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On the novel Hermite-Hadamard inequalities for composite inverse functions

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Abstract. The goal of this research is to discover some identities in the general form of the sum of left and right-sided weighted fractional integrals of a function concerning to another function. Using composite convex functions, several fractional Hermite-Hadamard inequalities are derived. The veracity of the inequalities established is demonstrated by drawing graphs of such relationships. Furthermore, our findings generalize a number of previously published outcomes. These findings will aid in the study of fractional differential equations and fractional boundary value problems with unique solutions.

1. Introduction

In mathematical analysis and inequality theory, convexity is very essential. Various researchers have defined many inequalities for convex functions, including the Ostrowski-type [1], Hardy-type [2], Olsen-type [3] and Gagliardo-Nirenberg-type [4]. Hermite-Hadamard's inequality [5] is a famous finding of Hermite and Hadamard for convex functions that gives us a necessary and sufficient condition for a function to be convex. This inequality for a convex function $F : U \to \mathbb{R}$ on $U \subseteq \mathbb{R}$ is defined by

$$F\left(\frac{\upsilon+\omega}{2}\right) \leq \frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega} F(\xi)d\xi \leq \frac{F(\upsilon)+F(\omega)}{2}. \tag{1}$$

In terms of the Cauchy Mean-Value Theorem for convex functions, Hadamard's inequality is sensitive. Hadamard's inequality can be used to discover upper and lower bounds for the mean value of a convex function. In regard to applications, this inequality is used to establish bounds in terms of trapezoid formulae [6]. It is also employed to explore certain means like in [7]. Similarly, convexity is utilized as a tool in optimization theory. Various researchers [8–15] have expended efforts to provide new bounds and estimations by using this inequality. The following fundamentals must be remembered.

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Definition 1.1. *[16] Let* $F : [\upsilon, \omega] \to \mathbb{R}$ *be termed as convex if for* $[\upsilon, \omega] \subseteq \mathbb{R}$, *we have*

$$F(\wp\vartheta + (1 - \wp)\varphi) \leq \wp F(\vartheta) + (1 - \wp)F(\varphi)$$

where $\vartheta, \varphi \in [\upsilon, \omega]$ *and* $\wp \in [0, 1]$.

Definition 1.2. *[17]Let* $h : U \subset \mathbb{R} \to \mathbb{R}^+$ *be a function such that* $F : U \subset \mathbb{R} \to \mathbb{R}$ *be h-convex, if* $F \geq 0$ *on* U *for* $\wp \in (0, 1)$, *we have*

$$F(\wp\upsilon + (1 - \wp)\omega) \leq h(\wp)F(\upsilon) + h(1 - \wp)F(\omega). \tag{2}$$

Silvestru Sever Dragomir present the definition of composite convex function in [18] as:

Definition 1.3. *A function* $\Phi : [\upsilon, \omega] \to \mathbb{R}$ *is composite-*$\mathcal{H}^{-1}$ *convex on* $[\upsilon, \omega]$, *if* $\Phi o \mathcal{H}^{-1} : [\mathcal{H}(\upsilon), \mathcal{H}(\omega)] \to \mathbb{R}$ *is convex on* $[\mathcal{H}(\upsilon), \mathcal{H}(\omega)]$ *i.e.,*

$$\Phi o \mathcal{H}^{-1}\Big((1 - \leftthreetimes)\mathcal{H}(x) + \leftthreetimes\mathcal{H}(y)\Big) \leq (1 - \leftthreetimes)\Phi(x) + \leftthreetimes\Phi(y)$$

for any $x, y \in [\upsilon, \omega]$ *and* $\leftthreetimes \in [0, 1]$.

Definition 1.4. *[19] The left-sided and right sided Rieman-Liouville fractional integral (RLFI)* $I^{\varsigma}_{\upsilon^+}F$ *and* $I^{\varsigma}_{\omega^-}F$ *of order* $\varsigma > 0$, *on a finite interval* $[\upsilon, \omega]$, *are defined as:*

$$I^{\varsigma}_{\upsilon^+}F(\nu) = \frac{1}{\Gamma(\varsigma)}\int_{\upsilon}^{\nu}(\nu - t)^{\varsigma-1}F(t)dt, \quad \nu > \upsilon$$

and

$$I^{\varsigma}_{\omega^-}F(\nu) = \frac{1}{\Gamma(\varsigma)}\int_{\nu}^{\omega}(t - \nu)^{\varsigma-1}F(t)dt, \quad \nu < \omega,$$

respectively. Here Γ *represents the usual Gamma function defined by*

$$\Gamma(t) = \int_{0}^{\infty}\nu^{t-1}\exp(-\nu)d\nu \ , \mathbb{R}(t) > 0.$$

Mubeen *et al.* present the following definition of k-RLFI in [20].

Definition 1.5. *Let* $F \in L_1[\upsilon, \omega]$. *Then the left and right-sided k-RLI* $I^{\varsigma}_{\upsilon^+,k}F$ *and* $I^{\varsigma}_{\omega^-,k}F$ *of order* $\varsigma > 0$ *with* $k > 0$, $\upsilon \geq 0$ *are defined by*

$$I^{\varsigma}_{\upsilon^+,k}F(\nu) = \frac{1}{k\Gamma_k(\varsigma)}\int_{\upsilon}^{\nu}(\nu - \mu)^{\frac{\varsigma}{k}-1}F(\mu)d\mu, \qquad \nu > \upsilon$$

and

$$I^{\varsigma}_{\omega^-,k}F(\nu) = \frac{1}{k\Gamma_k(\varsigma)}\int_{\nu}^{\omega}(\mu - \nu)^{\frac{\varsigma}{k}-1}F(\mu)d\mu, \qquad \nu < \omega,$$

respectively. Here $\Gamma_k(\varsigma)$ *is the k-Gamma function and* $I^{0}_{\upsilon^+,k}F$=$I^{0}_{\omega^-,k}F$=$F(\nu)$.

Definition 1.6. *[21] The weighted fractional integral (WFI) of a function F concerning to another function $\mathcal{H}$ of order ς can be written as*

$$J^{\varsigma}_{\Phi,\upsilon^+}F(\wp)=\frac{\Phi^{-1}(\wp)}{\Gamma(\varsigma)}\int_{\upsilon}^{\wp}(\mathcal{H}(\wp)-\mathcal{H}(\leftthreetimes))^{\varsigma-1}\Phi(\leftthreetimes)F(\leftthreetimes)\mathcal{H}'(\leftthreetimes)d\leftthreetimes,\qquad \wp>\upsilon.$$

The definition of RLFI of a function concerning to another function given in [19] defined as:

Definition 1.7. *Let $p:[\theta,\zeta]\to\mathbb{R}$ be a positive increasing function on $(\theta,\zeta]$, such that p' exists and continuous on (θ,ζ). The RLFI of F with concerning to the function p on $[\theta,\zeta]$ of order $\varsigma>0$ are defined by*

$$\Im^{\varsigma}_{\theta^+;p}F(\nu)=\frac{1}{\Gamma(\varsigma)}\int_{\theta}^{\nu}\frac{p'(\leftthreetimes)}{[p(\nu)-p(\leftthreetimes)]^{1-\varsigma}}F(\leftthreetimes)d\leftthreetimes,\ \nu>\theta,\tag{3}$$

$$\Im^{\varsigma}_{\zeta^-;p}F(\nu)=\frac{1}{\Gamma(\varsigma)}\int_{\nu}^{\zeta}\frac{p'(\leftthreetimes)}{[p(\leftthreetimes)-p(\nu)]^{1-\varsigma}}F(\leftthreetimes)d\leftthreetimes,\ \nu<\zeta,\tag{4}$$

provided that the integral exists. The integrals (3) and (4) are called left-sided and right-sided RLFI of a function with respect to another one.

2. Main Results

The following lemma presents the first key result of this section.

Lemma 2.1. *Let $F,\Phi:[\upsilon,\omega]\to\mathbb{R}$ both are differentiable composite-$\mathcal{H}^{-1}$-convex mappings on (υ,ω) with $\upsilon<\omega$. If $F,\Phi\in L_2[\upsilon,\omega]$, then the following identity for fractional integrals holds*

$$\frac{\Phi(\upsilon)\upsilon(\upsilon)+\Phi(\omega)\upsilon(\omega)}{2}-\frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))^{\varsigma}}\Big[\Phi(\omega)J^{\varsigma}_{\Phi,\upsilon^+}F(\omega)+\Phi(\upsilon)J^{\varsigma}_{\Phi,\omega^-}F(\upsilon)\Big]$$
$$=\frac{(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))}{2}\int_0^1\Big[(1-\leftthreetimes)^{\varsigma}-\leftthreetimes^{\varsigma}\Big]\frac{d}{d\leftthreetimes}$$
$$\times\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon)-\mathcal{H}(\omega))}\Big]d\leftthreetimes.$$

Proof. Consider

$$I=\int_0^1\Big[(1-\leftthreetimes)^{\varsigma}-\leftthreetimes^{\varsigma}\Big]\frac{d}{d\leftthreetimes}$$
$$\times\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon)-\mathcal{H}(\omega))}\Big]d\leftthreetimes$$
$$=\int_0^1(1-\leftthreetimes)^{\varsigma}\frac{d}{d\leftthreetimes}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon)-\mathcal{H}(\omega))}\Big]d\leftthreetimes$$
$$-\int_0^1\leftthreetimes^{\varsigma}\frac{d}{d\leftthreetimes}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon)-\mathcal{H}(\omega))}\Big]d\leftthreetimes$$
$$=I_1-I_2.\tag{5}$$

Integrating, we get

$$I_1 = \int_0^1 (1-\leftthreetimes)^{\varsigma} \frac{d}{d\leftthreetimes}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(v)-\mathcal{H}(\omega))}\Big]d\leftthreetimes$$

$$= (1-\leftthreetimes)^{\varsigma}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(v)-\mathcal{H}(\omega))}\Big]\Big|_0^1$$

$$-\int_0^1 \varsigma(-1)(1-\leftthreetimes)^{\varsigma-1}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(v)-\mathcal{H}(\omega))}\Big]d\leftthreetimes$$

$$= \frac{\Phi(\omega)F(\omega)}{(\mathcal{H}(\omega)-\mathcal{H}(v))} - \varsigma\int_v^{\omega}\Big(\frac{\mathcal{H}(u)-\mathcal{H}(v)}{\mathcal{H}(\omega)-\mathcal{H}(v)}\Big)^{\varsigma-1}\frac{\Phi(u)F(u)}{(\mathcal{H}(\omega)-\mathcal{H}(v))^2}\mathcal{H}'(u)du$$

$$= \frac{\Phi(\omega)F(\omega)}{(\mathcal{H}(\omega)-\mathcal{H}(v))} - \frac{\Phi(v)\Gamma(\varsigma+1)}{(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma+1}} J^{\varsigma}_{\Phi,\omega^-}F(v).$$

Similarly integrating I_2, we get

$$I_2 = \int_0^1 \leftthreetimes^{\varsigma} \frac{d}{d\leftthreetimes}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(v)-\mathcal{H}(\omega))}\Big]d\leftthreetimes$$

$$= \leftthreetimes^{\varsigma}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(v)-\mathcal{H}(\omega))}\Big]\Big|_0^1$$

$$-\int_0^1 \varsigma\leftthreetimes^{\varsigma-1}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(v)-\mathcal{H}(\omega))}\Big]d\leftthreetimes$$

$$= \frac{-\Phi(v)F(v)}{(\mathcal{H}(\omega)-\mathcal{H}(v))} + \varsigma\int_v^{\omega}\Big(\frac{\mathcal{H}(\omega)-\mathcal{H}(u)}{\mathcal{H}(\omega)-\mathcal{H}(v)}\Big)^{\varsigma-1}\frac{\Phi(u)F(u)}{(\mathcal{H}(\omega)-\mathcal{H}(v))^2}\mathcal{H}'(u)du$$

$$= \frac{-\Phi(v)F(v)}{(\mathcal{H}(\omega)-\mathcal{H}(v))} + \frac{\Phi(\omega)\Gamma(\varsigma+1)}{(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma+1}} J^{\varsigma}_{\Phi,v^+}F(\omega).$$

Substituting the values of I_1 and I_2 in (5), we have

$$\frac{\Phi(v)F(v)+\Phi(\omega)F(\omega)}{2} - \frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma}}\Big[\Phi(\omega)J^{\varsigma}_{\Phi,v^+}F(\omega)+\Phi(v)J^{\varsigma}_{\Phi,\omega^-}F(v)\Big]$$

$$= \frac{(\mathcal{H}(\omega)-\mathcal{H}(v))}{2}$$

$$\times\int_0^1\Big[(1-\leftthreetimes)^{\varsigma}-\leftthreetimes^{\varsigma}\Big]\frac{d}{d\leftthreetimes}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(v)-\mathcal{H}(\omega))}\Big]d\leftthreetimes.$$

This completes the proof. □

Remark 2.2. *If we take $\Phi(\wp)=1$, $\mathcal{H}(\wp)=\wp$ and $\varsigma=1$ in Lemma 2.1, then we get the result of Dragomir et al. [22, Lemma 2].*

Remark 2.3. *If we take $\Phi(\wp) = 1$ and $\mathcal{H}(\wp) = \wp$ in Lemma 2.1, then we get the result of Sarikaya et al. [23, Lemma 2].*

Lemma 2.4. *Let $F, \Phi : [\upsilon, \omega] \to \mathbb{R}$ both are twice differentiable composite $\mathcal{H}^{-1}$-convex mappings on (υ, ω). If $F, \Phi \in L_2[\upsilon, \omega]$, then the following equality for fractional integrals holds*

$$\frac{\Phi(\upsilon)F(\upsilon) + \Phi(\omega)F(\omega)}{2} - \frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega) - \mathcal{H}(\upsilon))^{\varsigma}}\Big[\Phi(\omega)J^{\varsigma}_{\Phi,\upsilon^+}F(\omega) + \Phi(\upsilon)J^{\varsigma}_{\Phi,\omega^-}F(\upsilon)\Big]$$
$$= \frac{(\mathcal{H}(\omega) - \mathcal{H}(\upsilon))^2}{2(\varsigma+1)} \int_0^1 \Big[1 - (1-\leftthreetimes)^{\varsigma+1} - \leftthreetimes^{\varsigma+1}\Big]$$
$$\times \frac{d^2}{d\leftthreetimes^2}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\omega) - \mathcal{H}(\upsilon))^2}\Big]d\leftthreetimes.$$

Proof. By using Lemma 2.1 and integrating, we have

$$\frac{\Phi(\upsilon)F(\upsilon) + \Phi(\omega)F(\omega)}{2} - \frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega) - \mathcal{H}(\upsilon))^{\varsigma}}\Big[\Phi(\omega)J^{\varsigma}_{\Phi,\upsilon^+}F(\omega) + \Phi(\upsilon)J^{\varsigma}_{\Phi,\omega^-}F(\upsilon)\Big]$$
$$= \frac{(\mathcal{H}(\omega) - \mathcal{H}(\upsilon))}{2} \int_0^1 \Big[(1-\leftthreetimes)^{\varsigma} - \leftthreetimes^{\varsigma}\Big]$$
$$\times \frac{d}{d\leftthreetimes}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon) - \mathcal{H}(\omega))}\Big]d\leftthreetimes$$
$$= \frac{(\mathcal{H}(\omega) - \mathcal{H}(\upsilon))}{2}\Big[\frac{d}{d\leftthreetimes}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon) - \mathcal{H}(\omega))}\Big]$$
$$\times\Big[\frac{-(1-\leftthreetimes)^{\varsigma+1}}{\varsigma+1} - \frac{\leftthreetimes^{\varsigma+1}}{\varsigma+1}\Big]\Big|_0^1$$
$$- \frac{1}{\varsigma+1}\int_0^1 \Big[-(1-\leftthreetimes)^{\varsigma+1} - \leftthreetimes^{\varsigma+1}\Big]$$
$$\times \frac{d^2}{d\leftthreetimes^2}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon) - \mathcal{H}(\omega))}\Big]d\leftthreetimes\Big]$$
$$= \frac{(\mathcal{H}(\omega) - \mathcal{H}(\upsilon))^2}{2(\varsigma+1)} \int_0^1 \Big[1 - (1-\leftthreetimes)^{\varsigma+1} - \leftthreetimes^{\varsigma+1}\Big]$$
$$\times \frac{d^2}{d\leftthreetimes^2}\Big[\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon) + (1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\omega) - \mathcal{H}(\upsilon))^2}\Big]d\leftthreetimes.$$

The proof is done. □

Remark 2.5. *If we take $\Phi(\tau) = 1$ and $\mathcal{H}(\tau) = \tau$ in Lemma 2.4, then we obtain the result of Wang et al. [24, Lemma 2.1].*

Theorem 2.6. *Let $F, \Phi : [\upsilon, \omega] \to \mathbb{R}$ both are composite $\mathcal{H}^{-1}$-convex mappings on (υ, ω) with $\upsilon < \omega$. If $F, \Phi \in L_2[\upsilon, \omega]$, then the following fractional integrals inequalities hold.*

$$
\begin{aligned}
&2\Phi o\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(\upsilon)+\mathcal{H}(\omega)}{2}\Big)Fo\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(\upsilon)+\mathcal{H}(\omega)}{2}\Big)-\frac{\varsigma}{2}\Omega\\
&\leq \frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))^{\varsigma}}\Big(\Phi(\omega)J^{\varsigma}_{\Phi,\upsilon^+}F(\omega)+\Phi(\upsilon)J^{\varsigma}_{\Phi,\omega^-}F(\upsilon)\Big)\\
&\leq M_1+M_2,
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
\Omega =& \int_0^1 \curlywedge^{\varsigma-1}\Big(\Phi o\mathcal{H}^{-1}\Big(\curlywedge\mathcal{H}(\upsilon)+(1-\curlywedge)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(\upsilon)+\curlywedge\mathcal{H}(\omega)\Big)\\
&+\Phi o\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(\upsilon)+\curlywedge\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\curlywedge\mathcal{H}(\upsilon)+(1-\curlywedge)\mathcal{H}(\omega)\Big)\Big)d\curlywedge,
\end{aligned}
$$

$$
M_1=\Big(\frac{\varsigma^2+\varsigma+2}{2(\varsigma+1)(\varsigma+2)}\Big)\Big(\Phi(\upsilon)F(\upsilon)+\Phi(\omega)F(\omega)\Big)
$$

and

$$
M_2=\Big(\frac{\varsigma}{(\varsigma+1)(\varsigma+2)}\Big)\Big(\Phi(\upsilon)F(\omega)+\Phi(\omega)F(\upsilon)\Big).
$$

Proof. By composite $\mathcal{H}^{-1}$-convex, we can write

$$
\Phi o\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(x)+\curlywedge\mathcal{H}(y)\Big)\leq(1-\curlywedge)\Phi(x)+\curlywedge\Phi(y),\tag{7}
$$

$$
Fo\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(x)+\curlywedge\mathcal{H}(y)\Big)\leq(1-\curlywedge)F(x)+\curlywedge F(y).\tag{8}
$$

Substituting $\curlywedge=\frac{1}{2}$ in (7) and (8), then multiplying the resulting equations, we have

$$
\begin{aligned}
&\Phi o\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(x)+\mathcal{H}(y)}{2}\Big)Fo\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(x)+\mathcal{H}(y)}{2}\Big)\\
&\leq\frac{1}{4}\Big(\Phi(x)F(x)+\Phi(y)F(x)+\Phi(x)F(y)+\Phi(y)F(y)\Big).
\end{aligned}
$$

Choose $x=\mathcal{H}^{-1}\Big(\curlywedge\mathcal{H}(\upsilon)+(1-\curlywedge)\mathcal{H}(\omega)\Big)$ and $y=\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(\upsilon)+\curlywedge\mathcal{H}(\omega)\Big)$, we get

$$
\begin{aligned}
&\Phi o\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(\upsilon)+\mathcal{H}(\omega)}{2}\Big)Fo\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(\upsilon)+\mathcal{H}(\omega)}{2}\Big)\\
&\leq\frac{1}{4}\Big(\Phi o\mathcal{H}^{-1}\Big(\curlywedge\mathcal{H}(\upsilon)+(1-\curlywedge)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\curlywedge\mathcal{H}(\upsilon)+(1-\curlywedge)\mathcal{H}(\omega)\Big)\\
&+\Phi o\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(\upsilon)+\curlywedge\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\curlywedge\mathcal{H}(\upsilon)+(1-\curlywedge)\mathcal{H}(\omega)\Big)\\
&+\Phi o\mathcal{H}^{-1}\Big(\curlywedge\mathcal{H}(\upsilon)+(1-\curlywedge)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(\upsilon)+\curlywedge\mathcal{H}(\omega)\Big)\\
&+\Phi o\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(\upsilon)+\curlywedge\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big((1-\curlywedge)\mathcal{H}(\upsilon)+\curlywedge\mathcal{H}(\omega)\Big)\Big).
\end{aligned}
\tag{9}
$$

Multiplying equation (9) with $\leftthreetimes^{\varsigma-1}$ and integrating over the interval $[0,1]$ we get

$$\Phi o\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(v)+\mathcal{H}(\omega)}{2}\Big)Fo\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(v)+\mathcal{H}(\omega)}{2}\Big)\int_0^1 \leftthreetimes^{\varsigma-1}d\leftthreetimes$$

$$\leq \frac{1}{4}\Big(\int_0^1 \leftthreetimes^{\varsigma-1}\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)d\leftthreetimes$$

$$+\int_0^1 \leftthreetimes^{\varsigma-1}\Phi o\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)d\leftthreetimes$$

$$+\int_0^1 \leftthreetimes^{\varsigma-1}\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)d\leftthreetimes$$

$$+\int_0^1 \leftthreetimes^{\varsigma-1}\Phi o\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)d\leftthreetimes\Big)$$

$$=\frac{1}{4}\Big(\frac{\Gamma(\varsigma)\Phi(\omega)}{(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma}}J^{\varsigma}_{\Phi,v^+}F(\omega)+\frac{\Gamma(\varsigma)\Phi(v)}{(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma}}J^{\varsigma}_{\Phi,\omega^-}F(v)$$

$$+\int_0^1 \leftthreetimes^{\varsigma-1}\Big(\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)$$

$$+\Phi o\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big(\leftthreetimes\mathcal{H}(v)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)\Big)d\leftthreetimes\Big)$$

$$=\frac{1}{4}\Big(\frac{\Gamma(\varsigma)\Phi(\omega)}{(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma}}J^{\varsigma}_{\Phi,v^+}F(\omega)+\frac{\Gamma(\varsigma)\Phi(v)}{(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma}}J^{\varsigma}_{\Phi,\omega^-}F(v)+\Omega\Big),$$

which gives

$$2\Phi o\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(v)+\mathcal{H}(\omega)}{2}\Big)Fo\mathcal{H}^{-1}\Big(\frac{\mathcal{H}(v)+\mathcal{H}(\omega)}{2}\Big)-\frac{\varsigma}{2}\Omega$$
$$\leq\frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma}}\Big(\Phi(\omega)J^{\varsigma}_{\Phi,v^+}F(\omega)+\Phi(v)J^{\varsigma}_{\Phi,\omega^-}F(v)\Big). \tag{10}$$

By substituting $x=v$ and $y=\omega$ in (7) and (8) respectively, then multiplying the resulting inequalities, we get

$$\Phi o\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)$$
$$\leq(1-\leftthreetimes)^2\Phi(v)F(v)+(\leftthreetimes-\leftthreetimes^2)\Phi(v)F(\omega)+(\leftthreetimes-\leftthreetimes^2)\Phi(\omega)F(v)+\leftthreetimes^2\Phi(\omega)F(\omega). \tag{11}$$

Multiplying the above inequality with $\leftthreetimes^{\varsigma-1}$ and then integrating over the interval $[0,1]$ we get

$$\int_0^1 \leftthreetimes^{\varsigma-1}\Phi o\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)Fo\mathcal{H}^{-1}\Big((1-\leftthreetimes)\mathcal{H}(v)+\leftthreetimes\mathcal{H}(\omega)\Big)d\leftthreetimes$$

$$\leq\int_0^1 \leftthreetimes^{\varsigma-1}(1-\leftthreetimes)^2\Phi(v)F(v)d\leftthreetimes+\int_0^1(\leftthreetimes^{\varsigma}-\leftthreetimes^{\varsigma+1})\Phi(v)F(\omega)d\leftthreetimes$$

$$+\int_0^1 (\leftthreetimes^{\varsigma}-\leftthreetimes^{\varsigma+1})\Phi(\omega)F(\upsilon)d\leftthreetimes+\int_0^1 \leftthreetimes^{\varsigma+1}\Phi(\omega)F(\omega)d\leftthreetimes,$$

which gives

$$\frac{\Gamma(\varsigma)\Phi(\upsilon)}{(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))^{\varsigma}}J^{\varsigma}_{\Phi,\omega^-}F(\upsilon)\leq L_1+L_2+L_3+L_4, \tag{12}$$

where

$$L_1=\frac{2\Phi(\upsilon)F(\upsilon)}{\varsigma(\varsigma+1)(\varsigma+2)},$$

$$L_2=\frac{\Phi(\upsilon)F(\omega)}{(\varsigma+1)(\varsigma+2)},$$

$$L_3=\frac{\Phi(\omega)F(\upsilon)}{(\varsigma+1)(\varsigma+2)}$$

and

$$L_4=\frac{\Phi(\omega)F(\omega)}{\varsigma+2}.$$

Multiplying the inequality (11) with $(1-\leftthreetimes)^{\varsigma-1}$ and then integrating over the interval $[0,1]$ we get

$$\frac{\Gamma(\varsigma)\Phi(\omega)}{(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))^{\varsigma}}J^{\varsigma}_{\Phi,\upsilon^+}F(\omega)\leq L_5+L_6+L_7+L_8, \tag{13}$$

where

$$L_5=\frac{\Phi(\upsilon)F(\upsilon)}{\varsigma+2},$$

$$L_6=\frac{\Phi(\upsilon)F(\omega)}{(\varsigma+1)(\varsigma+2)},$$

$$L_7=\frac{\Phi(\omega)F(\upsilon)}{(\varsigma+1)(\varsigma+2)}$$

and

$$L_8=\frac{2\Phi(\omega)F(\omega)}{\varsigma(\varsigma+1)(\varsigma+2)}.$$

Adding (12) and (13), we get

$$\frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))^{\varsigma}}\Big(\Phi(\omega)J^{\varsigma}_{\Phi,\upsilon^+}F(\omega)+\Phi(\upsilon)J^{\varsigma}_{\Phi,\omega^-}F(\upsilon)\Big)\leq M_1+M_2. \tag{14}$$

By combining (10) and (14), we get the result. □

Example 2.7. *The validity of the double inequality presented in (6) for a convex function $F(\wp)=\wp^2$ can be proved by plotting the graphs. It is well knowledge the WFI of this function for $\Phi(\wp)=1$ and $\mathcal{H}(\wp)=\wp$ are given by*

$$J^{\varsigma}_{\Phi,\upsilon^+}F(\omega)=\frac{1}{\Gamma(\varsigma)}\int_{\upsilon}^{\omega}(\omega-\leftthreetimes)^{\varsigma-1}\leftthreetimes^2\,d\leftthreetimes \tag{15}$$

and

$$J^{\varsigma}_{\Phi,\omega^-}F(\upsilon) = \frac{1}{\Gamma(\varsigma)}\int_{\upsilon}^{\omega}(\leftthreetimes-\upsilon)^{\varsigma-1}\leftthreetimes^2\,d\leftthreetimes. \tag{16}$$

By utilizing these expressions into the double inequality (6), we get

$$\begin{aligned}&2\Big(\frac{\upsilon+\omega}{2}\Big)^2-\frac{\varsigma}{2}\Big(\frac{2(\upsilon^2+\omega^2)(\varsigma^2+2\varsigma+2)}{\varsigma(\varsigma+1)(\varsigma+2)}+\frac{4\upsilon\omega}{(\varsigma+1)(\varsigma+2)}\Big)\\ &\leq\frac{\varsigma}{2(\omega-\upsilon)^{\varsigma}}\int_{\upsilon}^{\omega}[(\omega-\wp)^{\varsigma-1}+(\wp-\upsilon)^{\varsigma-1}]\wp^2d\wp\\ &\leq\frac{\upsilon^2+\omega^2}{2}.\end{aligned} \tag{17}$$

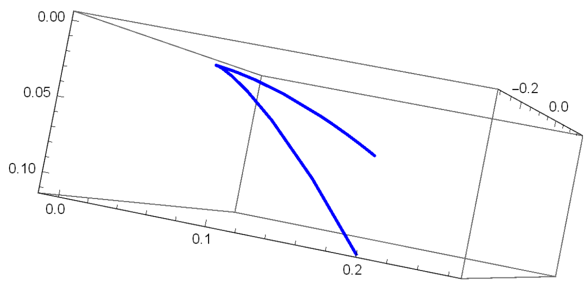

Figure 1: The validity of the double inequality (17) is illustrated by the graphs corresponding to $\upsilon = 0$ and $\omega = 1$.

The graphs of the dual inequality (17) are presented in Fig. 1 corresponding to $\varsigma \in (0, 1]$. *The Fig. 1 prove that the dual inequality (17) is true for the admissible choice of the parameters.*

Theorem 2.8. *Let the integrable synchronous functions* $\mathcal{K}$ *and* $\mathcal{H}$ *on* $[0,\infty)$, *then*

$$\Phi(x)J^{\varsigma}_{\Phi,\upsilon^+}\mathcal{K}(x)\geq\frac{\Gamma(\varsigma+1)}{(\mathcal{H}(x)-\mathcal{H}(\upsilon))^{\varsigma}}\mathfrak{I}^{\varsigma}_{\upsilon^+;\mathcal{H}}\mathcal{K}(x)\mathfrak{I}^{\varsigma}_{\upsilon^+;\mathcal{H}}\Phi(x),$$

where $\varsigma > 0$.

Proof. Since $\mathcal{K}$ and $\mathcal{H}$ are synchronous on $[0,\infty)$, we have

$$\mathcal{K}(u)\Phi(u)+\mathcal{K}(v)\Phi(v)\geq\mathcal{K}(u)\Phi(v)+\mathcal{K}(v)\Phi(u). \tag{18}$$

Multiplying both sides of (18) by $\frac{(\mathcal{H}(x)-\mathcal{H}(u))^{\varsigma-1}}{\Phi(x)\Gamma(\varsigma)}\mathcal{H}'(u)$, $\upsilon < u < x$
results in

$$\begin{aligned}&\frac{(\mathcal{H}(x)-\mathcal{H}(u))^{\varsigma-1}}{\Phi(x)\Gamma(\varsigma)}\mathcal{H}'(u)\mathcal{K}(u)\Phi(u)+\frac{(\mathcal{H}(x)-\mathcal{H}(u))^{\varsigma-1}}{\Phi(x)\Gamma(\varsigma)}\mathcal{H}'(u)\mathcal{K}(v)\Phi(v)\\ &\geq\frac{(\mathcal{H}(x)-\mathcal{H}(u))^{\varsigma-1}}{\Phi(x)\Gamma(\varsigma)}\mathcal{H}'(u)\mathcal{K}(u)\Phi(v)+\frac{(\mathcal{H}(x)-\mathcal{H}(u))^{\varsigma-1}}{\Phi(x)\Gamma(\varsigma)}\mathcal{H}'(u)\mathcal{K}(v)\Phi(u).\end{aligned}$$

Further integrating both sides with respect to u over υ to x, gives

$$\begin{aligned}&J^{\varsigma}_{\Phi,\upsilon^+}\mathcal{K}(x)+\frac{\Phi(v)\mathcal{K}(v)(\mathcal{H}(x)-\mathcal{H}(\upsilon))^{\varsigma}}{\Phi(x)\Gamma(\varsigma+1)}\\ &\geq\frac{\Phi(v)}{\Phi(x)}\mathfrak{I}^{\varsigma}_{\upsilon^+;\mathcal{H}}\mathcal{K}(x)+\frac{\mathcal{K}(v)}{\Phi(x)}\mathfrak{I}^{\varsigma}_{\upsilon^+;\mathcal{H}}\Phi(x).\end{aligned} \tag{19}$$

Multiplying both sides of (19) by $\frac{(\mathcal{H}(x)-\mathcal{H}(v))^{\varsigma-1}}{\Phi(x)\Gamma(\varsigma)}\mathcal{H}'(v)$ and integrating over v to x with respect to v, we get

$$\frac{(\mathcal{H}(x)-\mathcal{H}(v))^{\varsigma}}{\Phi(x)\Gamma(\varsigma+1)}J^{\varsigma}_{\Phi,v^+}\mathcal{K}(x)+\frac{(\mathcal{H}(x)-\mathcal{H}(v))^{\varsigma}}{\Phi(x)\Gamma(\varsigma+1)}J^{\varsigma}_{\Phi,v^+}\mathcal{K}(x)$$
$$\geq\frac{1}{(\Phi(x))^2}\mathfrak{I}^{\varsigma}_{v^+;\mathcal{H}}\Phi(x)\mathfrak{I}^{\varsigma}_{v^+;\mathcal{H}}\mathcal{K}(x)+\frac{1}{(\Phi(x))^2}\mathfrak{I}^{\varsigma}_{v^+;\mathcal{H}}\Phi(x)\mathfrak{I}^{\varsigma}_{v^+;\mathcal{H}}\mathcal{K}(x),$$

this implies

$$\Phi(x)J^{\varsigma}_{\Phi,v^+}\mathcal{K}(x)\geq\frac{\Gamma(\varsigma+1)}{(\mathcal{H}(x)-\mathcal{H}(v))^{\varsigma}}\mathfrak{I}^{\varsigma}_{v^+;\mathcal{H}}\mathcal{K}(x)\mathfrak{I}^{\varsigma}_{v^+;\mathcal{H}}\Phi(x).$$

The proof is done. □

Theorem 2.9. *Let $\mathcal{K},\Phi:[v,\omega]\to\mathbb{R}$ be such that their derivatives are composite $\mathcal{H}^{-1}$-convex mappings on (v,ω) with $v<\omega$. If $|\mathcal{K}|,|\Phi|\in L_2[v,\omega]$, then the following inequality for fractional integrals holds*

$$\left|\frac{\Phi(v)\mathcal{K}(v)+\Phi(\omega)\mathcal{K}(\omega)}{2}-\frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega)-\mathcal{H}(a))^{\varsigma}}\Big(\Phi(\omega)J^{\varsigma}_{\Phi,v^+}\mathcal{K}(\omega)+\Phi(v)J^{\varsigma}_{\Phi,\omega^-}\mathcal{K}(v)\Big)\right|$$
$$\leq\frac{\mathcal{H}(\omega)-\mathcal{H}(v)}{2}\Big\{\Big(|\Phi(v)||\mathcal{K}'(v)|+|\mathcal{K}(v)||\Phi'(v)|\Big)X$$
$$+\Big(|\Phi(v)||\mathcal{K}'(\omega)|+|\Phi(\omega)||\mathcal{K}'(v)|+|\mathcal{K}(v)||\Phi'(\omega)|+|\mathcal{K}(\omega)||\Phi'(v)|\Big)Y$$
$$+\Big(|\Phi(\omega)||\mathcal{K}'(\omega)|+|\mathcal{K}(\omega)||\Phi'(\omega)|\Big)Z\Big\},\tag{20}$$

where

$$X=\left(\frac{6+(\varsigma+1)(\varsigma+2)(\varsigma+3)}{(\varsigma+1)(\varsigma+2)(\varsigma+3)(\varsigma+4)}\right)\left|(\mathcal{H}^{-1})'(\mathcal{H}(v))\right|+\left(\frac{1}{(\varsigma+2)(\varsigma+3)}\right)\left|(\mathcal{H}^{-1})'(\mathcal{H}(\omega))\right|,$$
$$Y=\frac{|(\mathcal{H}^{-1})'(\mathcal{H}(v))|+|(\mathcal{H}^{-1})'(\mathcal{H}(\omega))|}{(\varsigma+2)(\varsigma+3)},$$
$$Z=\left(\frac{1}{(\varsigma+2)(\varsigma+3)}\right)\left|(\mathcal{H}^{-1})'(\mathcal{H}(v))\right|+\left(\frac{6+(\varsigma+1)(\varsigma+2)(\varsigma+3)}{(\varsigma+1)(\varsigma+2)(\varsigma+3)(\varsigma+4)}\right)\left|(\mathcal{H}^{-1})'(\mathcal{H}(\omega))\right|.$$

Proof. By using Lemma 2.1, modulus property and Definition 1.3, we have

$$\left|\frac{\Phi(v)\mathcal{K}(v)+\Phi(\omega)\mathcal{K}(\omega)}{2}-\frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega)-\mathcal{H}(v))^{\varsigma}}\Big(\Phi(\omega)J^{\varsigma}_{\Phi,v^+}\mathcal{K}(\omega)+\Phi(v)J^{\varsigma}_{\Phi,\omega^-}\mathcal{K}(v)\Big)\right|$$
$$=\left|\frac{(\mathcal{H}(\omega)-\mathcal{H}(v))}{2}\int_0^1\Big((1-\lambda)^{\varsigma}-\lambda^{\varsigma}\Big)\right.$$

$$\times \frac{d}{d\leftthreetimes}\Big(\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)\mathcal{K}o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon)-\mathcal{H}(\omega))}\Big)d\leftthreetimes\Big|$$

$$\leq \frac{(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))}{2}\int_0^1\Big((1-\leftthreetimes)^{\varsigma}+\leftthreetimes^{\varsigma}\Big)$$

$$\times\Big|\frac{d}{d\leftthreetimes}\Big(\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)\mathcal{K}o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon)-\mathcal{H}(\omega))}\Big)\Big|d\leftthreetimes$$

$$= \frac{(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))}{2}\int_0^1\Big((1-\leftthreetimes)^{\varsigma}+\leftthreetimes^{\varsigma}\Big)$$

$$\times\Big|\Big(\frac{\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)\frac{d}{d\leftthreetimes}\mathcal{K}o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon)-\mathcal{H}(\omega))}\Big)$$

$$+\Big(\frac{\frac{d}{d\leftthreetimes}\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)\mathcal{K}o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)}{(\mathcal{H}(\upsilon)-\mathcal{H}(\omega))}\Big)\Big|d\leftthreetimes$$

$$= \frac{(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))}{2}\int_0^1\Big((1-\leftthreetimes)^{\varsigma}+\leftthreetimes^{\varsigma}\Big)$$

$$\times\Big|\Big(\Phi o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)$$

$$\times\mathcal{K}'o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)(\mathcal{H}^{-1})'\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)\Big)$$

$$+\Big(\mathcal{K}o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)$$

$$\times\Phi'o\mathcal{H}^{-1}\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)(\mathcal{H}^{-1})'\Big(\leftthreetimes \mathcal{H}(\upsilon)+(1-\leftthreetimes)\mathcal{H}(\omega)\Big)\Big)\Big|d\leftthreetimes$$

$$\leq \frac{(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))}{2}\int_0^1\Big((1-\leftthreetimes)^{\varsigma}+\leftthreetimes^{\varsigma}\Big)$$

$$\times\Big\{\Big(\leftthreetimes|\Phi(\upsilon)|+(1-\leftthreetimes)|\Phi(\omega)|\Big)\Big(\leftthreetimes|\mathcal{K}'(\upsilon)|+(1-\leftthreetimes)|\mathcal{K}'(\omega)|\Big)$$

$$+\Big(\leftthreetimes|\mathcal{K}(\upsilon)|+(1-\leftthreetimes)|\mathcal{K}(\omega)|\Big)\Big(\leftthreetimes|\Phi'(\upsilon)|+(1-\leftthreetimes)|\Phi'(\omega)|\Big)\Big\}$$

$$\times\Big\{\leftthreetimes\Big|(\mathcal{H}^{-1})'\mathcal{H}(\upsilon)\Big|+(1-\leftthreetimes)\Big|(\mathcal{H}^{-1})'\mathcal{H}(\omega)\Big|\Big\}d\leftthreetimes.$$

By simple calculation, we get

$$\Big|\frac{\Phi(\upsilon)\mathcal{K}(\upsilon)+\Phi(\omega)\mathcal{K}(\omega)}{2}-\frac{\Gamma(\varsigma+1)}{2(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))^{\varsigma}}\Big(\Phi(\omega)J^{\varsigma}_{\Phi,\upsilon^{+}}\mathcal{K}(\omega)+\Phi(\upsilon)J^{\varsigma}_{\Phi,\omega^{-}}\mathcal{K}(\upsilon)\Big)\Big|$$

$$\leq \frac{(\mathcal{H}(\omega)-\mathcal{H}(\upsilon))}{2}\Big\{\Big(|\Phi(\upsilon)||\mathcal{K}'(\upsilon)|+|\mathcal{K}(\upsilon)||\Phi'(\upsilon)|\Big)$$

$$\times \int_0^1 \Big(\Big| (\mathcal{H}^{-1})' \mathcal{H}(\upsilon) \Big| \Big(\leftthreetimes^3 (1-\leftthreetimes)^\varsigma + \leftthreetimes^{\varsigma+3} \Big)$$
$$+ \Big| (\mathcal{H}^{-1})' \mathcal{H}(\omega) \Big| \Big(\leftthreetimes^2 (1-\leftthreetimes)^{\varsigma+1} + \leftthreetimes^{\varsigma+1}(1-\leftthreetimes) \Big) \Big) d\leftthreetimes$$
$$+ \Big(|\Phi(\upsilon)||\mathcal{K}'(\omega)| + |\Phi(\omega)||\mathcal{K}'(\upsilon)| + |\mathcal{K}(\upsilon)||\Phi'(\omega)| + |\mathcal{K}(\omega)||\Phi'(\upsilon)| \Big)$$
$$\times \int_0^1 \Big(\Big| (\mathcal{H}^{-1})' \mathcal{H}(\upsilon) \Big| \Big(\leftthreetimes^2 (1-\leftthreetimes)^{\varsigma+1} + \leftthreetimes^{\varsigma+2}(1-\leftthreetimes) \Big)$$
$$+ \Big| (\mathcal{H}^{-1})' \mathcal{H}(\omega) \Big| \Big(\leftthreetimes (1-\leftthreetimes)^{\varsigma+2} + \leftthreetimes^{\varsigma+1}(1-\leftthreetimes)^2 \Big) \Big) d\leftthreetimes$$
$$+ \Big(|\Phi(\omega)||\mathcal{K}'(\omega)| + |\mathcal{K}(\omega)||\Phi'(\omega)| \Big) \int_0^1 \Big(\Big| (\mathcal{H}^{-1})' \mathcal{H}(\upsilon) \Big| \Big(\leftthreetimes (1-\leftthreetimes)^{\varsigma+2} + \leftthreetimes^{\varsigma+1}(1-\leftthreetimes)^2 \Big)$$
$$+ \Big| (\mathcal{H}^{-1})' \mathcal{H}(\omega) \Big| \Big((1-\leftthreetimes)^{\varsigma+3} + \leftthreetimes^{\varsigma}(1-\leftthreetimes)^3 \Big) \Big) d\leftthreetimes \Big\}$$
$$= \frac{\mathcal{H}(\omega) - \mathcal{H}(\upsilon)}{2} \Big\{ \Big(|\Phi(\upsilon)||\mathcal{K}'(\upsilon)| + |\mathcal{K}(\upsilon)||\Phi'(\upsilon)| \Big) X$$
$$+ \Big(|\Phi(\upsilon)||\mathcal{K}'(\omega)| + |\Phi(\omega)||\mathcal{K}'(\upsilon)| + |\mathcal{K}(\upsilon)||\Phi'(\omega)| + |\mathcal{K}(\omega)||\Phi'(\upsilon)| \Big) Y$$
$$+ \Big(|\Phi(\omega)||\mathcal{K}'(\omega)| + |\mathcal{K}(\omega)||\Phi'(\omega)| \Big) Z \Big\}.$$

Hence the proof is done. □

Example 2.10. *The validity of the double inequality presented in Theorem 2.9 for the convex function $\mathcal{K}(\wp) = \wp^2$, $\Phi(\wp) = 1$ and $\mathcal{H}(\wp) = \wp$, can be proved by plotting the graphs. Substituting (15) and (16) into the inequality (20), we can write*

$$- \frac{(\omega-\upsilon)(2\upsilon+2\omega)}{2(\varsigma+1)}$$
$$\leq \frac{\upsilon^2+\omega^2}{2} - \frac{\varsigma}{2(\omega-\upsilon)^\varsigma} \int_\upsilon^\omega [(\omega-\wp)^{\varsigma-1} + (\wp-\upsilon)^{\varsigma-1}]\wp^2 d\wp$$
$$\leq \frac{(\omega-\upsilon)(2\upsilon+2\omega)}{2(\varsigma+1)} \tag{21}$$

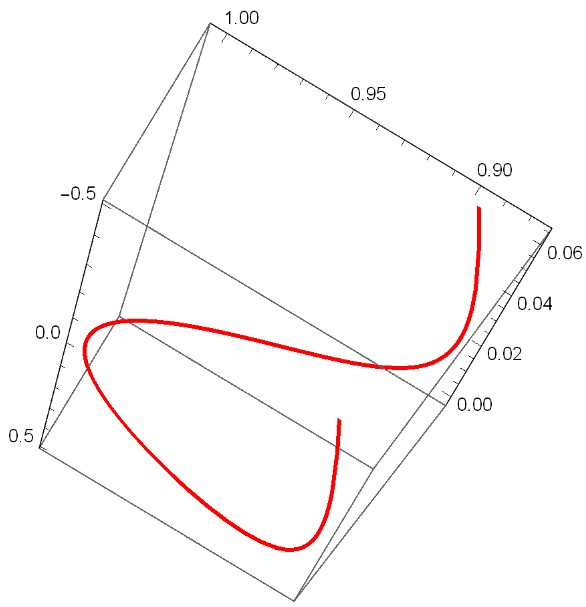

Figure 2: The result of double inequality (21) are illustrated by the graphs corresponding to $\upsilon = 0$ and $\omega = 1$.

The graphs of dual inequality (21) are plotted in Fig. 2 corresponding to $\varsigma \in (0, 1]$. *The Fig. 2 prove that the dual inequality (21) is true for the admissible choice of the parameters.*

3. Conclusions

Fractional calculus is an important branch of mathematics. It opens up new vistas of research for scientists. In continuous progress of fractional calculus, recently Jarad *et al.* introduced new WFI of a function concerning other function [21]. In this paper, we have established Hermite-Hadamard inequalities for the new introduced weighted fractional integral operators via rarely used composite-convex functions. The validity of established results proved by the graphs of certain special functions. This research may motivate the researchers working in mathematical sciences.

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