Filomat 37:10 (2023), 3063–3073
https://doi.org/10.2298/FIL2310063A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The strong BD property in Banach lattices

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Abstract. We introduce the concept of strong BD property in Banach lattices and we characterize Banach lattices with this property. Also, by introducing the class of almost limited (resp. Dunford-Pettis) weakly completely continuous operators from an arbitrary Banach lattice E to another F, we give some properties of them related to some well known classes of operators and related to the strong BD (resp. RDP*) property of the Banach lattice E.

1. Introduction

A subset A of a Banach space X is called limited (resp. Dunford–Pettis (DP)), if every weak* null (resp. weak null) sequence (x_n^*) in X^* converges uniformly on A, that is

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0.$$

A Banach space X has the BD (resp. RDP*) property, if each limited (resp. DP) set in X is relatively weakly compact. It is shown that if X is weakly sequentially complete or X does not contain the absolutely summable sequence space ℓ_1, then X has the BD property. If X is weakly sequentially complete, then it has the RDP* property. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand–Phillips (GP) property. If every DP subset of a Banach space X is relatively compact, then X has the $DP_{rc}P$ [7, 10–12].

A subset A of a Banach lattice E is called almost limited (almost DP), if every disjoint and weak* null (weak null) sequence (x_n^*) in E^* converges uniformly on A (see [6, 9]). A Banach lattice E has the weak DP property if each realatively weakly compact set in E is an almost DP set, or equivalently, for every weakly null sequence (x_n) in E and disjoint weakly null sequence (f_n) in E^*, we have $f_n(x_n) \to 0$ (see [25]).

Recently, the authors in [3, 4] introduced an alternative form of the GP property and $DP_{rc}P$ for Banach lattices. In fact, a Banach lattice E has the strong GP property (Strong $DP_{rc}P$) if every almost limited (almost DP) set in E is relatively compact. Then by introducing the concept of almost limited completely continuous (alcc) operators and almost DP completely continuous (aDPcc) operators between Banach lattices, they obtain

2020 *Mathematics Subject Classification*. 46B42, 46B50, 47B65.

Keywords. almost limited set, almost DP set, weakly completely continuous operator, RDP* property, BD property.
Received: 28 April 2022; Revised: 04 August 2022; Accepted: 16 August 2022
Communicated by Mohammad Sal Moslehian

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some characterizations of them and then the relation between the strong GP property and strong $DP_{rc}P$ of E and also alcc and aDPcc operators on E are treated.
In this article, we introduce an alternative form of the BD property for Banach lattices, so called, the strong BD property if every almost limited set in E is relatively weakly compact. Also, by introducing the concept of almost limited (resp. almost DP) weakly completely continuous operators between Banach lattices, we obtained some characterizations of them. Finally, some relations between the strong BD property (resp. RDP^* property) of Banach lattices with these operators on that spaces are treated.
A subset A of E is called solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$ and the solid hull of A is the smallest solid set containing A and is exactly the set $Sol(A) = \{y \in E : |y| \leq |x|, \text{ for some } x \in A\}$. The lattice operations in the dual Banach lattice E^* are weak* sequentially continuous if for every weak* null sequence (f_n) in E^*, $|f_n| \to 0$ for $\sigma(E^*, E)$. We refer the reader for undefined terminologies to the classical references [1, 2, 19].

2. Strong BD property in Banach lattices

Theorem 2.1. *Let E be a Banach lattice with order continuous norm. Then E has the strong BD property.*

Proof. Let A be an almost limited subset of E. By order continuity of the norm on E and [9, Theorem 2.6], A is L-weakly compact (i.e. $\|x_n\| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) contained in $Sol(A)$. Since every L-weakly compact set is relatively weakly compact, so A is relatively weakly compact, that is, E has the strong BD property. □

As an immediate consequence of the definition BD property one can conclude that the BD property is inherited by closed subspaces. In fact, if X is a Banach space with the BD property and Y is a closed subspace of X. Then each limited set A in Y is a limited set in X, hence relatively weakly compact in X. Then every sequence (x_n) in A has a subsequence weakly convergent to an element x in X. Since Y is closed, it is weakly closed. Hence x is in Y. Thus A is relatively weakly compact in Y, and Y has the BD property.

Theorem 2.2. *Let E be a σ–Dedekind complete Banach lattice without order continuous norm. Then E does not have the (strong) BD property.*

Proof. By [19, Corollary 2.4.3], every σ–Dedekind complete E without order continuous norm contains a sublattice isomorphic to ℓ_∞, which does not have the BD property [21, Example 1.1.8]. Since the BD property is inherited by closed subspaces [21], then E does not have the (strong) BD property. □

A Banach lattice E has the property (d) if the sequence $(|f_n|)$ is weak* null for every disjoint weak* null sequence (f_n) in E^*. Cleraly, each σ–Dedekind complete Banach lattice has the property (d).
In the following we characterize some classes of Banach lattices with the strong BD property.

Theorem 2.3. *Let E be a Banach lattice. Then the following are equivalent:*

(a) E has order continuous norm,

(b) E has the properties (d) and strong BD.

Proof. $(a) \Rightarrow (b)$. If E has order continuous norm, from Theorem 2.1, it the strong BD property. Also E is σ–Dedekind complete and so it has the property (d).
$(b) \Rightarrow (a)$. By defenition of almost limitedness, every order interval in a Banach lattice E is almost limited if and only if E has the property (d). So in a Banach lattice E with the property (d), every order interval is an almost limited set and because of another hypothesis of (b), it is weakly compact. Now by [2, Theorem 4.9], E has order continuous norm. □

Theorem 2.4. *Let E be a Grothendieck Banach lattice (i.e. weak* convergent sequences in its dual are weakly convergent). Then the following are equivalent:*

(a) E has the strong BD property,

(b) E has the BD property,

(c) E has order continuous norm.

Proof. $(a) \Rightarrow (b)$. It is obvious.
$(b) \Rightarrow (c)$. If E has the BD property, then each limited set in E is relatively weakly compact. Since E is Grothendieck, so each DP set in E is relatively weakly compact. Now by [6, Theorem 2.10], E is a KB-space. Therefore E has order continuous norm.
$(c) \Rightarrow (a)$. It is obvious. □

Theorem 2.5. *Let E be a σ–Dedekind complete Banach lattice. Then the following are equivalent:*

(a) E has the strong BD property,

(b) E has the BD property,

(c) E has order continuous norm,

(d) E has the GP property.

Proof. $(a) \Rightarrow (b)$. It is obvious.
$(b) \Rightarrow (c)$. If E has the BD property, then it does not contain any sublattice isomorphic to ℓ_∞. Since E is σ–Dedekind complete, by [19, Corollary 2.4.3], E has order continuous norm.
$(c) \Rightarrow (a)$. It follows from Theorem 2.1.
Also by [23, Theorem 4.5], (c) and (d) are equivalent. □

Theorem 2.6. *Let E be a Banach lattice such that E^* has the weak* sequentially continuous lattice operations, the following are equivalent:*

(a) E has the strong BD property,

(b) E has the BD property.

Proof. $(a) \Rightarrow (b)$. It is obvious.
$(b) \Rightarrow (a)$. Let A be an almost limited set in E. Since E^* has the weak* sequentially continuous lattice operations, so E has the property (d) and by [20, Lemma 2.1], $B = Sol(A)$ is almost limited. By hypothesis and from [18, Theorem 2.6], B is limited. Since E has the BD property, $B = Sol(A)$ (and hence A itself) is relatively weakly compact. Hence E has the strong BD property.
As an alternative proof, if E^* has the weak* sequentially continuous lattice operations, then by [18, Proposition 2.1], each order interval in E is limited and so it is weakly compact, thanks to the BD property of E. So [2, Theorem 4.9], implies that E has order continuous norm and so it has the strong BD property. □

The following theorem is proved by a similar method of Theorem 2.6.

Theorem 2.7. *Let E be a Banach lattice such that E^* has the weak* sequentially continuous lattice operations, the following are equivalent:*

(a) E has the strong GP property,

(b) E has the GP property.

Proof. $(a) \Rightarrow (b)$. It is obvious.
$(b) \Rightarrow (a)$. Let A be an almost limited set in E. Since E^* has the weak* sequentially continuous lattice operations, so E has the property (d) and by [20], $B = Sol(A)$ is almost limited. By the hypothesis and from [18, Theorem 2.6], B is limited. Since E has the GP property, $B = Sol(A)$ (and hence A itself) is relatively compact. Hence E has the strong GP property.
Also, we give an alternative proof for it. In fact, if E^* has the weak* sequentially continuous lattice operations, then by [18, Proposition 2.1] order interval in E is limited and so it is relatively compact, by the GP property of E. So by [1, Theorem 21.13], E is discrete with order continuous norm and so by [3, Theorem 2.1] it has the strong GP property. □

Every Banach lattice with the strong GP property has the strong BD property, but the converse is false. In fact, each non discrete Banach lattice with order continuous norm has the strong BD property, but from [3, Theorem 2.3], it does not have the strong GP property. For examples, $L^1[0,1]$ and the dual Banach lattice $(C[0,1])^*$ are two cases of such Banach lattices.

Example 2.8. Consider $C(K)$, where K is the Cantor set. Then $C(K)$ has the GP property, but it does not have the strong BD property.

Proof. Since $C(K)$, where K is the Cantor set, is isomorphic to $C[0,1]$, then it is seperable and so it has the GP property. Now we show that it does not have the strong BD (strong GP) property. Consider the Rademacher sequence in $C(K)$:

$$\begin{aligned}
r_1 &= 1, \quad \text{on } [0, \tfrac{1}{3}] \cap K, \\
r_1 &= -1, \quad \text{on } [\tfrac{2}{3}, 1] \cap K, \\
r_2 &= 1, \quad \text{on } ([0, \tfrac{1}{9}] \cup [\tfrac{2}{3}, \tfrac{7}{9}]) \cap K, \\
r_2 &= -1, \quad \text{on } ([\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{8}{9}, 1]) \cap K, \\
& \textit{etc.}
\end{aligned}$$

This sequence is not compact (weakly compact), since it is equivalent to the unit vector basis of ℓ_1. In fact, let (a_n) be a real sequence. One can find a point $t \in [0,1]$ so that, $\operatorname{sign} a_n = \operatorname{sign} r_n(t)$ for all n. Then

$$\|\sum_{n=1}^{\infty} a_n r_n\| \geq \sum_{n=1}^{\infty} (a_n r_n(t)) = \sum_{n=1}^{\infty} |a_n|.$$

But the sequence (r_n) is almost limited. Indeed if (x_n) is a convergent sequence of distinct points in K, then the point functionals at (x_n) converges weak* to the point functional at 0. So the the sequence of functionals $\delta(x_{2n}) - \delta(x_{2n-1})$ is weak* null, disjoint and converges uniformly to 0 on (r_n). Since $\operatorname{ext}(B_{C(K)^*}) = \{\pm\delta_t : t \in K\}$, so every weak* null and disjoint sequenec (μ_n) in $C(K)^*$ converges uniformly to 0 on (r_n); that is the sequence (r_n) is almost limited. □

Recall from [18] that a Banach lattice E has the dual Schur property if every disjoint and weak* null sequence in E^* is norm null. In other words, the closed unit ball B_E is almost limited.
A Banach space X has the DP* property whenever every relatively weakly compact set in X is limited [5]. Also a Banach lattice E has the weak DP* property if every relatively weakly compact set in E is almost limited [9]. Since every Banach lattice with order continuous norm is a GP space, E has the Schur property if and only if E has both order continuous norm and the DP* property. From [9, Prposition 3.3], every Banach lattice with the positive Schur (i.e., if every weakly null sequence with positive terms is norm null, or equivalently, every disjoint weakly null sequence is norm null [22]) and without Schur has the weak DP* property, but it does not have the DP* property.

Theorem 2.9. *If a Banach lattice E has the dual Schur and strong BD properties, then E is finite dimensional.*

Proof. If E has the dual Schur property, then the closed unit ball B_E is almost limited and so it is relatively weakly compact, by the strong BD property of E. Hence E is reflexive and so E has order continuous norm. But, dual Schur property of E implies that each disjoint weak* null sequence $(f_n) \subset (E^*)^+$ is norm null and then by [24, Propositions 2.3, Proposition 2.1], the norm on E is not order continuous when E is infinite dimensional. Hence E must be finite dimensional. □

The first part (a) of the following lemma follows from [8] and the second part (b) is proved similar to (a).

Lemma 2.10. *Let E be a Banach lattice. Then*

(a) *E has the DP* property iff conditionally weakly compact sets and limited sets in X coincide,*

(b) *E has the weak DP* property iff every conditionally weakly compact set in E is almost limited.*

Theorem 2.11. *For a Banach lattice E, the following are equivalent:*

(a) *E has the BD and DP* properties,*

(b) *E is a KB-space with the DP* property,*

(c) *E has order continuous norm and the DP* property,*

(d) *E has the Schur property.*

Proof. $(a) \Rightarrow (b)$. Let (x_n) be a weakly Cauchy sequence in E. Since E has the DP* property, then by Lemma 2.10, (x_n) is limited and so it is relatively weakly compact, by the BD property of E. Hence E is a KB-space.
$(b) \Rightarrow (c)$. Each KB-space has order continuous norm.
$(c) \Rightarrow (d)$. Let A be a relatively weakly compact set in E. Then by the DP* property of E, A is limited. Since E has order continuous norm, hence it has the GP properrty, and then A is relatively compact; that is, E has the Schur property.
$(d) \Rightarrow (a)$. It is clear that, every Banach lattice with the Schur property is a KB-space with the DP* property. □

We note that every KB-space has the BD property, but the converse is false. In fact c_0 has the BD property, but it is not a KB-space.

Theorem 2.12. *For a Banach lattice E, the following are equivalent:*

(a) *E has the strong BD and weak DP* properties,*

(b) *E is a KB-space with the weak DP* property,*

(c) *E has order continuous norm and the weak DP* property,*

(d) *E has the positive Schur property.*

Proof. $(a) \Rightarrow (b)$. Let (x_n) be a weakly Cauchy sequence in E. Since E has the weak DP* property, then by Lemma 2.10, (x_n) is almost limited and so it is relatively weakly compact by the strong BD property of E. Hence E is a KB-space.
$(b) \Rightarrow (c)$. Each KB-space has order continuous norm.
$(c) \Rightarrow (d)$. Let A be a relatively weakly compact set in E. Then by the weak DP* property of E, A is almost limited. Since E has order continuous norm, then by [9, Theorem 2.6] A is L-weakly compact. So by [19, Corollary 3.6.8], E has the positive Schur property.
$(d) \Rightarrow (a)$. It is clear. Since every Banach lattice with the positive Schur property is a KB- space with the weak DP* property. □

Theorem 2.13. *Let E be a Banach lattice such that E^* has a weak unit. Then the following are equivalent:*

(a) *E has the dual Schur property,*

(b) *E has the weak DP* property and E does not contain ℓ_1.*

Proof. $(a) \Rightarrow (b)$. If E has the dual Schur property, then the closed unit ball B_E is almost limited and so by [3, Theorem 2.14], it is conditionally weakly compact. Now by Rosenthal's ℓ_1-Theorem, E does not contain ℓ_1. Clearly, every relatively weakly compact set in E is almost limited; that is, E has the weak DP* property.
$(b) \Rightarrow (a)$. If E does not contain ℓ_1, by Rosenthal's ℓ_1-Theorem, the closed unit ball B_E is conditionally weakly compact and so it is almost limited by the weak DP* property and Lemma 2.10 (b). Hence E has the dual Schur property. □

We note that ℓ_∞ has the dual Schur and weak DP* properties, but ℓ_∞ contain ℓ_1. In fact ℓ_∞ does not have order continuous norm and $(\ell_\infty)^*$ dose not have any weak unit.
If a Banach lattice E has the strong BD property, then every weakly Cauchy and almost limited sequence in E is weakly convergent. The following theorem shows that the converse is false, in general.

Theorem 2.14. *Let K be a compact metric space. Then every weakly Cauchy and almost limited sequence in $C(K)$ is weakly convergent (actually norm convergent).*

Proof. Let (f_n) be a weakly Cauchy and almost limited sequence in $C(K)$. Then the sequence $(f_n - f_m)$ is weakly null and almost limited, so by [3, Theorem 2.12] it is norm null; that is, the sequence (f_n) is Cauchy and so it is norm convergent in $C(K)$. □

Theorem 2.15. *Let E be a Banach lattice such that E^* has a weak unit. Then E has the strong BD property if and only if every weakly Cauchy and almost limited sequence in E is weakly convergent.*

Proof. The direct implication is clear. For the converse, let A be an almost limited subset of E. Then by [3] every sequence (x_n) in A has a subsequence, denoted again by (x_n), that is weakly Cauchy. Since, the sequence (x_n) is almost limited, by the hypothesis, it is weakly convergent. Thus A is relatively weakly compact. □

It should be noted that every weakly Cauchy and almost limited sequence in a Banach lattice is not weakly convergent, in general. For example, ℓ_∞ is not a KB-space and so there is a weakly Cauchy sequence in ℓ_∞ which is not weakly convergent. But since ℓ_∞ has the dual Schur property, the closed unit ball B_{ℓ_∞} is almost limited. Therefore there is a weakly Cauchy and almost limited sequence in ℓ_∞ which is not weakly convergent.

Theorem 2.16. *Let E be a Banach lattice such that E^* has a weak unit or E has order continuous norm. Then the following are equivalent:*

(a) E^* has the positive Schur property,

(b) E has the weak DP property and E does not contain ℓ_1.

Proof. $(a) \Rightarrow (b)$. If E^* has the positive Schur property, then the closed unit ball B_E is almost DP and by [4, Theorem 2.8] it is conditionally weakly compact. Now by Rosenthal's ℓ_1-Theorem, E does not contain ℓ_1. Clearly, every relatively weakly compact set in E is almost DP; that is, E has the weak DP property.
$(b) \Rightarrow (a)$. If E does not contain ℓ_1, by Rosenthal's ℓ_1-Theorem, the closed unit ball B_E is conditionally weakly compact and so it is almost DP by the weak DP property. Then E^* has the positive Schur property. □

We know that by [24, Propositin 2.5], $(\ell_\infty)^*$ has the positive Schur property. Also ℓ_∞ has the weak DP property and it contains ℓ_1. As we mentioned before, ℓ_∞ does not have order continuous norm and $(\ell_\infty)^*$ does not have any weak unit. So the assumptions of Theorem 2.16 are essential.

Theorem 2.17. *For a Banach lattice E, the following assertions are equivalent:*

(a) E has the RDP property,*

(b) every weakly Cauchy and almost DP sequence in E is weakly convergent,

(c) every weakly Cauchy and DP sequence in E is weakly convergent.

Proof. $(a) \Rightarrow (b)$. If E has the RDP* property, then it is a KB-space and so by [19, Theorem 2.5.6], every weakly Cauchy (and almost DP) sequence in E is weakly convergent.
$(b) \Rightarrow (c)$. Since each DP set is almost DP, it is clear.
$(c) \Rightarrow (a)$. If every weakly Cauchy and DP sequence in E is weakly convergent, then E does not contain a copy of c_0. Since for standard basis sequence (e_n) in c_0, the sequence $f_n = \sum_{i=1}^{n} e_i$ is weakly Cauchy and it is a DP sequenece, but it is not weakly convergent. Then by [19, Theorem 2.5.6], E is a KB-space; that is, it has the RDP* property. □

3. Almost limited weakly completely continuous operators

An operator $T : X \to Y$ is called weakly completely continuous (wcc), if T carries weakly Cauchy sequences in X to weakly convergent ones, and the class of them is denoted by $Wcc(X, Y)$. Also, T is limited weakly completely continuous (lwcc), if T carries weakly Cauchy and limited sequences to weakly convergent ones and the class of them is denoted by $Lwcc(X, Y)$ [15].
We say that an operator $T : X \to Y$ is DP weakly completely continuous (DPwcc), if T carries weakly Cauchy and DP sequences in X to weakly convergent ones, and we use the class of DPwcc operators from X to Y by $DPwcc(X, Y)$. Also we define almost limited weakly completely continuous (alwcc) and almost DP weakly completely continuous (aDPwcc) operators and establish some additional properties of them.

Definition 3.1. An operator $T : E \to Y$ is alwcc (aDPwcc), if T carries weakly Cauchy and almost limited (almost DP) sequences in E to weakly convergent ones, and the class of alwcc and aDPwcc operators are denoted by $L^a wcc(E, Y)$ and $DP^a wcc(E, Y)$.

If a Banach lattice E has the DP^* property, then, for every Banach space Y, $L^a wcc(E, Y) = Lwcc(E, Y) = Wcc(E, Y)$.
Also, if a Banach lattice E has the DP property, then for every Banach space Y, $DP^a wcc(E, Y) = DPwcc(E, Y) = Wcc(E, Y)$.

Theorem 3.2. *An operator T on E is an alwcc (aDPwcc) operator if, for each almost limited (almost DP) set $A \subset E$, the set $T(A)$ is relatively weakly compact.*

Proof. Let (x_n) be a weakly Cauchy and almost limited (almost DP) sequence in E. Then by hypothesis, the sequence (Tx_n) is relatively weakly compact, and so it has a weakly convergent subsequence (Tx_{n_k}). Since T carries weakly Cauchy sequences to weakly Cauchy ones, the sequence (Tx_n) is weakly Cauchy. This shows that the sequence (Tx_n) is weakly convergent, and then T is alwcc (aDPwcc). $\square$

It should be noted that the converse of the theorem is false, in general. In fact, let K be a compact metric space. Then by Theorem 2.14, every weakly Cauchy and almost limited sequence in $C(K)$ is weakly convergent (actually norm convergent) and so the identity operator $Id_{C(K)}$ is alwcc, but as we said before, there exists an almost limited set A in $C(K)$ such that A is not relatively weakly compact. The following theorem garanties that under the containment a weak unit of E^*, the converse is true.

Theorem 3.3. *Let $T : E \to Y$ be an alwcc (aDPwcc) operator such that E has order continuous norm or E^* has a weak unit. Then, for every almost limited (almost DP) set A in E, $T(A)$ is relatively weakly compact.*

Proof. Let $A \subset E$ be almost limited (almost DP). Then by hypothesis, every sequence (x_n) in A has a subsequence, denoted again by (x_n), that is weakly Cauchy. On the other hand (x_n) is almost limited (almost DP) and then the sequence (Tx_n) is weakly convergent. Thus $T(A)$ is relatively weakly compact. $\square$

In the following theorem, we give a characterization of the strong BD property, with respect to alwcc operators.

Theorem 3.4. *If E^* has a weak unit, then the following are equivalent:*

- *(a) E has the strong BD property,*
- *(b) $L(E, Y) = L^a wcc(E, Y)$, for each Banach space Y,*
- *(c) $L(E, \ell_\infty) = L^a wcc(E, \ell_\infty)$.*

Proof. $(a) \Rightarrow (b)$. Let $A \subset E$ be almost limited and $T \in L(E, Y)$. Since E has the strong BD property, A and so $T(A)$ are relatively weakly compact. Now by Theorem 3.2, T is alwcc; that is $L(E, Y) = L^a wcc(E, Y)$.
$(b) \Rightarrow (c)$. It is clear.
$(c) \Rightarrow (a)$. Assume that E does not have the strong BD property. Since E^* has a weak unit, then there is a weakly Cauchy and almost limited sequence (x_n) in E which is not weakly convergent. Let F be the closed

subspace of E spanned by this sequence. Then F is separable and so it is isomorphic to a closed subspace of ℓ_∞. Let T be such an isomorphic embedding from F to ℓ_∞. Since ℓ_∞ is injective, we can extend T to all of E. Hence T is not wcc and this completes the proof of the theorem. □

Note that if E^* does not have a weak unit, then Theorem 3.4 may be false. In fact, every weakly Cauchy and almost limited sequence in $C(K)$, where K is the compact metric space, is norm convergent and so every operator on $C(K)$ is alwcc, but $C(K)$ does not have the strong BD property.
Also note that continuous linear images of limited sets or sequences are limited, but the same conclusion is false for almost limited sets or sequences, in general. In fact, for the operator $T : L^1[0,1] \to c_0$ defined by $Tf = (\int_0^1 f(t)r_n(t)dt)$ for all $f \in L^1[0,1]$ the Rademacher sequence $f_n(t) = r_n(t)$ is weakly null and so it is almost limited, by the weak DP* property of $L^1[0,1]$, but $(Tf_n) = (e_n)$ is not almost limited.
The next theorem is a characterization of Banach lattices with the RDP* property using aDPwcc operators. In [6, Theorem 2.10] and [13, Proposition 3.14], it has been shown that, E has the RDP* property iff E is a KB-space.

Theorem 3.5. *For a Banach lattice E, the following are equivalent:*

(a) E has the RDP property,*

(b) $L(E,F) = Wcc(E,F)$, for each Banach lattice F,

(c) $L(E,F) = DP^a wcc(E,F)$, for each Banach lattice F,

(d) $L(E,F) = DPwcc(E,F)$, for each Banach lattice F,

Proof. $(a) \Rightarrow (b)$. If E has the RDP* property, then each weakly Cauchy sequence in E is weakly convergent, and so $L(E,F) = Wcc(E,F)$, for each Banach lattice F.
$(b) \Rightarrow (c)$. It is obvious.
$(c) \Rightarrow (d)$. It is obvious.
$(d) \Rightarrow (a)$. If $F = E$, then (d) implies that the identity operator on E is DPwcc; that is, every weakly Cauchy and DP sequence in E is weakly convergent. Then E does not contain a copy of c_0 and so it has the RDP* property. □

A similar proof shows that if one replace the role of E and F in cases (b), (c) and (d) of theorem, then the conclusion holds.

Corollary 3.6. *Every bounded linear operator on a Banach lattice E with order continuous norm is an alwcc operator. Also if E is a KB-space, then every operator on E is an aDPwcc operator.*

Recall from [19] that a positive linear operator $T : E \to F$ is almost interval preserving, if $T[0,x]$ is dense in $[0,Tx]$, for every $x \in E^+$. In [3, Theorem 3.8] and [4, Theorem 3.12], we establish some conditions which guarantees the continuous linear images of almost limited (resp. almost DP) sets are also almost limited (resp. almost DP).

Corollary 3.7. *(a) Let F be a Banach lattice with the strong BD property. Then every almost interval preserving operator $T : E \to F$ is alwcc.*

(b) Let $T : E \to F$ and $S : F \to G$ be almost interval preserving operators such that S or T is alwcc (resp. aDPwcc). Then the composition operator ST is alwcc (resp. aDPwcc).

Proposition 3.8. *(a) Let E and F have the property (d), and let $T : E \to F$ be an order bounded operator. If F has the strong BD property, then T is an alwcc operator.*

(b) Let $T : E \to F$ be an order bounded linear operator between two Banach lattices with the property (d) and let $S : F \to G$ be an operator such that S or T is alwcc. Then the composition operator ST is alwcc.

Proof. (a). Suppose that (x_n) is a weakly Cauchy and almost limited sequence in E and $A = sol((x_n)_n)$. From [20, Lemma 2.1, Theorem 2.3], A and also $T(A)$ are almost limited. Since the sequence $(Tx_n) \subset T(A)$, then (Tx_n) is weakly Cauchy and almost limited. Since F has the strong BD property, (Tx_n) is weakly convergent and so T is alwcc.
(b). Let S be an alwcc operator and let (x_n) be a weakly Cauchy and almost limited sequence in E. Let $A = Sol((x_n)_n)$. As we said in the proof of (a), (Tx_n) is weakly cauchy and almost limited. Since S is an alwcc operator, $(ST(x_n))$ is weakly convergent and so the composition operator ST is alwcc. If T is alwcc, then clearly the operator ST is alwcc. □

Theorem 3.9. *If E^* has a weak unit or E has order continuous norm and $T : X \to E$ is an operator. Then for the following assertions:*

(a) T is almost limited; that is, $T(B_X)$ is almost limited in E,

(b) for each Banach space Y and each alwcc operator $S : E \to Y$, the operator ST is weakly compact,

(c) for each almost interval preserving operator $S : E \to F$, where F has the strong BD property, the operator ST is weakly compact,

the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) valid.

Proof. (a) $\Rightarrow$ (b). If T is almost limited, then $T(B_X)$ is almost limited in E. Since $S : E \to Y$ is alwcc, then by Theorem 3.3, $S(T(B_X))$ is relatively weakly compact. So the operator ST is weakly compact.
(b) $\Rightarrow$ (c). By Corolllary 3.7, almost interval preserving operators are alwcc and from (b) the result is clear. □

In the same way, the following theorem can be expressed for RDP* property using almost DP sets.

Theorem 3.10. *Let $T : X \to E$ be an operator and E^* has a weak unit or E has order continuous norm. Then for the following assertions:*

(a) T is almost weakly limited; that is, $T(B_X)$ is almost DP in E,

(b) for each Banach space Y and each aDPwcc operator $S : E \to Y$, the operator ST is weakly compact,

(c) for each operator $S : E \to F$, where F has the RDP property, the operator ST is weakly compact,*

the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) valid.

Proof. (a) $\Rightarrow$ (b). If T is almost weakly limited, then $T(B_X)$ is almost DP in E. Since $S : E \to Y$ is aDPwcc, then by Theorem 3.3, $S(T(B_X))$ is relatively weakly compact. So the operator ST is weakly compact.
(b) $\Rightarrow$ (c). Since F has the RDP* property, then by Theorem 3.5, each operator $S : E \to F$ is aDPwcc. From (b), the operator ST is weakly compact. □

Recall from [7] that $A \subset X$ is limited iff for every operator $S : X \to c_0$, $S(A)$ is a relatively compact set in c_0. Recently, using the concept of disjoint operators from E into c_0 (are uniquely determined by disjoint weak* null sequences in E^*) a similar characterization for almost limited sets is obtained in [14]. In fact, $A \subset E$ is almost limited iff for each disjoint operator $S : E \to c_0$, $S(A)$ of c_0 is a relatively compact subset of c_0. To prove the following proposition, let us recall that, an operator is regular if it can be written as difference of two positive operators and regular operators are order bounded.

Proposition 3.11. *For each operator $T : E \to F$ such that E^* has a weak unit or the norm of E is order continuous and F has the property (d), the following are equivalent:*

(a) T is almost limited,

(b) For each Banach space Z and each alcc (resp. alwcc) operator $S : F \to Z$, the composition operator ST is compact (resp. weakly compact),

(c) For each disjoint operator $S : F \to c_0$, the operator ST is compact (weakly compact).

Proof. $(a) \Rightarrow (b)$. If T is almost limited, then by Theorem 3.3, $S(T(B_X))$ is relatively compact (resp. relatively weakly compact). So the operator ST is compact (resp. weakly compact).
$(b) \Rightarrow (c)$. From [20, Proposition 3.10], each disjoint operator $S : F \to c_0$ is alcc (resp. alwcc) and apply (b).
$(c) \Rightarrow (a)$. For every disjoint operator $S : F \to c_0$, the operator ST is compact (resp. weakly compact); that is, $S(T(B_X))$ is relatively compact (resp. relatively weakly compact) and so by [14], $T(B_X)$ is almost limited in F. So T is almost limited. □

Theorem 3.12. *Continuous linear image of an almost limited set under a quotient map is almost limited.*

Proof. Let F be a closed ideal of a Banach lattice E. Consider the quotient map $Q : E \to E/F$. If $A \subset E$ is almost limited, we show that $Q(A)$ is an almost limited set in E/F; that is, for all weak* null and disjoint sequence (f_n) in $(E/F)^* = F^\perp$, $\sup_{x\in A} |\langle f_n, x\rangle| \to 0$.
For each $x \in E$, $x + F \in E/F$. Let (f_n) be a disjoint weak* unll sequence in $F^\perp$. So $\langle f_n, x\rangle = \langle f_n, x + F\rangle \to 0$, as $n \to \infty$; that is (f_n) is weak* null in E^*. Also for all $m \neq n$, $\langle f_n \wedge f_m, x\rangle = \langle f_n \wedge f_m, x + F\rangle = 0$. Then (f_n) in E^* is disjoint and so

$$\sup_{x\in A} |\langle f_n, Q(x)\rangle| = \sup_{x\in A} |\langle f_n, x\rangle| \to 0.$$

□

To prove the theorem 3.14, we need the following lemma of [17]:

Lemma 3.13. *Let X be a Banach space and Y be a reflexive subspace of X. Let $Q : X \to X/Y$ denotes the quotient map. Let (x_n) be a bounded sequence in X such that $(Q(x_n))$ converges weakly to some $Q(x) \in X/Y$. Then (x_n) is relatively weakly compact.*

Theorem 3.14. *Let E be a Banach lattice and F be a reflexive closed ideal of E. If E/F has the strong BD property, then E has the same property.*

Proof. Let (x_n) be an almost limited sequence in E. If $Q : E \to E/F$ denotes the quotient map, then by Theorem 3.12, $(Q(x_n))$ is an almost limited sequence in E/F and, by our assumption, it must be relatively weakly compact; hence some subsequence $(Q(x_{n_k}))$ has to converge weakly to some element of E/F, say $Q(x)$. Thanks Lemma 3.13 a further subsequence of (x_{n_k}) has to converge weakly in E. □

We conclude this section with some results about the strong BD property of a Banach lattice $\mathcal{M}$ of some operator spaces. If $\mathcal{M} \subset L(X, Y)$ is a Banach lattice with the strong BD (resp., RDP*) property, then all of evaluation operators ϕ_x and ψ_{y^*} are alwcc (resp., aDPwcc), where $\phi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$ for $x \in X$, $y^* \in Y^*$ and $T \in \mathcal{M}$. In the rest of this section, we provide some conditions such that the alwcc ness of evaluation operators is a sufficient condition for the strong BD property of their domain.

Theorem 3.15. *Let $L(X, Y) = K(X, Y)$ and $\mathcal{M} \subset K(X, Y)$ be a Banach lattice and $\mathcal{M}^*$ has a weak unit and all of ϕ_x and ψ_{y^*} are alwcc (resp. aDPwcc) operators, then $\mathcal{M}$ has the strong BD (resp. RDP*) property.*

Proof. Let H be an almost limited (resp. almost DP) set in $\mathcal{M}$. Since $\mathcal{M}^*$ has a weak unit, then by Theorem 3.3, $\phi_x(H) = Hx$ and $\psi_{y^*}(H) = H^*y^*$ are relatively weakly compact, for all $x \in X$ and $y^* \in Y^*$. Also by Theorem 2.14 of [3] (resp. Theorem 2.8 of [4]), H is conditionally weakly compact. Hence by [16, Theorem 4.10], H is relatively weakly compact; that is, $\mathcal{M}$ has the strong BD (resp. RDP*) property. □

Theorem 3.16. *Let E has the strong GP property and $\mathcal{M} \subset K_{w^*}(E^*, Y)$ be a Banach lattice such that $\mathcal{M}^*$ has a weak unit. If for each $x^* \in E^*$, the evaluation operator ϕ_{x^*} on $\mathcal{M}$ is alwcc, then $\mathcal{M}$ has the strong BD property.*

Proof. Let H be an almost limited set in $\mathcal{M}$ and let (T_n) be a sequence in H. Since $\mathcal{M}^*$ has a weak unit, then by Theorem 2.14 of [3], H is conditionally weakly compact. Without loss of generality, we can assume that (T_n) is weakly Cauchy. Let $x^* \in E^*$. Since ϕ_{x^*} on $\mathcal{M}$ is alwcc, then $(\phi_{x^*}T_n) = (T_nx^*)$ is weakly convergent. Define $T : E^* \to Y$ by $Tx^* = w\text{-}\lim T_nx^*$, for each $x^* \in E^*$. Note that T is w^*-w continuous. We show that

$T^*(B_{Y^*})$ is almost limited in E, or equivalently, for every disjoint and weak* null sequence $(x_i^*) \subset E^*$ and every sequence $(y_i^*) \subset B_{Y^*}$, $\langle x_i^*, T^*(y_i^*)\rangle \to 0$ as $i \to \infty$. Note that $(x_i^* \otimes y_i^*)$ is a disjoint and weak* null sequence in $\mathcal{M}^*$ (see [2, page, 75]) and so the operator $L : \mathcal{M} \subset K_{w^*}(E^*, Y) \to c_0$ defined by

$$L(S) = (\langle x_i^* \otimes y_i^*, S\rangle)_i = (\langle x_i^*, S^* y_i^*\rangle)_i,\ S \in M$$

is a bounded well-defined operator. Since (T_n) is an almost limited sequence in $\mathcal{M}$, then

$$\sup_n |L(T_n)(i)| = \sup_n |\langle x_i^*, T_n^* y_i^*\rangle| = \sup_n |\langle x_i^* \otimes y_i^*, T_n\rangle| \to 0,$$

as $i \to \infty$. Now by [2, page, 168], the sequence $L(T_n)$ is relatively compact in c_0 (one can also use [14, Proposition 2.3]). Note that $\langle x_i^*, T_n^* y_i^*\rangle \to \langle x_i^*, T^* y_i^*\rangle$ as $n \to \infty$. So $\langle x_i^*, T^* y_i^*\rangle \to 0$ as $i \to \infty$. Then $T^*(B_{Y^*})$ is an almost limited set in E, and so it is relatively compact, by the strong GP property of E. This proves that T^* and so T is compact. Hence $T_n \to T$ weakly by [16, Lemma 4.7]. Thus H is relatively weakly compact, and $\mathcal{M}$ has the strong BD property. □

Under the same assumptions on E and $\mathcal{M}$, a similar result by a similar proof can be inferred for a Banach lattice $\mathcal{M} \subset K(E, Y)$, (with E^* instead of E) and the isometry $K_{w^*}(E^{**}, Y) \cong K(E, Y)$:

Theorem 3.17. *Let E^* has the strong GP property and $\mathcal{M} \subset K(E, Y)$ be a Banach lattice and $\mathcal{M}^*$ has a weak unit. If for every $x \in E$, ϕ_x on $\mathcal{M}$ is alwcc, then $\mathcal{M}$ has the strong BD property.*

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