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Existence of solutions for first order impulsive periodic boundary value problems on time scales

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Abstract. In this paper we study a class of first order impulsive periodic boundary value problems on time scales. We give conditions under which the considered problem has at least one and at least two solutions. The arguments are based upon recent fixed point index theory in cones of Banach spaces for a k-set contraction perturbed by an expansive operator. An example is given to illustrate the obtained result.

1. Introduction

Dynamic equations on time scales have attracted great interest since S. Hilger introduced the concept of time scales [13] to create an idea that could combine continuous and discrete analysis. The theory of impulsive differential equations has received significant attention recently as the differential equations with impulses are more amenable to modeling [2, 3, 14]. Naturally, some authors have focused their attention on the study of the existence of solutions for boundary value problems of impulsive dynamic equations on time scales [4, 15]. In this work, we focus on the existence of at least one and at least two solutions for first-order impulsive periodic boundary value problems on time scales.

In [11], Wang and Guan studied the following nonlinear first-order periodic boundary value problem on time scales:

$$\begin{cases} x^{\Delta}(t) + p(t)x(\sigma(t)) = \lambda f(t, x(\sigma(t))), \quad t \in J\backslash\{t_1, \dots, t_m\}, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k \in \{1, \dots, m\}, \\ x(0) = x(\sigma(T)), \end{cases} \tag{1}$$

where $\lambda > 0$ is a positive parameter. By using the Leggett-Williams fixed point theorem, they provided sufficient conditions for existence of three positive solutions of (1).

In [12], Guan, Li and Ma studied the following nonlinear first-order periodic boundary value problem on time scales:

$$\begin{cases} x^{\Delta}(t) + p(t)x(\sigma(t)) = 0, \quad t \in J = [0, T]_{\mathbb{T}} \quad t \neq t_k, \quad k = 1, 2 \dots m, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k \in \{1, \dots, m\}, \\ x(0) = x(\sigma(T)). \end{cases} \tag{2$$

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By using the fixed point theorem in cones, they provided existence criteria for single and multiple positive solutions to the class of nonlinear first-order periodic boundary value problems of impulsive dynamic equations on time scales (2).

In [16], Li and Shu are concerned with the following first-order nonlinear impulsive integral boundary value problem on time scales:

$$\begin{cases} x^{\Delta}(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))), \quad t \in J\backslash\{t_1, \ldots, t_m\}, \\ x(t_i^+) - x(t_i^-) = I_k(x(t_k^-)), \quad i \in \{1, \ldots, m\}, \\ \alpha x(0) - \beta x(\sigma(t)) = \int_0^{\sigma(t)} g(s)x(s)\Delta s. \end{cases} \tag{3}$$

They obtained some sufficient conditions for existence of at least one, two, or three positive solutions for BVP (3) using Guo-Krasnoselskii and Legget-Williams fixed point theorem, respectively.

Motivated by the above studies, in this work we deal with the first-order multi-point boundary value problem for impulsive dynamic equations of the form:

$$\begin{aligned} x^{\Delta}(t) + p(t)x(\sigma(t)) &= f(t, x(\sigma(t))), \quad t \in J\backslash\{t_1, \ldots, t_m\}, \\ x(t_k^+) - x(t_k^-) &= I_k(x(t_k^-)), \quad k \in \{1, \ldots, m\}, \\ x(0) &= x(\sigma(T)), \end{aligned} \tag{4}$$

where

(H1) $T > 0$, $0 < t_1 < \ldots < t_m < t_{m+1} = \sigma(T)$ are right-dense, $p \in \mathcal{R}_+$, $J = [0, \sigma(T)]$,

(H2) $f \in C([0, \sigma(T)] \times \mathbb{R})$,

$$|f(t,z)| \le a_1(t) + a_2(t)|z|^l, \quad t \in [0, \sigma(T)], \quad z \in \mathbb{R},$$

$a_1, a_2 \in C(J)$, $0 \le a_1, a_2 \le B$ on J, $l \ge 0$, for some positive constant B,

(H3) $I_k \in C(\mathbb{R})$,

$$|I_k(z)| \le b_k|z|^{p_k} + c_k, \quad z \in \mathbb{R},$$

$b_k, c_k \ge 0$, $k \in \{1, \ldots, m\}$, are constants.

Let $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k \in \{1, \ldots, m\}$, and

$$\begin{aligned} PC(J) \;=\; \{x : J \to \mathbb{R} : x \in C(J_k), \quad \exists x(t_k^+), \quad x(t_k^-), \\ x(t_k^-) = x(t_k), \quad k \in \{1, \ldots, m\}\}. \end{aligned}$$

We will investigate the PBVP (4) for existence of solutions in $PC(J) \cap C^1(J\backslash\{t_1, \ldots, t_m\})$. Our main results are as follows.

Theorem 1.1. *Suppose* $(H1)$-$(H3)$. *Then the PBVP* (4) *has at least one solution in* $PC(J) \cap C^1(J\backslash\{t_1, \ldots, t_m\})$.

Theorem 1.2. *Suppose* $(H1)$-$(H3)$. *Then the PBVP* (4) *has at least two solutions in* $PC(J) \cap C^1(J\backslash\{t_1, \ldots, t_m\})$.

In [10], the PBVP (4) is investigated under the following conditions:

(G1) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist nonnegative constants α and K so that for any $\lambda \in (0, 1)$

$$\lambda|f(t,x)| \le \alpha(\lambda f(t,x) - p(t)x) + K, \quad t \in J, \quad x \in \mathbb{R},$$

(G2) $I_k : \mathbb{R} \to \mathbb{R}$ is continuous and there exist nonnegative constants β_k and N_k such that

$$|I_k(z)| \le \beta_k |z| + N_k, \quad k \in \{1, \ldots, m\}, \quad z \in \mathbb{R},$$

(G3) $m\beta^* e_p(\sigma(T), 0) < e_p(\sigma(T), 0) - 1$, where $\beta^* = \max_{k \in \{1,\ldots,m\}} \beta_k$,

and it is proved that the PBVP (4) has at least one solution in $PC(J) \cap C^1(J \backslash \{t_1, \ldots, t_m\})$. When $p_k = 1$, $k \in \{1, \ldots, m\}$, in ($H3$), we get that ($G2$) and ($H3$) coincide. Note that ($H2$) is valid for any $p_k \ge 0, k \in \{1, \ldots, m\}$. Therefore our results are valid for more classes I_k, $k \in \{1, \ldots, m\}$, than the results in [10]. Moreover, our condition ($H2$) for f is different than the condition ($G1$) in [10]. Thus, we can consider the results in this paper as complementary results to the results in [10]. Next, our conditions ($H1$)-($H3$) ensure nonuniqueness of the solutions of the PBVP (4). Also, the results in [10] are proved using the Schaefer fixed point theorem. In this paper we propose new arguments based upon recent fixed point index theory in cones of Banach spaces for a k-set contraction perturbed by an expansive operator.

The paper is organized as follows. In the next section, we make a short overview on time scale calculus. In Section 3, we give some auxiliary results. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2, and in Section 6, we give an example.

2. Elements of Time Scale Calculus

Before giving further details, we give some of the main definitions for time scales extracted from [1], [5], and [6] that we need in the sequel.

A time scale $\mathbb{T}$ is any closed subset of the real numbers. The forward and backward jump operators $\sigma, \rho \colon \mathbb{T} \to \mathbb{T}$ are defined, respectively as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

The point $t \in \mathbb{T}$ is left dense, left scattered, right dense, right scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.

Definition 2.1. *If $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^\kappa = \mathbb{T}$. In other words,*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \mathbb{T} \end{cases}.$$

Definition 2.2. *For a function $f \colon \mathbb{T} \to \mathbb{R}$ we define $f^\sigma \colon \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$.*

Definition 2.3. *We define $f^\Delta(t)$ to be the number with the property that given any $\epsilon > 0$ there is a neighborhood U of t such that*

$$\left| [f(\sigma(t)) - f(s)] - f^\Delta(t) [\sigma(t) - s] \right| \le \epsilon \, |\sigma(t) - s|$$

for all $s \in U$. Here, $f^\Delta(t)$ is called the delta derivative of f at t, and f is called delta differentiable in $\mathbb{T}^\kappa$ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 2.4. *A function $f \colon \mathbb{T} \to \mathbb{R}$ is called regulated provided that its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.*

We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f.

Definition 2.5. *A function $F\colon \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f\colon \mathbb{T} \to \mathbb{R}$ if*

$$F^{\Delta}(t) = f(t) \text{ for all } t \in \mathbb{T}^{\kappa}.$$

Definition 2.6. *$f\colon \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided that f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left-dense limit at each left-dense point of $\mathbb{T}$. The set of rd-continuous functions will be denoted by $C_{rd}(\mathbb{T})$ and the set of differentiable functions that posseses rd-continuous derivatives is denoted by $C^1_{rd}(\mathbb{T})$.*

Theorem 2.7. *Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g \in C_{rd}$. Then,*

(i) $\displaystyle\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$

(ii) $\displaystyle\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$

(iii) $\displaystyle\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$

(iv) *if* $\left|f(t)\right| \leq g(t)$ *on* $[a, b)$, *then* $\displaystyle\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t;$

(v) *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\displaystyle\int_a^b f(t)\Delta t \geq 0;$

(vi) *if* $t \in \mathbb{T}^{\kappa}$, *then* $\displaystyle\int_t^{\sigma(t)} f(\mathcal{T})\Delta(\mathcal{T}) = \mu(t)f(t);$

(vii) $\displaystyle\int_a^b [f(t)g(t)]\Delta t \leq \int_a^b \left|f(t)g(t)\right| \Delta t \leq \sup_{t\in[a,b)} \left|f(t)\right| \int_a^b \left|g(t)\right| \Delta t.$

Definition 2.8. *A function $p\colon \mathbb{T} \to \mathbb{R}$ is regressive provided that $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$ where $\mu(t) = \sigma(t) - t$ is the graininess function.*

The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}$.

Definition 2.9. *If $p \in \mathcal{R}$, then we define the exponential function by*

$$e_p(t, s) = exp\left(\int_s^t \xi_{\mu(\mathcal{T})}(p(\mathcal{T}))\Delta\mathcal{T}\right),$$

where

$$\xi_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0, \end{cases}$$

is the cylinder transformation for $s, t \in \mathbb{T}$.

Theorem 2.10. *[5] If $p, q \in \mathcal{R}$, then*

(i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1;$

(ii) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(t, s);$

(iii) $e_p(t, u)e_p(u, s) = e_p(t, s);$

(iv) $e^{\Delta}_p(t, t_0) = p(t)e_p(t, t_0)$ *for* $t \in \mathbb{T}^{\kappa}$ *and* $t_0 \in \mathbb{T}$.

3. Auxiliary Results

To prove our existence result we will use the following fixed point theorem.

Theorem 3.1. *Let $\epsilon > 0$, $B > 0$, E be a Banach space and $X = \{x \in E : \|x\| \leq B\}$. Let also, $Tx = -\epsilon x$, $x \in X$, $S : X \to E$ is continuous, $(I - S)(X)$ resides in a compact subset of E and*

$$\{x \in E : x = \lambda(I - S)x, \quad \|x\| = B\} = \emptyset \tag{5}$$

for any $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, there exists $x^* \in X$ so that

$$Tx^* + Sx^* = x^*.$$

Proof. Define

$$r\left(-\frac{1}{\epsilon}x\right) = \begin{cases} -\frac{1}{\epsilon}x & \text{if} \quad \|x\| \leq B\epsilon \\ \frac{Bx}{\|x\|} & \text{if} \quad \|x\| > B\epsilon. \end{cases}$$

Then $r\left(-\frac{1}{\epsilon}(I - S)\right) : X \to X$ is continuous and compact. Hence, by Schauder fixed point theorem, it follows that there exists $x^* \in X$ so that

$$r\left(-\frac{1}{\epsilon}(I - S)x^*\right) = x^*.$$

Assume that $-\frac{1}{\epsilon}(I - S)x^* \notin X$. Then

$$\left\|(I - S)x^*\right\| > B\epsilon, \quad \frac{B}{\|(I - S)x^*\|} < \frac{1}{\epsilon}$$

and

$$x^* = \frac{B}{\|(I - S)x^*\|}(I - S)x^* = r\left(-\frac{1}{\epsilon}(I - S)x^*\right)$$

and hence, $\|x^*\| = B$. This contradicts with (5). Therefore $-\frac{1}{\epsilon}(I - S)x^* \in X$ and

$$x^* = r\left(-\frac{1}{\epsilon}(I - S)x^*\right) = -\frac{1}{\epsilon}(I - S)x^*$$

or

$$-\epsilon x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

This completes the proof. □

Let X be a real Banach space.

Definition 3.2. *A mapping $K : X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for l-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 3.3. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0,\infty)$ is defined by*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad diam(Y_j) \le \delta, \quad j \in \{1,\dots,m\}\right\},$$

where $diam(Y_j) = \sup\{\|x-y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1,\dots,m\}$.

For the main properties of measure of noncompactness we refer the reader to [7].

Definition 3.4. *A mapping $K : X \to X$ is said to be l-set contraction if it is continuous, bounded and there exists a constant $l \ge 0$ such that*

$$\alpha(K(Y)) \le l\alpha(Y),$$

for any bounded set $Y \subset X$. The mapping K is said to be a strict set contraction if $l < 1$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction (see [9]).

Definition 3.5. *Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|Kx - Ky\|_Y \ge h\|x-y\|_X$$

for any $x, y \in X$.

Definition 3.6. *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \ge 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$.

Lemma 3.7. *Let X be a closed convex subset of a Banach space E and $U \subset X$ a bounded open subset with $0 \in U$. Assume there exists $\varepsilon > 0$ small enough and that $K : \overline{U} \to X$ is a strict k-set contraction that satisfies the boundary condition:*

$$Kx \notin \{x, \ \lambda x\} \ \text{for all} \ x \in \partial U \text{ and } \lambda \ge 1+\varepsilon.$$

Then the fixed point index $i(K, U, X) = 1$.

Proof. Consider the homotopic deformation $H : [0,1] \times \overline{U} \to X$ defined by

$$H(t,x) = \frac{1}{\varepsilon+1}\, tKx.$$

The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is a strict set contraction for each $t \in [0,1]$. In addition, $H(t,.)$ has no fixed point on ∂U. On the contrary,
• if $t = 0$, there exists some $x_0 \in \partial U$ such that $x_0 = 0$, contradicting $x_0 \in U$.
• if $t \in (0,1]$, there exists some $x_0 \in \mathcal{P} \cap \partial U$ such that $\frac{1}{\varepsilon+1}\, tKx_0 = x_0$; then $Kx_0 = \frac{1+\varepsilon}{t}x_0$ with $\frac{1+\varepsilon}{t} \ge 1+\varepsilon$, contradicting the assumption. From the invariance under homotopy and the normalization properties of the index, we deduce

$$i\left(\frac{1}{\varepsilon+1}K, U, X\right) = i(0, U, X) = 1.$$

Now, we show that

$$i\,(K,U,X) = i\,(\frac{1}{\varepsilon+1}\,K,U,X).$$

We have

$$\frac{1}{\varepsilon+1}Kx \neq x,\ \forall\, x \in \partial U. \tag{6}$$

Then, there exists $\gamma > 0$ such that

$$\|x - \frac{1}{\varepsilon+1}\,Kx\| \geq \gamma,\ \forall\, x \in \partial U.$$

On the other hand, we have $\frac{1}{\epsilon+1}Kx \to Kx$ as $\epsilon \to 0$, for $x \in \overline{U}$. So, for ε small enough

$$\|Kx - \frac{1}{\varepsilon+1}\,Kx\| < \frac{\gamma}{2},\ \forall\, x \in \partial U.$$

Define the convex deformation $G : [0,1] \times \overline{U} \to X$ by

$$G(t,x) = tKx + (1-t)\frac{1}{\varepsilon+1}Kx.$$

The operator G is continuous and uniformly continuous in t for each x, and the mapping $G(t,.)$ is a strict set contraction for each $t \in [0,1]$, since $t + \frac{1}{\varepsilon+1}(1-t) < t + 1 - t = 1$. In addition, $G(t,.)$ has no fixed point on ∂U. In fact, for all $x \in \partial U$, we have

$$\begin{aligned}\|x - G(t,x)\| &= \|x - tKx - (1-t)\frac{1}{\varepsilon+1}Kx\| \\ &\geq \|x - \frac{1}{\varepsilon+1}Kx\| - t\|Kx - \frac{1}{\varepsilon+1}Kx\| \\ &> \gamma - \frac{\gamma}{2} > \frac{\gamma}{2}.\end{aligned}$$

Then, our claim follows, from the invariance property by homotopy of the index.
□

Proposition 3.8. *Let $\mathcal{P}$ be a cone in a Banach space E. Let also, U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$, $S : \overline{U} \to E$ is a l-set contraction with $0 \leq l < h - 1$, and $S(\overline{U}) \subset (I-T)(\Omega)$. If there exists $\varepsilon \geq 0$ such that*

$$Sx \neq \{(I-T)(x),\quad (I-T)(\lambda x)\}\ \textit{for all}\ x \in \partial U \cap \Omega\ \textit{and}\ \lambda \geq 1+\varepsilon,$$

then the fixed point index $i_*\,(T+S, U \cap \Omega, \mathcal{P}) = 1$.

Proof. The mapping $(I-T)^{-1}S : \overline{U} \to \mathcal{P}$ is a strict set contraction and it is readily seen that the following condition is satisfied

$$(I-T)^{-1}Sx \notin \{x, \lambda x\} \quad \text{for} \quad \text{all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1+\epsilon.$$

Our claim then follows from the definition of i_* and Lemma 3.7. □

The following result will be used to prove our main result.

Theorem 3.9. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \leq k < h - 1$ and $S(\overline{U}_3) \subset (I-T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \neq (I-T)(x-\lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\epsilon \geq 0$ such that $Sx \neq (I-T)(\lambda x)$, for all $\lambda \geq 1+\epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) *$Sx \neq (I-T)(x-\lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T+S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If $Sx = (I-T)x$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_2 \cap \Omega$ of the operator $T+S$. Suppose that $Sx \neq (I-T)x$ for any $x \in \partial U_2 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$ and $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$, otherwise the conclusion has been proved. By [8, Proposition 2.11 and Proposition 2.16] and Proposition 3.8, we have

$$i_*(T+S, U_1 \cap \Omega, \mathcal{P}) = i_*(T+S, U_3 \cap \Omega, \mathcal{P}) = 0 \text{ and } i_*(T+S, U_2 \cap \Omega, \mathcal{P}) = 1.$$

The additivity property of the index yields

$$i_*(T+S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{P}) = 1 \text{ and } i_*(T+S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{P}) = 1.$$

Consequently, by the existence property of the index, $T+S$ has at least two fixed points $x_1 \in (U_2 \setminus U_1) \cap \Omega$ and $x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. □

Let

$$G(t,s) = \begin{cases} \frac{e_p(s,t)e_p(\sigma(T),0)}{e_p(\sigma(T),0)-1}, & 0 \leq s \leq t \leq \sigma(T), \\ \frac{e_p(s,t)}{e_p(\sigma(T),0)-1}, & 0 \leq t < s \leq \sigma(T). \end{cases}$$

We have

$$\sup_{t,s\in[0,\sigma(T)]} |G(t,s)| \leq \frac{e_p(\sigma(T),0)}{e_p(\sigma(T),0)-1} = G_1.$$

In [17], it is proved that the function G is the Green function for the PBVP (4) and $x \in PC(J) \cap C^1(J \backslash \{t_1, \ldots, t_m\})$ is a solution to the PBVP (4) if and only if it is a solution to the integral equation

$$u(t) = \int_0^{\sigma(T)} G(t,s) f(s, u(\sigma(s)))\Delta s + \sum_{k=1}^m G(t,t_k) I_k(u(t_k)), \quad t \in J.$$

In $X = PC(J)$ define the norm

$$\|u\| = \max_{k\in\{1,\ldots,m\}} \{ \sup_{t\in(t_k,t_{k+1}]} |u(t)| \}.$$

For $u \in X$, define the operator

$$S_1 u(t) = u(t) - \int_0^{\sigma(T)} G(t,s) f(s, u(\sigma(s)))\Delta s - \sum_{k=1}^m G(t,t_k) I_k(u(t_k)),$$

$t \in J$. Note that if $u \in X$ satisfies the equation

$$S_1 u(t) = 0, \quad t \in J, \tag{7}$$

then u is a solution to the PBVP (4). Set

$$B_1 = B + G_1 B(1 + B^l)\sigma(T) + G_1 \sum_{k=1}^{m} (b_k B^{p_k} + c_k).$$

Lemma 3.10. *Suppose* (H1)-(H3). *If* $u \in X$, $\|u\| \leq B$, *then*

$$\|S_1 u\| \leq B_1.$$

Proof. We have

$$\begin{aligned} |S_1 u(t)| &= \left| u(t) - \int_0^{\sigma(T)} G(t,s) f(s, u(\sigma(s)))\Delta s - \sum_{k=1}^{m} G(t,t_k) I_k(u(t_k)) \right| \\ &\leq |u(t)| + \int_0^{\sigma(T)} |G(t,s)||f(s,u(\sigma(s)))|\Delta s + \sum_{k=1}^{m} |G(t,t_k)||I_k(u(t_k))| \\ &\leq B + G_1 \int_0^{\sigma(T)} \left(a_1(s) + a_2(s)|u(\sigma(s))|^l\right)\Delta s + G_1 \sum_{k=1}^{m} (b_k |u(t_k)|^{p_k} + c_k) \\ &\leq B + G_1 B(1 + B^l)\sigma(T) + G_1 \sum_{k=1}^{m} (b_k B^{p_k} + c_k) \\ &= B_1, \quad t \in J. \end{aligned}$$

This completes the proof. □

4. Proof of Theorem 1.1

Below, suppose

(H4) $\epsilon \in (0,1)$, A and B satisfy the inequalities $\epsilon B_1(1 + A) < B$ and $AB_1 < 1$.

For $u \in X$, define the operator

$$S_2 u(t) = \frac{A}{\sigma(T)} \int_0^t S_1 u(s)\Delta s, \quad t \in J.$$

Lemma 4.1. *Suppose that* (H1)-(H3) *hold. If* $u \in X$ *satisfies the equation*

$$S_2 u(t) = C, \quad t \in J, \tag{8}$$

where C *is an arbitrary constant, then* u *is a solution to the PBVP* (4). *Moreover, if* $u \in X$ *and* $\|u\| \leq B$, *then* $\|S_2 u\| \leq AB_1$.

Proof. Let $u \in X$ be a solution to the equation (9). Then

$$\frac{A}{\sigma(T)} \int_0^t S_1 u(s)\Delta s = C, \quad t \in J.$$

We differentiate with respect to t the last equation and we find

$$\frac{A}{\sigma(T)} S_1 u(t) = 0, \quad t \in J,$$

whereupon $S_1 u(t) = 0$, $t \in J$. Hence, from (7), we get that u satisfies (4). Next, let $u \in X$ and $\|u\| \le B$. Then, by the definition of the operator S_2 and Lemma 3.10, we arrive at

$$\begin{aligned} \|S_2 u\| &\le \frac{A}{\sigma(T)} \int_0^{\sigma(T)} \|S_1 u\| \Delta s \\ &\le AB_1. \end{aligned}$$

This completes the proof. □

Let $\tilde{\tilde{\tilde{Y}}}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $\tilde{\tilde{Y}} = \overline{\tilde{\tilde{\tilde{Y}}}}$ be the closure of $\tilde{\tilde{\tilde{Y}}}$,

$$Y = \{u \in \tilde{\tilde{Y}} : \|u\| \le B\}.$$

Note that Y is a compact set in X. For $u \in X$, define the operators

$$\begin{aligned} Tu(t) &= -\epsilon u(t, x), \\ Su(t) &= u(t, x) + \epsilon u(t) + \epsilon S_2 u(t), \quad t \in J. \end{aligned}$$

For $u \in Y$, we have

$$\begin{aligned} \|(I - S)u\| &= \|\epsilon u - \epsilon S_2 u\| \\ &\le \epsilon \|u\| + \epsilon \|S_2 u\| \\ &\le \epsilon B_1 + \epsilon AB_1 \\ &= \epsilon B_1 (1 + A) \\ &< B \end{aligned}$$

Thus, $S : Y \to X$ is continuous and $(I - S)(Y)$ resides in a compact subset of X. Now, suppose that there is a $u \in X$ so that $\|u\| = B$ and

$$u = \lambda (I - S) u$$

or

$$\frac{1}{\lambda} u = (I - S) u = -\epsilon u - \epsilon S_2 u,$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right) u = -\epsilon S_2 u$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2 u\| \le AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right) B = \left(\frac{1}{\lambda} + \epsilon\right)\|u\| = \epsilon\|S_2 u\| < \epsilon B,$$

which is a contradiction. By Theorem 3.1, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t) &= Tu^*(t) + Su^*(t) \\ &= -\epsilon u^*(t) + u^*(t) + \epsilon u^*(t) + \epsilon S_2 u^*(t), \quad t \in J, \end{aligned}$$

whereupon

$$0 = S_2 u^*(t), \quad t \in J,$$

and

$$0 = S_1 u^*(t), \quad t \in J.$$

From here, it follows that u is a solution to the PBVP (4). This completes the proof.

5. Proof of Theorem 1.2

Let X be the space used in the previous section. Suppose the following.

(H5) Let $m_1 > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m_1} + 1\right)L,$$

$$AB_1 < \frac{L}{5}.$$

Let

$$\widetilde{P} = \{u \in X : u \ge 0 \quad \text{on} \quad J\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t) &= (1 + m_1\epsilon)v(t) - \epsilon\frac{L}{10}, \\ S_3 v(t) &= -\epsilon S_2 v(t) - m_1\epsilon v(t) - \epsilon\frac{L}{10}, \end{aligned}$$

$t \in J$. By Lemma 4.1, it follows that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the PBVP (4). Define

$$\begin{aligned} U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\ U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}, \\ R_2 &= R_1 + \frac{A}{m_1}B_1 + \frac{L}{5m_1}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \le R_2\}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have
$$\|T_1v_1 - T_1v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|,$$
whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.
2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get
$$\begin{aligned}\|S_3v\| &\leq \varepsilon\|S_2v\| + m_1\varepsilon\|v\| + \varepsilon\frac{L}{10}\\ &\leq \varepsilon\Big(AB_1 + m_1R_1 + \frac{L}{10}\Big).\end{aligned}$$
Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.
3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set
$$v_2 = v_1 + \frac{1}{m}S_2v_1 + \frac{L}{5m_1}.$$
Note that $S_2v_1 + \frac{L}{5} \geq 0$ on J because $\|S_2v_1\| \leq AB_1$ and $AB_1 < \frac{L}{5}$. We have $v_2 \geq 0$ on J and
$$\begin{aligned}\|v_2\| &\leq \|v_1\| + \frac{1}{m_1}\|S_2v_1\| + \frac{L}{5m_1}\\ &\leq R_1 + \frac{A}{m_1}B_1 + \frac{L}{5m_1}\\ &= R_2.\end{aligned}$$
Therefore $v_2 \in \Omega$ and
$$-\varepsilon m_1v_2 = -\varepsilon m_1v_1 - \varepsilon S_2v_1 - \varepsilon\frac{L}{10} - \varepsilon\frac{L}{10}$$
or
$$\begin{aligned}(I - T_1)v_2 &= -\varepsilon m_1v_2 + \varepsilon\frac{L}{10}\\ &= S_3v_1.\end{aligned}$$
Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.
4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $x \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $x \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that
$$S_3x = (I - T_1)(x - \lambda u_0).$$
Then
$$-\epsilon S_2x - m_1\epsilon x - \epsilon\frac{L}{10} = -m_1\epsilon(x - \lambda u_0) + \epsilon\frac{L}{10}$$
or
$$-S_2x = \lambda m_1u_0 + \frac{L}{5}.$$
Hence,
$$\|S_2x\| = \left\|\lambda m_1u_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$
This is a contradiction.

5. Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1+\epsilon_1$ such that $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ and

$$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{9}$$

In particular, for $\epsilon_1 > \dfrac{2}{5m_1}$, we have $x_1 \in \partial\mathcal{P}_L$, $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1+\epsilon_1$ and (9) holds. Since $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, it follows that

$$\left(\frac{2}{5m_1}+1\right)L < \lambda_1 L = \lambda_1\|x_1\| \leq R_1.$$

Moreover,

$$-\epsilon S_2 x_1 - m_1\epsilon x_1 - \epsilon\frac{L}{10} = -\lambda_1 m_1 \epsilon x_1 + \epsilon\frac{L}{10},$$

or

$$S_2 x_1 + \frac{L}{5} = (\lambda_1 - 1)m_1 x_1.$$

From here,

$$2\frac{L}{5} \geq \left\|S_2 x_1 + \frac{L}{5}\right\| = (\lambda_1 - 1)m_1\|x_1\| = (\lambda_1 - 1)m_1 L$$

and

$$\frac{2}{5m_1} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 3.9 hold. Hence, the PBVP (4) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

6. Example

Let $\mathbb{T} = \bigcup_{i=1}^{4}[2i, 2i+1]$, where each interval $[2i, 2i+1]$, $i \in \{1, \ldots 4\}$, is real-valued. Let also, $T = 7$, $m = 3$, $p(t) = a_2(t) = 1$, $a_1(t) = 0$, $t \in J$, $l = 1$, $p_k = 2$, $b_k = 1$, $c_k = 0$, $k \in \{1, 2, 3\}$, and

$$R_1 = 10, \quad B = \frac{1}{10^{30}}, \quad L = 5, \quad r = 4, \quad m_1 = 10^{50}, \quad A = \frac{1}{10B_1}, \quad \epsilon = \frac{1}{5B_1(1+A)},$$

and

$$t_0 = 0, \quad t_1 = 2, \quad t_2 = 4, \quad t_3 = 6,$$

Then

$$\epsilon B_1(1+A) < 1, \quad AB_1 < 1, \quad r < L < R_1, \quad R_1 > \left(\frac{2}{5m_1}+1\right)L.$$

Thus, $(H4)$ and $(H5)$ hold. Consider the following PBVP

$$\begin{cases} x^{\Delta}(t) + x(\sigma(t)) & = \dfrac{x(\sigma(t))}{1 + (x(\sigma(t)))^2}, \quad t \in J \backslash \{t_1, t_2, t_3\}, \\ x(t_k^+) - x(t_k) & = \dfrac{(x(t_k))^2}{2 + 4(x(t_k))^2}, \quad k \in \{1, \dots, 3\}, \\ x(0) & = x(8). \end{cases} \tag{10}$$

Here

$$f(t,x) = \frac{x}{1+x^2} \leq 1,$$

$$I_k(x) = \frac{x^2}{2+4x^2} \leq 1, \quad k \in \{1, \dots, 3\}.$$

Thus, for the PBVP (10) Theorem 1.1 and Theorem 1.2 hold.

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