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Isosymmetric composition operators on H^2

Rong Yang[a], Songxiao Li[b,*], Lian Hu[a]

[a]*Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, 610054, Chengdu, Sichuan, P. R. China.*
[b]*Department of Mathematics, Shantou University, 515063, Shantou, Guangdong, P.R. China*

Abstract. In this paper, we give some necessary and sufficient conditions for the composition operator C_φ to be isosymmetric on H^2 when φ is a linear fractional self-map of $\mathbb{D}$.

1. Introduction

Let H be a complex Hilbert space, $B(H)$ be the space of all bounded linear operators defined in H. An operator $T \in B(H)$ is called normal if $[T, T^*] = 0$, where $[T, T^*] = TT^* - T^*T$. An operator $T \in B(H)$ is called hyponormal if $T^*T \geq TT^*$. An operator $T \in B(H)$ is subnormal if there is a Hilbert space K containing H and a normal operator N on K such that $NH \subset H$ and $T = N|H$. An operator $T \in B(H)$ is quasinormal if T commutes with T^*T, that is, $[T, T^*T] = 0$. Quasimormal operators were first proposed, studied by Brown in [1]. $T \in B(H)$ is said to be binormal if $[T^*T, TT^*] = 0$ (see [8]). An operator T is said to belong to Θ class if $[T^*T, T + T^*] = 0$. From [2, 7], we obtain that

$$\text{quasinormal} \subset \text{binormal}.$$

$$\text{normal} \subset \text{quasinormal} \subset \text{subnormal} \subset \text{hyponormal}.$$

Stankus in [10] introduced and studied isosymmetric operators. According to [10] or [11], an operator $T \in B(H)$ is said to be isosymmetric if

$$T^{*2}T - T^*T^2 - T^* + T = 0.$$

See [10, 11] for more properties of isosymmetric operators.

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of those analytic functions on $\mathbb{D}$. A function $f \in H(\mathbb{D})$ belongs to the Hardy space H^2 if

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

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* Corresponding author: Songxiao Li
Email addresses: yangrong071428@163.com (Rong Yang), jyulsx@163.com (Songxiao Li), hl152808@163.com (Lian Hu)

In other words, the space H^2 consists of all analytic functions on $\mathbb{D}$ having power series representation with square summable complex coefficients. For any $a \in \mathbb{D}$, the function $K_a(z) = \frac{1}{1-\bar{a}z}$ is called the reproducing kernel for a in H^2 such that

$$f(a) = \langle f, K_a \rangle$$

for any $f \in H^2$. The space H^∞ denote the space of all bounded analytic functions f in $\mathbb{D}$.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ on H^2 is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H^2.$$

It is well known that $C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z)$ for any $\alpha \in \mathbb{D}$. In addition, for $\psi \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$, the weighted composition operator $W_{\psi,\varphi}$ is defined by

$$W_{\psi,\varphi} f = \psi \cdot f \circ \varphi.$$

The Toeplitz operator T_f on H^2 is defined by

$$T_f(g) = P(fg)$$

for $f \in L^\infty(\partial\mathbb{D})$ and $g \in H^2$, where P denotes the orthogonal projection of L^2 onto H^2. It is easy to check that

$$T_f^* K_\alpha = \overline{f(\alpha)} K_\alpha$$

for $\alpha \in \mathbb{D}$ and $f \in H^\infty$.

Schwarz [9] showed that C_φ is normal on H^2 if and only if $\varphi(z) = az$ with $|a| \leq 1$. In [3], Cowen studied the subnormality and hyponormality of composition operators on H^2 (in a more limited way). When φ is an automorphism of $\mathbb{D}$ or $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map into $\mathbb{D}$ with $c = 0$, Jung, Kim, and Ko [7] proved that $C_\varphi \in \Theta$ if and only if C_φ is normal. In addition, if C_φ is a p-hyponormal operator in Θ, then it must be normal. In [8], Jung, Kim, and Ko proved that C_φ is quasinormal if and only if C_φ is normal when $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map into $\mathbb{D}$. When $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map into $\mathbb{D}$ and $\varphi(0) = 0$, they proved that C_φ is binormal if and only if C_φ is normal or subnormal. When $\varphi(z) = \frac{z}{uz+v}$ with $u \neq 0$ and $|v| \geq 1 + |u|$, they also proved that C_φ is binormal if and only if C_φ is hyponormal or p-hyponormal for $0 < p < 1$. If $C_\varphi \in \Theta$, then C_φ is binormal if and only if it is normal. For more results on composition operators on H^2, see [3–9].

In this paper, we study isosymmetric composition operators with linear fractional symbols on H^2. We give some necessary and sufficient conditions for the composition operator C_φ to be isosymmetric when φ is automorphism or $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map into $\mathbb{D}$. In particular, we prove that C_φ is isosymmetric on H^2 if and only if C_φ is normal when $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map into $\mathbb{D}$ with $c = 0$.

2. Main results and proofs

2.1. Auxiliary results

To prove our main results in this paper, we need some lemmas.

Lemma 1. *[9] Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is normal if and only if $\varphi(z) = \alpha z$ with $|\alpha| \leq 1$.*

Let $\varphi(z) = \frac{az+b}{cz+d}$ be a nonconstant linear fractional self-map of $\mathbb{D}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Cowen in [3] proved that

$$C_\varphi^* = T_g C_\sigma T_h^*,$$

where

$$\sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}, \quad g(z) = \frac{1}{-\bar{b}z + \bar{d}}, \quad h(z) = cz + d.$$

From now on, unless otherwise stated, we assume that σ, g and h are given as above.

Using this adjoint formula, Jung, Kim, and Ko in [8] obtained the following basic lemmas.

Lemma 2. *[8] Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$. Then*

$$C_\varphi^* C_\varphi K_\alpha(z) = -c\frac{\overline{g(\alpha)}}{\sigma(\alpha)}K_{\varphi(0)}(z) + h\left(\frac{1}{\sigma(\alpha)}\right)\overline{g(\alpha)}K_{\varphi(\sigma(\alpha))}(z)$$

for any $\alpha \in \mathbb{D}$ with $a\overline{\alpha} \neq c$.

The following lemma can be directly obtained by [8, Lemma 2.3].

Lemma 3. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$. Then*

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) =& \frac{c(a+d)}{c(a+d)-(a^2+bc)\overline{\alpha}}K_{\varphi(0)}(z) \\ &+ \left(\frac{bc+d^2}{bc+d^2-b(a+d)\overline{\alpha}} - \frac{c(a+d)}{c(a+d)-(a^2+bc)\overline{\alpha}}\right)K_{\varphi(\sigma_2(\alpha))}(z) \end{aligned}$$

for any $\alpha \in \mathbb{D}$ with $(a^2+bc)\overline{\alpha} \neq c(a+d)$.

Lemma 4. *[7, Lemma 2.1] Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$. Then*

$$C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) = -c\frac{\overline{g(\alpha)}}{\sigma(\alpha)}K_{\varphi_2(0)}(z) + h(\frac{1}{\sigma(\alpha)})\overline{g(\alpha)}K_{\varphi_2(\sigma(\alpha))}(z)$$

for any $\alpha \in \mathbb{D}$ with $a\overline{\alpha} \neq c$.

Lemma 5. *[7, Lemma 2.3] Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map into $\mathbb{D}$ and $c = 0$. Then*

$$C_\varphi = C_{\tilde{\sigma}}^* T_{\tilde{g}}^*,$$

where $\tilde{\sigma}(z) = \frac{\overline{a}z}{-\overline{b}z+\overline{d}}$ and $\tilde{g}(z) = \frac{\overline{d}}{-\overline{b}z+\overline{d}}$.

2.2. Automorphism

Theorem 1. *Let φ be an automorphism of $\mathbb{D}$. Then C_φ is isosymmetric on H^2 if and only if $\varphi(z) = -\lambda z$, $|\lambda| = 1$.*

Proof. Assume that C_φ is isosymmetric and $\varphi(z) = \frac{\lambda(z-a)}{\overline{a}z-1}$, where $a \in \mathbb{D}$ and $|\lambda| = 1$. Let $\sigma(z) = \frac{\overline{\lambda}z-a}{\overline{\lambda a}z-1}$. We note that $(\sigma \circ \varphi)(z) = (\varphi \circ \sigma)(z) = z$ and

$$\varphi_2(z) = \frac{\lambda(\lambda-|a|^2)z - \lambda a(\lambda-1)}{(\lambda-1)\overline{a}z - (\lambda|a|^2-1)}.$$

From Lemma 4, if $\alpha \in \mathbb{D}$ with $\lambda\overline{\alpha} \neq \overline{a}$, then

$$\begin{aligned} &C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) \\ =& \frac{\overline{a}}{\overline{a}-\lambda\overline{\alpha}}K_{\varphi_2(0)}(z) + \left(-1 - \frac{\overline{a}(-1+\lambda a\overline{\alpha})}{\overline{a}-\lambda\overline{\alpha}}\right)\frac{1}{-1+\lambda a\overline{\alpha}}K_{\varphi_2(\sigma(\alpha))}(z) \\ =& \frac{\overline{a}}{\overline{a}-\lambda\overline{\alpha}}\frac{1}{1-\overline{\varphi_2(0)}z} + \frac{\lambda\overline{\alpha}(1-|a|^2)}{(\overline{a}-\lambda\overline{\alpha})(\lambda a\overline{\alpha}-1)}\frac{1}{1-\overline{\varphi_2(\sigma(\alpha))}z} \\ =& \frac{\overline{a}}{\overline{a}-\lambda\overline{\alpha}}\frac{1}{1-\frac{\overline{\lambda a}(\overline{\lambda}-1)}{\overline{\lambda}|a|^2-1}z} + \frac{\lambda\overline{\alpha}(1-|a|^2)}{(\overline{a}-\lambda\overline{\alpha})(\lambda a\overline{\alpha}-1)}\frac{1}{1-\frac{\overline{\lambda}(\overline{\alpha}-\overline{a})}{a\overline{\alpha}-1}z} \\ =& \frac{\overline{a}}{\overline{a}-\lambda\overline{\alpha}}\frac{\overline{\lambda}|a|^2-1}{\overline{\lambda}|a|^2-1-\overline{\lambda a}(\overline{\lambda}-1)z} + \frac{\lambda\overline{\alpha}(1-|a|^2)}{(\overline{a}-\lambda\overline{\alpha})(\lambda a\overline{\alpha}-1)}\frac{a\overline{\alpha}-1}{a\overline{\alpha}-1-\overline{\lambda}(\overline{\alpha}-\overline{a})z}. \end{aligned} \tag{1}$$

Also, we obtain from Lemma 3 that for any $\alpha \in \mathbb{D}$ with $\lambda(\lambda - |a|^2)\overline{\alpha} \neq \overline{a}(\lambda - 1)$

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) =& \frac{\overline{a}(\lambda-1)}{\overline{a}(\lambda-1)-\lambda(\lambda-|a|^2)\overline{\alpha}} K_{\varphi(0)}(z) \\ &+ \left(\frac{1-\lambda|a|^2}{1-\lambda|a|^2+\lambda a(\lambda-1)\overline{\alpha}} - \frac{\overline{a}(\lambda-1)}{\overline{a}(\lambda-1)-\lambda(\lambda-|a|^2)\overline{\alpha}} \right) K_{\varphi(\sigma_2(\alpha))}(z) \\ =& \frac{\overline{a}(\lambda-1)}{\overline{a}(\lambda-1)-\lambda(\lambda-|a|^2)\overline{\alpha}} \frac{1}{1-\overline{\lambda a}z} \\ &+ \left(\frac{1-\lambda|a|^2}{1-\lambda|a|^2+\lambda a(\lambda-1)\overline{\alpha}} - \frac{\overline{a}(\lambda-1)}{\overline{a}(\lambda-1)-\lambda(\lambda-|a|^2)\overline{\alpha}} \right) \frac{1}{1-\frac{\lambda\overline{\alpha}-\overline{a}}{\lambda a\overline{\alpha}-1}z}. \end{aligned} \tag{2}$$

In addition,

$$C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z) = \frac{1}{1-\overline{\varphi(\alpha)}(z)} = \frac{1}{1-\frac{\overline{\lambda}(\overline{\alpha}-\overline{a})}{a\overline{\alpha}-1}z} = \frac{a\overline{\alpha}-1}{a\overline{\alpha}-1-\overline{\lambda}(\overline{\alpha}-\overline{a})z} \tag{3}$$

and

$$C_\varphi K_\alpha(z) = K_\alpha(\varphi(z)) = \frac{1}{1-\overline{\alpha}\varphi(z)} = \frac{1}{1-\frac{\lambda(z-a)}{\overline{a}z-1}\overline{\alpha}} = \frac{\overline{a}z-1}{\overline{a}z-1-\lambda(z-a)\overline{\alpha}}. \tag{4}$$

Taking $z = 0$ in (1), (2), (3), and (4), we obtain that

$$C_\varphi^* C_\varphi^* C_\varphi K_\alpha(0) = \frac{\overline{a}}{\overline{a}-\lambda\overline{\alpha}} + \frac{\lambda\overline{\alpha}(1-|a|^2)}{(\overline{a}-\lambda\overline{\alpha})(\lambda a\overline{\alpha}-1)},$$

$$C_\varphi^* C_\varphi C_\varphi K_\alpha(0) = \frac{1-\lambda|a|^2}{1-\lambda|a|^2+\lambda a(\lambda-1)\overline{\alpha}}, \quad C_\varphi^* K_\alpha(0) = 1, \quad C_\varphi K_\alpha(0) = \frac{-1}{\lambda a\alpha - 1}.$$

Since C_φ is isosymmetric, we have

$$C_\varphi^* C_\varphi^* C_\varphi K_\alpha(0) - C_\varphi^* C_\varphi C_\varphi K_\alpha(0) - C_\varphi^* K_\alpha(0) + C_\varphi K_\alpha(0) = 0$$

for any $\alpha \in \mathbb{D}$, i.e.,

$$\frac{\overline{a}}{\overline{a}-\lambda\overline{\alpha}} + \frac{\lambda\overline{\alpha}(1-|a|^2)}{(\overline{a}-\lambda\overline{\alpha})(\lambda a\overline{\alpha}-1)} - \frac{1-\lambda|a|^2}{1-\lambda|a|^2+\lambda a(\lambda-1)\overline{\alpha}} - 1 - \frac{1}{\lambda a\alpha-1} = 0$$

for any $\alpha \in \mathbb{D}$, which implies that

$$\frac{1+\lambda a\overline{\alpha}}{1-\lambda a\overline{\alpha}} = \frac{1-\lambda|a|^2}{1-\lambda|a|^2-\lambda a(\lambda-1)\overline{\alpha}}$$

for any $\alpha \in \mathbb{D}$. After a calculation, we get

$$\lambda^2 a^2(\lambda-1)\overline{\alpha}^2 + \lambda a(1+\lambda-2\lambda|a|^2)\overline{\alpha} = 0$$

for any $\alpha \in \mathbb{D}$. By the assumption that $|\lambda| = 1$, we get $a = 0$. Hence, $\varphi(z) = -\lambda z, |\lambda| = 1$.

Conversely, let $\varphi(z) = -\lambda z, |\lambda| = 1$. Set $\sigma(z) = -\overline{\lambda}z$. From Lemmas 3 and 4, we have

$$C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) = K_{\varphi_2(\sigma(\alpha))}(z) = K_{\varphi(\alpha)}(z),$$

$$C_\varphi^* C_\varphi C_\varphi K_\alpha(z) = K_{\varphi(\sigma_2(\alpha))}(z) = K_{\sigma(\alpha)}(z) = \frac{1}{1-\overline{\sigma(\alpha)}z} = \frac{1}{1+\lambda\overline{\alpha}z},$$

$$C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z),$$

$$C_\varphi K_\alpha(z) = K_\alpha(\varphi(z)) = \frac{1}{1-\overline{\alpha}\varphi(z)} = \frac{1}{1+\overline{\alpha}\lambda z}.$$

From the above equalities, we get that

$$C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) - C_\varphi^* C_\varphi C_\varphi K_\alpha(z) - C_\varphi^* K_\alpha(z) + C_\varphi K_\alpha(z) = 0$$

for any $\alpha \in \mathbb{D}$ and $z \in \mathbb{D}$. It is well known that the linear span of uncountably many reproducing kernels is dense in H^2. Hence, C_φ is isosymmetric on H^2. □

2.3. Linear fractional self-maps with $c = 0$

Theorem 2. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map into $\mathbb{D}$ with $c = 0$. Then C_φ is isosymmetric on H^2 if and only if C_φ is normal.*

Proof. Sufficiency. Since C_φ is normal, by Lemma 1 we see that $\varphi(z) = \lambda z$, $|\lambda| \le 1$. Similarly to the second part proof of Theorem 1, we see that

$$C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) - C_\varphi^* C_\varphi C_\varphi K_\alpha(z) - C_\varphi^* K_\alpha(z) + C_\varphi K_\alpha(z) = 0,$$

for any $\alpha \in \mathbb{D}$ and $z \in \mathbb{D}$. Hence C_φ is isosymmetric.

Necessity. Since $c = 0$, set $\varphi(z) = sz + t$, where $s = \frac{a}{d}$, $t = \frac{b}{d}$ and $|s| + |t| \le 1$. Put $\sigma(z) = \frac{\bar{s}z}{1-\bar{t}z}$, $g(z) = \frac{1}{1-\bar{t}z}$. According to the proof of [7, Theorem 2.4], we obtain

$$\begin{aligned} C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) &= C_\varphi^* C_\varphi^* C_\sigma^* T_g^* K_\alpha(z) = \overline{g(\alpha)} K_{\varphi_2(\sigma(\alpha))}(z) \\ &= \frac{1}{(1-t\bar{\alpha}) - [\bar{t}(\bar{s}+1) + (|s|^2\bar{s} - |t|^2\bar{s} - |t|^2)\,\bar{\alpha}]\,z} \end{aligned} \tag{5}$$

and

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) &= C_\varphi^* C_\sigma^* T_g^* C_\sigma^* T_g^* K_\alpha(z) = \overline{g(\alpha) g(\sigma(\alpha))} K_{\varphi(\sigma_2(\alpha))}(z) \\ &= \frac{1}{1 - t(s+1)\overline{\alpha} - \left[\bar{t} + (|s|^2 s - |t|^2 s - |t|^2)\,\overline{\alpha}\right] z} \end{aligned} \tag{6}$$

for any $\alpha, z \in \mathbb{D}$. In addition,

$$C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z) = \frac{1}{1-\overline{\varphi(\alpha)}z} = \frac{1}{1-(\overline{s\alpha}+\bar{t})z}, \tag{7}$$

and

$$C_\varphi K_\alpha(z) = K_\alpha(\varphi(z)) = \frac{1}{1-\overline{\alpha}\varphi(z)} = \frac{1}{1-\overline{\alpha}(sz+t)} \tag{8}$$

for any $\alpha, z \in \mathbb{D}$. Taking $\alpha = 0$ in (5), (6), (7), and (8), we get that

$$C_\varphi^* C_\varphi^* C_\varphi K_0(z) = \frac{1}{1-\bar{t}(\bar{s}+1)z}, \quad C_\varphi^* K_0(z) = \frac{1}{1-\bar{t}z},$$

$$C_\varphi^* C_\varphi C_\varphi K_0(z) = \frac{1}{1-\bar{t}z}, \quad C_\varphi K_0(z) = 1.$$

Since C_φ is isosymmetric, we get that

$$C_\varphi^* C_\varphi^* C_\varphi K_0(z) - C_\varphi^* C_\varphi C_\varphi K_0(z) - C_\varphi^* K_0(z) + C_\varphi K_0(z) = 0$$

for any $z \in \mathbb{D}$, which implies that

$$\frac{1}{1-\bar{t}(\bar{s}+1)z} - \frac{2}{1-\bar{t}z} + 1 = 0.$$

So

$$\bar{t}^2(\bar{s}+1)z^2 + \bar{t}(\bar{s}-1)z = 0$$

for any $z \in \mathbb{D}$. Hence $t = 0$. So $\varphi(z) = sz, |s| \le 1$. Therefore, C_φ is normal by Lemma 1. □

From the last Theorem and the results in [7, 8], we get the following corollary.

Corollary 1. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map into $\mathbb{D}$ with $c = 0$. Then the following statements are equivalent.*

(i) C_φ is normal;
(ii) C_φ is subnormal;
(iii) C_φ is binormal;
(iv) $C_\varphi \in \Theta$;
(v) C_φ is quasinormal;
(vi) C_φ is isosymmetric.

Theorem 3. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map into $\mathbb{D}$ with $c = 0$. If C_φ^* is isosymmetric on H^2, then either $b = 0$ or $|\frac{b}{d}| \neq 1 - \frac{a}{d}$.*

Proof. Assume that C_φ^* is isosymmetric and $b \neq 0$. We suppose that $|\frac{b}{d}| = 1 - \frac{a}{d}$. Set $\varphi(z) = sz + t$, where $s = \frac{a}{d}$ and $t = \frac{b}{d}$. Then $t \neq 0$ and $|t| = 1 - s$. For a real number η. Define $(U_\eta f)(z) = f(e^{i\eta}z)$ for any $z \in \mathbb{D}$ and $f \in H^2(\mathbb{D})$. Then U_η is unitary and

$$(U_\eta^* C_\varphi U_\eta f)(z) = U_\eta^* C_\varphi f(e^{i\eta}z) = U_\eta^* f(e^{i\eta}\varphi(z)) = f(e^{i\eta}\varphi(e^{-i\eta}z)) = f(\chi(z)) = (C_\chi f)(z)$$

for any $z \in \mathbb{D}$ and $f \in H^2(\mathbb{D})$. Hence, we know that C_φ is unitary equivalent to C_χ, where $\chi(z) = sz + (1-s)$. Since C_φ^* is isosymmetric, so does C_χ^*, which means that

$$C_\chi C_\chi C_\chi^* - C_\chi C_\chi^* C_\chi^* - C_\chi + C_\chi^* = 0.$$

From Lemma 5, we get that

$$C_\chi = C_\sigma^* T_g^*,$$

where $\sigma(z) = \frac{sz}{1-(1-s)z}$ and $g(z) = \frac{1}{1-(1-s)z}$. Hence,

$$\begin{aligned} C_\chi C_\chi C_\chi^* K_\alpha(z) &= C_\sigma^* T_g^* C_\sigma^* T_g^* K_{\chi(\alpha)}(z) = \overline{g(\chi(\alpha))g(\sigma(\chi(\alpha)))} K_{\sigma_2(\chi(\alpha))}(z) \\ &= \frac{1}{1-(1-s)\overline{\chi(\alpha)}} \frac{1}{1-(1-s)\overline{\sigma(\chi(\alpha))}} \frac{1}{1-\frac{s\overline{\sigma(\chi(\alpha))}}{1-(1-s)\overline{\sigma(\chi(\alpha))}}z} \\ &= \frac{1}{1-(1-s)\overline{\chi(\alpha)}} \frac{1}{1-(1-s)\overline{\sigma(\chi(\alpha))} - s\overline{\sigma(\chi(\alpha))}z} \\ &= \frac{1}{1-(1-s)\overline{\chi(\alpha)}} \frac{1}{1-\frac{s(1-s)\overline{\chi(\alpha)}}{1-(1-s)\overline{\chi(\alpha)}} - \frac{s^2\overline{\chi(\alpha)}}{1-(1-s)\overline{\chi(\alpha)}}z} \\ &= \frac{1}{1-(1-s)\overline{\chi(\alpha)} - s(1-s)\overline{\chi(\alpha)} - s^2\overline{\chi(\alpha)}z} \\ &= \frac{1}{s+s^2-s|s|^2+s(s^2-1)\overline{\alpha} - s^2(s\overline{\alpha}+1-s)z}, \end{aligned} \tag{9}$$

$$
\begin{aligned}
C_\chi C_\chi^* C_\chi^* K_\alpha(z) =& C_\sigma^* T_g^* K_{\chi_2(\alpha)}(z) = \overline{g(\chi_2(\alpha))} K_{\sigma(\chi_2(\alpha))}(z) \\
=& \frac{1}{1-(1-s)\overline{\chi_2(\alpha)}} \frac{1}{1-\frac{s\overline{\chi_2(\alpha)}}{1-(1-s)\overline{\chi_2(\alpha)}} z} \\
=& \frac{1}{1-(1-s)\overline{\chi_2(\alpha)} - s\overline{\chi_2(\alpha)} z} \\
=& \frac{1}{s+s^2-s|s|^2+s^2(s-1)\overline{\alpha} - s(s^2\overline{\alpha}+1-|s|^2)z},
\end{aligned} \tag{10}
$$

$$
C_\chi K_\alpha(z) = \frac{1}{1-\overline{\alpha}\chi(z)} = \frac{1}{1-\overline{\alpha}(sz+1-s)}, \tag{11}
$$

and

$$
C_\chi^* K_\alpha(z) = \frac{1}{1-\overline{\chi(\alpha)}z} = \frac{1}{1-(s\overline{\alpha}+1-s)z} \tag{12}
$$

for any $\alpha, z \in \mathbb{D}$. Since C_χ^* is isosymmetric, we have

$$
\begin{aligned}
&\frac{1}{s+s^2-s|s|^2+s(s^2-1)\overline{\alpha} - s^2(s\overline{\alpha}+1-s)z} + \frac{1}{1-(s\overline{\alpha}+1-s)z} \\
=&\frac{1}{s+s^2-s|s|^2+s^2(s-1)\overline{\alpha} - s(s^2\overline{\alpha}+1-|s|^2)z} + \frac{1}{1-\overline{\alpha}(sz+1-s)}
\end{aligned} \tag{13}
$$

for any $\alpha, z \in \mathbb{D}$. Take $z = 0$ in (13), we get that

$$
\frac{1}{s+s^2-s|s|^2+s(s^2-1)\overline{\alpha}} + 1 = \frac{1}{s+s^2-s|s|^2+s^2(s-1)\overline{\alpha}} + \frac{1}{1+\overline{\alpha}(s-1)}
$$

for any $\alpha \in \mathbb{D}$. Let $A = s+s^2-s|s|^2$. We get

$$
\frac{1}{A+s(s^2-1)\overline{\alpha}} + 1 = \frac{1}{A+s^2(s-1)\overline{\alpha}} + \frac{1}{1+(s-1)\overline{\alpha}}
$$

for any $\alpha \in \mathbb{D}$, which implies that

$$
\frac{A+s(s^2-1)\overline{\alpha}+1}{A+s(s^2-1)\overline{\alpha}} = \frac{1+(s-1)\overline{\alpha}+A+s^2(s-1)\overline{\alpha}}{[A+s^2(s-1)\overline{\alpha}][1+(s-1)\overline{\alpha}]}. \tag{14}
$$

Therefore,

$$
\begin{aligned}
&[A+s(s^2-1)\overline{\alpha}+1][A+s^2(s-1)\overline{\alpha}][1+(s-1)\overline{\alpha}] \\
=&[1+(s-1)\overline{\alpha}+A+s^2(s-1)\overline{\alpha}][A+s(s^2-1)\overline{\alpha}]
\end{aligned}
$$

for any $\alpha \in \mathbb{D}$. Comparing the coefficients of $\overline{\alpha}^3$ and constant term in above equation, we obtain that

$$
s^3(s+1)(s-1)^3 = 0.
$$

Hence $s = \pm 1 (s \neq 0)$. If $s = -1$, the equation (14) is true for any $\alpha \in \mathbb{D}$. Then

$$
1 = \frac{1}{1-2\overline{\alpha}}
$$

for any $\alpha \in \mathbb{D}$. This is a contradiction. In addition, since $s \neq 0$, which yields $s = 1$ and hence $t = 0$, a contradiction. Hence $|\frac{b}{d}| \neq 1 - \frac{a}{d}$. □

2.4. Linear fractional self-maps with $\varphi(0) \neq 0$ and $a = 0$

Lemma 6. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a constant function. Then C_φ is isosymmetric on H^2 if and only if φ is identically zero on $\mathbb{D}$.*

Proof. Sufficiency. It is obvious.

Necessity. Let $\varphi(z) \equiv b$ for some $b \in \mathbb{D}$. From Lemma 2, we obtain that

$$C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) = C_\varphi^* \frac{1}{1 - b\overline{\alpha}} K_{\varphi(0)}(z) = \frac{1}{1 - b\overline{\alpha}} K_{\varphi_2(0)}(z) = \frac{1}{1 - b\overline{\alpha}} \frac{1}{1 - \overline{b}z},$$

$$C_\varphi^* C_\varphi C_\varphi K_\alpha(z) = C_\varphi^* K_\alpha(\varphi_2(z)) = \frac{1}{1 - b\overline{\alpha}} K_{\varphi(0)}(z) = \frac{1}{1 - b\overline{\alpha}} \frac{1}{1 - \overline{b}z},$$

$$C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z) = \frac{1}{1 - \overline{\varphi(\alpha)}z} = \frac{1}{1 - \overline{b}z},$$

and

$$C_\varphi K_\alpha(z) = K_\alpha(\varphi(z)) = \frac{1}{1 - \overline{\alpha}\varphi(z)} = \frac{1}{1 - \overline{\alpha}b}$$

for any $\alpha, z \in \mathbb{D}$. Since C_φ is isosymmetric, we have

$$C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) - C_\varphi^* C_\varphi C_\varphi K_\alpha(z) - C_\varphi^* K_\alpha(z) + C_\varphi K_\alpha(z) = 0$$

for any $\alpha, z \in \mathbb{D}$. Hence

$$\frac{1}{1 - b\overline{\alpha}} \frac{1}{1 - \overline{b}z} - \frac{1}{1 - b\overline{\alpha}} \frac{1}{1 - \overline{b}z} - \frac{1}{1 - \overline{b}z} + \frac{1}{1 - \overline{\alpha}b} = 0.$$

The above equation holds for any $\alpha, z \in \mathbb{D}$ if and only if $b = 0$. □

Theorem 4. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$ with $\varphi(0) \neq 0$ and $a = 0$. Then C_φ is not isosymmetric on H^2.*

Proof. Since $\varphi(0) \neq 0$ and $a = 0$, we can set $\varphi(z) = \frac{1}{uz+v}$, where $u = \frac{c}{b}$ and $v = \frac{d}{b}$. If $u = 0$, then $\varphi(z) = \frac{1}{v} \neq 0$ and so C_φ is not isosymmetric from Lemma 6.

Now we assume that $u \neq 0$ and C_φ is isosymmetric. From Lemma 4, we obtain that

$$\begin{aligned} C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) &= K_{\varphi_2(0)}(z) + \frac{\overline{\alpha}}{v - \overline{\alpha}} K_{\varphi_2(\sigma(\alpha))}(z) \\ &= \frac{1}{1 - \overline{\varphi_2(0)}z} + \frac{\overline{\alpha}}{v - \overline{\alpha}} \frac{1}{1 - \overline{\varphi_2(\sigma(\alpha))}z} \\ &= \frac{1}{1 - \frac{\overline{v}z}{\overline{u}+\overline{v}^2}} + \frac{\overline{\alpha}}{v - \overline{\alpha}} \frac{1}{1 - \frac{(|u|^2-|v|^2+\overline{\alpha v})z}{(\overline{u}+\overline{v}^2)(\overline{\alpha}-v)+|u|^2\overline{v}}} \\ &= \frac{\overline{u} + \overline{v}^2}{\overline{u} + \overline{v}^2 - \overline{v}z} + \frac{\overline{\alpha}}{v - \overline{\alpha}} \frac{(\overline{u} + \overline{v}^2)(\overline{\alpha} - v) + |u|^2\overline{v}}{(\overline{u} + \overline{v}^2)(\overline{\alpha} - v) + |u|^2\overline{v} - (|u|^2 - |v|^2 + \overline{\alpha v})z} \end{aligned} \tag{15}$$

for any $\alpha \in \mathbb{D}$. Since $u \neq 0$ and $|v| = |\varphi(0)|^{-1} > 1$, by Lemma 3,

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) &= \frac{v}{v-\overline{\alpha}} K_{\varphi(0)}(z) + \left(\frac{u+v^2}{u+v^2-v\overline{\alpha}} - \frac{v}{v-\overline{\alpha}} \right) K_{\varphi(\sigma_2(\alpha))}(z) \\ &= \frac{v}{v-\overline{\alpha}} \frac{1}{1-\overline{\varphi(0)}z} + \left(\frac{u+v^2}{u+v^2-v\overline{\alpha}} - \frac{v}{v-\overline{\alpha}} \right) \frac{1}{1-\overline{\varphi(\sigma_2(\alpha))}z} \\ &= \frac{v}{v-\overline{\alpha}} \frac{\overline{v}}{\overline{v}-z} + \left(\frac{u+v^2}{u+v^2-v\overline{\alpha}} - \frac{v}{v-\overline{\alpha}} \right) \frac{1}{1-\frac{u-v(\overline{\alpha}-v)}{(|u|^2-|v|^2)(\overline{\alpha}-v)+u\overline{v}}z} \\ &= \frac{v}{v-\overline{\alpha}} \frac{\overline{v}}{\overline{v}-z} + \left(\frac{u+v^2}{u+v^2-v\overline{\alpha}} - \frac{v}{v-\overline{\alpha}} \right) \frac{(|u|^2-|v|^2)(\overline{\alpha}-v)+u\overline{v}}{(|u|^2-|v|^2)(\overline{\alpha}-v)+u\overline{v}-[u-v(\overline{\alpha}-\overline{v})]z} \end{aligned} \tag{16}$$

for any $\alpha \in \mathbb{D}$ with $u\overline{\alpha} \neq uv$. In addition,

$$C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z) = \frac{1}{1-\overline{\varphi(\alpha)}(z)} = \frac{1}{1-\frac{z}{\overline{u\alpha}+\overline{v}}} = \frac{\overline{u\alpha}+\overline{v}}{\overline{u\alpha}+\overline{v}-z}, \tag{17}$$

$$C_\varphi K_\alpha(z) = K_\alpha(\varphi(z)) = \frac{1}{1-\overline{\alpha}\varphi(z)} = \frac{1}{1-\frac{\overline{\alpha}}{uz+v}} = \frac{uz+v}{uz+v-\overline{\alpha}}. \tag{18}$$

Taking $\alpha = 0$ in (15), (16), (17) and (18), we obtain

$$C_\varphi^* C_\varphi^* C_\varphi K_0(z) = \frac{\overline{u}+\overline{v}^2}{\overline{u}+\overline{v}^2-\overline{v}z}, \qquad C_\varphi^* C_\varphi C_\varphi K_0(z) = \frac{\overline{v}}{\overline{v}-z},$$

$$C_\varphi^* K_0(z) = \frac{\overline{v}}{\overline{v}-z}, \qquad C_\varphi K_0(z) = 1.$$

Since

$$C_\varphi^* C_\varphi^* C_\varphi K_0(z) - C_\varphi^* C_\varphi C_\varphi K_0(z) - C_\varphi^* K_0(z) + C_\varphi K_0(z) = 0$$

for any $z \in \mathbb{D}$, we have

$$\frac{\overline{u}+\overline{v}^2}{\overline{u}+\overline{v}^2-\overline{v}z} - \frac{2\overline{v}}{\overline{v}-z} + 1 = 0$$

for any $z \in \mathbb{D}$. This implies that $v = 0$. Moreover, $|v| > 1$ since $|\varphi(0)| = |\frac{1}{v}| < 1$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$, which is a contradiction. Therefore, C_φ is not isosymmetric. □

2.5. Linear fractional self-maps with $\varphi(0) \neq 0$, $a \neq 0$ and $c \neq 0$

Theorem 5. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-maps with $\varphi(0) \neq 0$, $a \neq 0$ and $c \neq 0$. Then C_φ is not isosymmetric on H^2.*

Proof. We prove it by contradiction. Assume that C_φ is isosymmetric. From Lemma 4, if $\alpha \in \mathbb{D}$ with $a\overline{\alpha} \neq c$, then

$$
\begin{aligned}
C_\varphi^* C_\varphi^* C_\varphi K_\alpha(z) =& \frac{c}{c-a\overline{\alpha}} K_{\varphi_2(0)}(z) + \left(d - \frac{c(d-b\overline{\alpha})}{c-a\overline{\alpha}}\right)\frac{1}{d-b\overline{\alpha}} K_{\varphi_2(\sigma(\alpha))}(z) \\
=& \frac{c}{c-a\overline{\alpha}} \frac{1}{1-\overline{\varphi_2(0)}z} + \left(d - \frac{c(d-b\overline{\alpha})}{c-a\overline{\alpha}}\right)\frac{1}{d-b\overline{\alpha}} \frac{1}{1-\overline{\varphi_2(\sigma(\alpha))}z} \\
=& \frac{c}{c-a\overline{\alpha}} \frac{1}{1-\frac{\overline{ab}+\overline{bd}}{\overline{bc}+\overline{d}^2}z} + \left(d - \frac{c(d-b\overline{\alpha})}{c-a\overline{\alpha}}\right)\frac{1}{d-b\overline{\alpha}} \\
& \cdot \frac{1}{1-\frac{(\overline{a}^2+\overline{bc})(a\overline{\alpha}-c)+(\overline{ab}+\overline{bd})(-b\overline{\alpha}+d)}{(\overline{ac}+\overline{cd})(a\overline{\alpha}-c)+(\overline{bc}+\overline{d}^2)(-b\overline{\alpha}+d)}z} \\
=& \frac{c}{c-a\overline{\alpha}} \frac{\overline{bc}+\overline{d}^2}{\overline{bc}+\overline{d}^2-(\overline{ab}+\overline{bd})z} + \left(d - \frac{c(d-b\overline{\alpha})}{c-a\overline{\alpha}}\right)\frac{1}{d-b\overline{\alpha}} \\
& \cdot \frac{1}{1-\frac{(\overline{a}^2+\overline{bc})(a\overline{\alpha}-c)+(\overline{ab}+\overline{bd})(-b\overline{\alpha}+d)}{(\overline{ac}+\overline{cd})(a\overline{\alpha}-c)+(\overline{bc}+\overline{d}^2)(-b\overline{\alpha}+d)}z}.
\end{aligned} \tag{19}
$$

From Lemma 3, for any $\alpha \in \mathbb{D}$ with $(a^2+bc)\overline{\alpha} \neq c(a+d)$, we get

$$
\begin{aligned}
C_\varphi^* C_\varphi C_\varphi K_\alpha(z) =& \frac{c(a+d)}{c(a+d)-(a^2+bc)\overline{\alpha}} \frac{1}{1-\overline{\varphi(0)z}} \\
&+ \left(\frac{bc+d^2}{bc+d^2-b(a+d)\overline{\alpha}} - \frac{c(a+d)}{c(a+d)-(a^2+bc)\overline{\alpha}}\right) \frac{1}{1-\overline{\varphi(\sigma_2(\alpha))z}} \\
=& \frac{c(a+d)}{c(a+d)-(a^2+bc)\overline{\alpha}} \frac{\overline{d}}{\overline{d}-\overline{b}z} \\
&+ \left(\frac{bc+d^2}{bc+d^2-b(a+d)\overline{\alpha}} - \frac{c(a+d)}{c(a+d)-(a^2+bc)\overline{\alpha}}\right) \frac{1}{1-\frac{(|a|^2-|b|^2)(a\overline{\alpha}-c)+(\overline{b}d-\overline{a}c)(-b\overline{\alpha}+d)}{(a\overline{c}-b\overline{d})(a\overline{\alpha}-c)+(|d|^2-|c|^2)(-b\overline{\alpha}+d)}z}.
\end{aligned} \tag{20}
$$

In addition,

$$
C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z) = \frac{1}{1-\frac{\overline{a\alpha}+\overline{b}}{\overline{c\alpha}+\overline{d}}z} = \frac{\overline{c\alpha}+\overline{d}}{\overline{c\alpha}+\overline{d}-(\overline{a\alpha}+\overline{b})z}. \tag{21}
$$

$$
C_\varphi K_\alpha(z) = K_\alpha(\varphi(z)) = \frac{1}{1-\overline{\alpha}\varphi(z)} = \frac{1}{1-\overline{\alpha}\frac{az+b}{cz+d}} = \frac{cz+d}{cz+d-\overline{\alpha}(az+b)}. \tag{22}
$$

Taking $\alpha = 0$ in (19), (20), (21), and (22), we obtain that

$$
C_\varphi^* C_\varphi^* C_\varphi K_0(z) = \frac{\overline{bc}+\overline{d}^2}{\overline{bc}+\overline{d}^2-(\overline{ab}+\overline{bd})z},
$$

$$
C_\varphi^* C_\varphi C_\varphi K_0(z) = \frac{\overline{d}}{\overline{d}-\overline{b}z}, \quad C_\varphi^* K_0(z) = \frac{\overline{d}}{\overline{d}-\overline{b}z}, \quad C_\varphi K_0(z) = 1.
$$

By the assumption, we get

$$
C_\varphi^* C_\varphi^* C_\varphi K_0(z) - C_\varphi^* C_\varphi C_\varphi K_0(z) - C_\varphi^* K_0(z) + C_\varphi K_0(z) = 0
$$

for any $z \in \mathbb{D}$, i.e.,

$$\frac{\overline{bc} + \overline{d}^2}{\overline{bc} + \overline{d}^2 - (\overline{ab} + \overline{bd})z} - \frac{2\overline{d}}{\overline{d} - \overline{b}z} + 1 = 0$$

for any $z \in \mathbb{D}$. That is,

$$\frac{\overline{bc} + \overline{d}^2}{\overline{bc} + \overline{d}^2 - (\overline{ab} + \overline{bd})z} = \frac{\overline{d} + \overline{b}z}{\overline{d} - \overline{b}z}$$

for any $z \in \mathbb{D}$. So if we cross multiply, we get that

$$(\overline{ab} + \overline{bd})\overline{b}z^2 + [(\overline{ab} + \overline{bd})\overline{d} - 2(\overline{bc} + \overline{d}^2)\overline{b}]z = 0$$

for any $z \in \mathbb{D}$, which is a contradiction. Hence C_φ is not isosymmetric. The proof is complete. □

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Conflicts of Interest The authors declare that they have no conflicts of interest.

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