, dx\right).\end{aligned}$$

Then recalling the fact that the balls $\{B_{\rho_i}(x_i)\}$ are disjoint and

$$\bigcup_{i \geq 1} B_{5\rho_i}(x_i) \cup \text{negligible set} \supset E(u_\lambda, V, 1) \quad \text{for any} \quad \lambda > 0,$$

we have

$$\begin{aligned}&\left|\{x \in \mathbb{R}^n : [\Phi(u, V)]^p > 2^p N_3 \mu\}\right| \\ &\leq \sum_i \left|\left\{x \in B_{5\rho_i}(x_i) : [\Phi(u, V)]^p > 2^p N_3 \mu\right\}\right| \\ &\leq \frac{C\epsilon^p}{\mu}\left(\int_{\left\{x \in \mathbb{R}^n :\, [\Phi(u,V)]^p > \frac{\mu}{3}\right\}} [\Phi(u, V)]^p \, dx + \frac{1}{\delta}\int_{\left\{x \in \mathbb{R}^n :\, |\mathbf{f}|^p > \frac{\delta\mu}{3}\right\}} |\mathbf{f}|^p \, dx\right).\end{aligned} \tag{31}$$

Now we make use of Lemma 2.4 (2) and the above inequality (31) to compute

$$\begin{aligned}\int_{\mathbb{R}^n}\phi\left([\Phi(u,V)]^p\right)dx &= \int_0^\infty \left|\{x\in\mathbb{R}^n:[\Phi(u,V)]^p>2^pN_3\mu\}\right| d\left[\phi\left(2^pN_3\mu\right)\right]\\ &\le C\epsilon^p\int_0^\infty\frac{1}{\mu}\left\{\int_{\left\{x\in\mathbb{R}^n:[\Phi(u,V)]^p>\frac{\mu}{3}\right\}}[\Phi(u,V)]^p\,dx\right\}d\left[\phi\left(2^pN_3\mu\right)\right]\\ &\quad +C\epsilon^p\int_0^\infty\frac{1}{\mu}\left\{\int_{\left\{x\in\mathbb{R}^n:|\mathbf{f}|^p>\frac{\delta\mu}{3}\right\}}|\mathbf{f}|^p\,dx\right\}d\left[\phi\left(2^pN_3\mu\right)\right].\end{aligned}$$

Furthermore, Lemma 2.5 implies that

$$\int_{\mathbb{R}^n}\phi\left([\Phi(u,V)]^p\right)dx\le C_1\epsilon^p\int_{\mathbb{R}^n}\phi\left([\Phi(u,V)]^p\right)dx+C_2\epsilon^p\int_{\mathbb{R}^n}\phi\left(|\mathbf{f}|^p\right)dx,$$

where $C_1=C_1(n,\phi)$ and $C_2=C_2(n,\phi,\delta,\epsilon)$. Here we may as well assume that $[\Phi(u,V)]^p\in L^\phi(\mathbb{R}^n)$ via a similar approximation argument in §3.1 of [47]. Finally, by choosing a suitable $\epsilon>0$ such that $C_1\epsilon^p\le 1/2$ we obtain

$$\int_{\mathbb{R}^n}\phi\left([\Phi(u,V)]^p\right)dx\le C\int_{\mathbb{R}^n}\phi\left(|\mathbf{f}|^p\right)dx,$$

which completes the proof. □

Acknowledgments

The author wishes to thank the anonymous reviewer for many valuable comments and suggestions to improve the expressions.

References

[1] E. Acerbi, G. Mingione, Gradient estimates for a class of parabolic systems, Duke Mathematical Journal 136 (2) (2007) 285–320.
[2] R. A. Adams, J. J. F. Fournier, Sobolev spaces (2nd edition), Academic Press, New York, 2003.
[3] V. Ambrosio, V. D. Rădulescu, Fractional double-phase patterns: concentration and multiplicity of solutions, Journal de Mathématiques Pures et Appliquées. Neuvième Série 142 (2020) 101–145.
[4] A. Balci, L. Diening, M. Weimar, Higher order Calderón-Zygmund estimates for the p-Laplace equation, Journal of Differential Equations 268 (2) (2020) 590–635.
[5] P. Baroni, Lorentz estimates for degenerate and singular evolutionary systems, Journal of Differential Equations 255 (9) (2013) 2927–2951.
[6] P. Baroni, C. Lindfors, The Cauchy-Dirichlet problem for a general class of parabolic equations, Annales de l'Institut Henri Poincaré C. Analyse Non Linéaire 34 (3) (2017) 593–624.
[7] P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, Calculus of Variations and Partial Differential Equations 57 (2) (2018) Paper No. 62 48 pp.
[8] V. Bögelein, F. Duzaar, G. Mingione, Degenerate problems with irregular obstacles, Journal für die Reine und Angewandte Mathematik 650 (2011) 107–160.
[9] V. Bögelein, F. Duzaar, J. Habermann, C. Scheven, Partial Hölder continuity for discontinuous elliptic problems with VMO-coefficients, Proceedings of the London Mathematical Society. Third Series 103 (3) (2011) 371–404.
[10] D. Breit, A. Cianchi, L. Diening, T. Kuusi, S. Schwarzacher, Pointwise Calderón-Zygmund gradient estimates for the p-Laplace system, Journal de Mathématiques Pures et Appliquées. Neuvième Série 114 (2018) 146–190.
[11] S. Byun, H. Kwon, H. So, L. Wang, Nonlinear gradient estimates for elliptic equations in quasiconvex domains, Calculus of Variations and Partial Differential Equations 54 (2) (2015) 1425–1453.
[12] S. Byun, D. K. Palagachev, P. Shin, Global Sobolev regularity for general elliptic equations of p-Laplacian type, Calculus of Variations and Partial Differential Equations 57 (5) (2018) Paper No. 135 19 pp.
[13] S. Byun, S. Ryu, Weighted Orlicz estimates for general nonlinear parabolic equations over nonsmooth domains, Journal of Functional Analysis 272 (10) (2017) 4103–4121.
[14] S. Byun, L. Wang, Nonlinear gradient estimates for elliptic equations of general type, Calculus of Variations and Partial Differential Equations 45 (3-4) (2012) 403–419.
[15] S. Byun, F. Yao, S. Zhou, Gradient estimates in Orlicz space for nonlinear elliptic equations, Journal of Functional Analysis 255 (8) (2008) 1851–1873.

[16] A. Carbonaro, G. Metafune, C. Spina, Parabolic Schrödinger operators, Journal of Mathematical Analysis and Applications 343 (2) (2008) 965–974.
[17] M. Colombo, G. Mingione, Calderón-Zygmund estimates and non-uniformly elliptic operators, Journal of Functional Analysis 270 (4) (2016) 1416–1478.
[18] M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, Archive for Rational Mechanics and Analysis 218 (1) (2015) 219–273.
[19] S. Dahlke, L. Diening, C. Hartmann, B. Scharf, M. Weimar, Besov regularity of solutions to the p-Poisson equation, Nonlinear Analysis. Theory, Methods & Applications 130 (2016) 298–329.
[20] E. DiBenedetto, J. Manfredi, On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems, American Journal of Mathematics 115 (5) (1993) 1107–1134.
[21] L. Diening, S. Schwarzacher, Global gradient estimates for the $p(\cdot)$-Laplacian, Nonlinear Analysis. Theory, Methods & Applications 106 (2014) 70–85.
[22] F. Duzaar, G. Mingione, Local Lipschitz regularity for degenerate elliptic systems, Annales de l'Institut Henri Poincaré C. Analyse Non Linéaire 27 (6) (2010) 1361–1396.
[23] F. Duzaar, G. Mingione, Gradient estimates via non-linear potentials, American Journal of Mathematics 133 (4) (2011) 1093–1149.
[24] L. Esposito, G. Mingione, C. Trombetti, On the Lipschitz regularity for certain elliptic problems, Forum Mathematicum 18 (2) (2006) 263–292.
[25] M. Eleuteri, A. Passarelli di Napoli, Higher differentiability for solutions to a class of obstacle problems, Calculus of Variations and Partial Differential Equations 57 (5) (2018) Paper No. 115 29 pp.
[26] W. Gao, Y. Jiang, L^p estimate for parabolic Schrödinger operator with certain potentials, Journal of Mathematical Analysis and Applications 310 (1) (2005) 128–143.
[27] T. Iwaniec, Projections onto gradient fields and L^p-estimates for degenerated elliptic operators, Studia Mathematica 75 (3) (1983) 293–312.
[28] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[29] J. Kinnunen, S. Zhou, A local estimate for nonlinear equations with discontinuous coefficients, Communications in Partial Differential Equations 24 (11-12) (1999) 2043–2068.
[30] V. Kokilashvili, M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces, World Scientific Publishing Co., Inc., River Edge, NJ, 1991.
[31] M. Lee, J. Ok, Interior and boundary $W^{1,q}$-estimates for quasi-linear elliptic equations of Schrödinger type, Journal of Differential Equations 269 (5) (2020) 4406–4439.
[32] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, Communications in Partial Differential Equations 16 (2-3) (1991) 311–361.
[33] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Analysis. Theory, Methods & Applications 12 (11) (1988) 1203–1219.
[34] T. Mengesha, N. Phuc, Quasilinear Riccati type equations with distributional data in Morrey space framework, Journal of Differential Equations 260 (6) (2016) 5421–5449.
[35] T. Mengesha, N. Phuc, Global estimates for quasilinear elliptic equations on Reifenberg flat domains, Archive for Rational Mechanics and Analysis 203 (1) (2012) 189–216.
[36] T. Mengesha, N. Phuc, Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains, Journal of Differential Equations 250 (5) (2011) 2485–2507.
[37] G. Mingione, The Calderón-Zygmund theory for elliptic problems with measure data, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V 6 (2) (2007) 195–261.
[38] W. Orlicz, Üeber eine gewisse Klasse von Räumen vom Typus **B**, Bulletin of the Polish Academy of Sciences. Mathematics 8 (1932) 207–220.
[39] G. Pan, L. Tang, Solvability for Schrödinger equations with discontinuous coefficients, Journal of Functional Analysis 270 (1) (2016) 88–133.
[40] M. Rao, Z. Ren, Applications of Orlicz spaces, Marcel Dekker, Inc., New York, 2002.
[41] Z. Shen, On the Neumann problem for Schrödinger operators in Lipschitz domains, Indiana University Mathematics Journal 43 (1) (1994) 143–176.
[42] Z. Shen, L^p estimates for Schrödinger operators with certain potentials, Université de Grenoble. Annales de l'Institut Fourier 45 (2) (1995) 513–546.
[43] E. M. Stein, Harmonic Analysis, Princeton University Press, Princeton, NJ, 1993.
[44] L. Tang, J. Han, L^p boundedness for parabolic Schrödinger type operators with certain nonnegative potentials, Forum Mathematicum 23 (1) (2011) 161–179.
[45] S. Thangavelu, Riesz transforms and the wave equation for the Hermite operator, Communications in Partial Differential Equations 15 (8) (1990) 1199–1215.
[46] M. Tran, T. Nguyen, G. Nguyen, Lorentz gradient estimates for a class of elliptic p-Laplacian equations with a Schrödinger term, Journal of Mathematical Analysis and Applications 496 (1) (2021) Paper No. 124806 18 pp.
[47] L. Wang, F. Yao, S. Zhou, H. Jia, Optimal regularity for the poisson equation, Proceedings of the American Mathematical Society 137 (6) (2009) 2037–2047.
[48] M. Xiang, B. Zhang, V. D. Rădulescu, Existence of solutions for perturbed fractional p-Laplacian equations, Journal of Differential Equations 260 (2) (2016) 1392–1413.
[49] S. Yang, D. Yang, L^p Neumann problem for some Schrödinger equations in (semi-)convex domains, Communications in Contemporary Mathematics 22 (2) (2020) 1950007 20 pp.
[50] F. Yao, Optimal regularity for Schrödinger equations, Nonlinear Analysis. Theory, Methods & Applications 71 (11) (2009)

5144–5150.

[51] F. Yao, S. Zhou, Global estimates in Orlicz spaces for p-Laplacian systems in $\mathbb{R}^N$, Journal of Partial Differential Equations 25 (2) (2012) 103–114.

[52] C. Zhang, S. Zhou, Global weighted estimates for quasilinear elliptic equations with non-standard growth, Journal of Functional Analysis 267 (2) (2014) 605–642.

[53] J. Zhong, Harmonic analysis for some Schrödinger type operators, Ph.D. Thesis, Princeton University, 1993.