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Tauberian theorems for the Cesàro summability method of regularly generated double integrals

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Abstract. For a continuous function g over $\mathbb{R}^2_+ := [1,\infty)\times[1,\infty)$, we denote its integral over $[1,x]\times[1,y]$ by $h(x,y) = \int_1^x\int_1^y g(u,v)dudv$ and its $(C,1,1)$ mean, the average of $h(x,y)$ over $[1,x]\times[1,y]$, by $t(h(x,y)) = (xy)^{-1}\int_1^x\int_1^y h(u,v)dudv$. Analogously, the other means $(C,1,0)$ and $(C,0,1)$ can be defined. In this paper, we introduce the concept of regularly generated double integrals in senses $(1,1)$, $(1,0)$ and $(0,1)$ and obtain Tauberian conditions in terms of the regularly generated double integrals in senses $(1,1)$, $(1,0)$ and $(0,1)$ under which convergence of $h(x,y)$ follows from that of $t(h(x,y))$.

1. Introduction

Tauberian conditions based on regularly generated integrals for functions of one variable for Cesàro summability methods have been recently introduced by Çanak and Totur [3, 4, 6]. A Tauberian-like theorem asserting that slow oscillation of improper integral follows from Cesàro summability of a generator of the improper integral is proved by Totur and Çanak [13]. We refer the papers [1, 2, 5, 7, 8, 11, 12, 14] for Tauberian theorems given in terms of the regularly generated sequences of single and double sequences.

Since there aren't studies for regularly generated integrals of functions of two variables, we contribute to this area by extending results given for functions of one variable to functions of two variables. In this paper, the notion of regularly generated integrals for improper double integrals is introduced and a number of Tauberian theorems for Cesàro summability method of improper double integrals by using regularly generated integrals are proved.

The paper is organized as follows. Section 2 is dedicated to recall some basic notions and auxiliary lemmas. Regularly generated integrals in different senses are introduced and well-known Kronecker identities are presented in section 3. Section 4 covers some Tauberian theorems for Cesàro summability method of double integrals by using the notion of regularly generated integrals.

2. Preliminaries

Assume that g is a continuous function on $\mathbb{R}^2_+ := [1,\infty)\times[1,\infty)$ and $h(x,y) = \int_1^x\int_1^y g(u,v)dudv$ for $1 < x, y < \infty$.

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The mean $(C,1,1)$ of $h(x,y)$ is defined by

$$\begin{aligned} t(h(x,y)) &= t_{11}(h(x,y)) := \frac{1}{xy}\int_1^x\int_1^y h(u,v)dudv \\ &= \int_1^x\int_1^y \left(1-\frac{u}{x}\right)\left(1-\frac{v}{y}\right)g(u,v)dudv \end{aligned} \tag{1}$$

for $x,y>1$. The integral

$$\int_1^\infty\int_1^\infty g(u,v)dudv \tag{2}$$

is said to be $(C,1,1)$ summable to s if

$$\lim_{x,y\to\infty} t(h(x,y)) = s. \tag{3}$$

The mean $(C,1,0)$ of $h(x,y)$ is defined by

$$t_{10}(h(x,y)) := \frac{1}{x}\int_1^x h(u,y)du = \int_1^x\int_1^y \left(1-\frac{u}{x}\right)g(u,v)dudv \tag{4}$$

for $x,y>1$. The integral (2) is said to be $(C,1,0)$ summable to s if

$$\lim_{x,y\to\infty} t_{10}(h(x,y)) = s.$$

Similarly, the mean $(C,0,1)$ of $h(x,y)$ is defined by

$$t_{01}(h(x,y)) = \frac{1}{y}\int_1^y h(x,v)dv = \int_1^x\int_1^y \left(1-\frac{v}{y}\right)g(u,v)dudv \tag{5}$$

for $x,y>1$. The integral (2) is said to be $(C,0,1)$ summable to s if

$$\lim_{x,y\to\infty} t_{01}(h(x,y)) = s.$$

Note that we use convergence in Pringsheim's sense [10] throughout this paper.

A function $h(x,y)$ is bounded if there exists a real number $C>0$ such that $\left|h(x,y)\right| \leq C$ for all $x,y>0$. In this case, we write $h(x,y)=O(1)$.

We denote the set of all double integrals which are P-convergent to 0, bounded, one-sided bounded, both P-convergent to 0 and bounded by $\mathfrak{N}$, $\mathfrak{B}$, $\mathfrak{B}^{>}$ and $\mathfrak{N}\mathfrak{B}$, respectively.

The backward difference in sense $(1,1)$ of $h(x,y)$ is defined by

$$\Delta_{11}h(x,y) := \frac{\partial^2 h(x,y)}{\partial x \partial y} = g(x,y)$$

for $x,y>1$.

The $(C,1,1)$ means of $xy\Delta_{11}h(x,y)$ is defined by

$$V_{11}(\Delta_{11}h(x,y)) := \frac{1}{xy}\int_1^x\int_1^y xy\Delta_{11}(h(u,v))dudv = \frac{1}{xy}\int_1^x\int_1^y uvg(u,v)dudv. \tag{6}$$

The backward difference in sense $(1,0)$ of $h(x,y)$ is defined by

$$\Delta_{10}h(x,y) := \frac{\partial h(x,y)}{\partial x} = \int_1^y g(x,v)dv$$

for $x,y>1$.

The $(C,1,0)$ means of $x\Delta_{10}h(x,y)$ is defined by

$$V_{10}(\Delta_{10}h(x,y)) := \frac{1}{x}\int_1^x u\Delta_{10}(h(u,y))du = \frac{1}{x}\int_1^x\int_1^y ug(u,v)dudv. \tag{7}$$

The backward difference in sense $(0,1)$ of $h(x,y)$ is defined by

$$\Delta_{01}h(x,y) := \frac{\partial h(x,y)}{\partial y} = \int_1^x g(u,y)du$$

for $x, y > 1$.

The $(C,0,1)$ means of $y\Delta_{01}h(x,y)$ is defined by

$$V_{01}(\Delta_{01}h(x,y)) := \frac{1}{y}\int_1^y v\Delta_{01}(h(x,v))dv = \frac{1}{y}\int_1^x\int_1^y vg(u,v)dudv. \tag{8}$$

The Kronecker identities for double integrals are given as follows. For $x, y > 1$, we have

$$h(x,y) - t_{10}(h(x,y)) - t_{01}(h(x,y)) + t_{11}(h(x,y)) = V_{11}(\Delta_{11}h(x,y)), \tag{9}$$

$$h(x,y) - t_{10}(h(x,y)) = V_{10}(\Delta_{10}h(x,y)), \tag{10}$$

$$h(x,y) - t_{01}(h(x,y)) = V_{01}(\Delta_{01}h(x,y)). \tag{11}$$

A function $h(x,y)$ on $\mathbb{R}^2_+$ is said to be slowly oscillating in sense $(1,0)$ for $\mu > 1$ (see [9]) if

$$\lim_{\mu\to 1^+}\limsup_{x,y\to\infty}\max_{x\le u\le \mu x}\left|h(u,y) - h(x,y)\right| = 0.$$

Analogously, a function $h(x,y)$ on $\mathbb{R}^2_+$ is said to be slowly oscillating in sense $(0,1)$ for $\mu > 1$ (see [9]) if

$$\lim_{\mu\to 1^+}\limsup_{x,y\to\infty}\max_{y\le v\le \lambda v}\left|h(x,v) - h(x,y)\right| = 0.$$

We denote the sets of all double integrals which are slowly oscillating in senses $(1,0)$ and $(0,1)$ by S_{10} and S_{01}, respectively.

A function $h(x,y)$ on $\mathbb{R}^2_+$ is said to be strong slowly oscillating in sense $(1,0)$ for $\mu > 1$ (see [9]) if

$$\lim_{\mu\to 1^+}\limsup_{x,y\to\infty}\max_{\substack{x\le u\le \mu x\\ y\le v\le \mu y}}|h(u,v) - h(x,v)| = 0.$$

Analogously, a function $h(x,y)$ on $\mathbb{R}^2_+$ is said to be strong slowly oscillating in sense $(0,1)$ for $\mu > 1$ (see [9]) if

$$\lim_{\mu\to 1^+}\limsup_{x,y\to\infty}\max_{\substack{x\le u\le \mu x\\ y\le v\le \mu y}}|h(u,v) - h(u,y)| = 0.$$

We denote the sets of all double integrals which are strong slowly oscillating in senses $(1,0)$ and $(0,1)$ by SS_{10} and SS_{01}, respectively.

Slow oscillation and strong slow oscillation in senses $(1,0)$ and $(0,1)$ for $h(x,y)$ can be analogously defined for $0 < \mu < 1$.

In the following lemma $(C,1,1)$ means of $xy\Delta_{11}h(x,y)$, $(C,1,0)$ means of $x\Delta_{10}h(x,y)$, and $(C,0,1)$ means of $y\Delta_{01}h(x,y)$ are given in terms of $(C,1,1)$ means of $h(x,y)$, $(C,1,0)$ means of $h(x,y)$, and $(C,0,1)$ means of $h(x,y)$, respectively.

Lemma 2.1. *Let $h(x,y)$ be a double integral over $[1,x]\times[1,y]$. Then we have*

$$xy\Delta_{11}(t_{11}(h(x,y))) = V_{11}(\Delta_{11}h(x,y)) \tag{12}$$

$$x\Delta_{10}(t_{10}(h(x,y))) = V_{10}(\Delta_{10}h(x,y)) \tag{13}$$

$$y\Delta_{01}(t_{01}(h(x,y))) = V_{01}(\Delta_{01}h(x,y)) \tag{14}$$

for $x,y>1$.

Proof. First, we prove (12). Taking the backward difference of $t_{11}(h(x,y))$ in sense $(1,1)$, we get

$$\begin{aligned}\Delta_{11}(t_{11}(h(x,y))) &= \frac{\partial^2 t_{11}(h(x,y))}{\partial x\partial y}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\int_1^x\int_1^y\left(1-\frac{u}{x}\right)\left(1-\frac{v}{y}\right)g(u,v)dudv\right)\\ &= \frac{\partial}{\partial x}\int_1^x\int_1^y\left(1-\frac{u}{x}\right)\frac{v}{y^2}g(u,v)dudv\\ &= \int_1^x\int_1^y\frac{u}{x^2}\frac{v}{y^2}g(u,v)dudv. \end{aligned}\tag{15}$$

From (15) we deduce that

$$xy\Delta_{11}(t_{11}(h(x,y))) = \frac{1}{xy}\int_1^x\int_1^y uvg(u,v)dudv = V_{11}(\Delta_{11}h(x,y)).$$

Now, we prove (13). Taking the backward difference of $t_{10}(h(x,y))$ in sense $(1,0)$, we get

$$\begin{aligned}\Delta_{10}(t_{10}(h(x,y))) &= \frac{\partial\sigma_{10}(h(x,y))}{\partial x} = \frac{\partial}{\partial x}\int_1^x\int_1^y\left(1-\frac{u}{x}\right)g(u,v)dudv\\ &= \int_1^x\int_1^y\frac{u}{x^2}g(u,v)dudv. \end{aligned}\tag{16}$$

From (16) we deduce that

$$x\Delta_{10}(t_{10}(h(x,y))) = \frac{1}{x}\int_1^x\int_1^y ug(u,v)dudv. \tag{17}$$

Similarly, it can be easily seen that (14) holds. □

The relationships between different Cesàro means of $h(x,y)$ are given below.

Lemma 2.2. *Let $h(x,y)$ be a double integral over $[1,x]\times[1,y]$. Then we have*

$$t_{10}(t_{01}(h(x,y))) = t_{01}(t_{10}(h(x,y))) = t_{11}(h(x,y)), \tag{18}$$

$$t_{10}(t_{11}(h(x,y))) = t_{11}(t_{10}(h(x,y))), \tag{19}$$

$$t_{01}(t_{11}(h(x,y))) = t_{11}(t_{01}(h(x,y))) \tag{20}$$

for $x,y>1$.

We omit the proof of Lemma 2.2. It can be easily seen from definitions of Cesàro means in different senses.

3. Regularly generated double integrals in different senses

Suppose that $\mathfrak{L}$ is a set of all continuous functions on $[1,x]\times[1,y]$ for each $x,y>0$ and $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ are any nonempty subsets of $\mathfrak{L}$.

A double integral $h(x,y)$ is said to be regularly generated in sense $(1,1)$ if

$$h(x,y)=\xi(x,y)+\nu(x,y)+\int_1^x\int_1^y\frac{\eta(u,v)}{uv}dudv-\eta(x,y) \tag{21}$$

for some $\xi(x,y)\in\mathfrak{D}$, $\nu(x,y)\in\mathfrak{E}$ and $\eta(x,y)\in\mathfrak{F}$. In this case, $\xi(x,y)$, $\nu(x,y)$ and $\eta(x,y)$ are said to be the generators of $h(x,y)$.

The set of all regularly generated integrals by some $\xi=\xi(x,y)$, $\nu=\nu(x,y)$ and $\eta=\eta(x,y)$ is denoted by $C_{11}(\mathfrak{D},\mathfrak{E},\mathfrak{F})$.

A double integral $h(x,y)$ is said to be regularly generated in sense $(1,0)$ if

$$h(x,y)=\xi(x,y)+\int_1^x\frac{\xi(u,y)}{u}du \tag{22}$$

for some $\xi(x,y)\in\mathfrak{D}$.

In this case, $\xi(x,y)$ is said to be the generator of $h(x,y)$. The set of all regularly generated integrals by some $\xi=\xi(x,y)$ is denoted by $C_{10}(\mathfrak{D})$.

A double integral $h(x,y)$ is said to be regularly generated in sense $(0,1)$ if

$$h(x,y)=\nu(x,y)+\int_1^y\frac{\nu(x,v)}{v}dv \tag{23}$$

for some $\nu(x,y)\in\mathfrak{E}$.

In this case, $\nu(x,y)$ is said to be the generator of $h(x,y)$. The set of all regularly generated integrals by some $\nu=\nu(x,y)$ is denoted by $C_{01}(\mathfrak{E})$.

For a double integral $h(x,y)$ over $[1,x]\times[1,y]$, we have by Lemma 2.1

$$t_{11}(h(x,y))=\int_1^x\int_1^y\frac{V_{11}(\Delta_{11}h(u,v))}{uv}dudv, \tag{24}$$

$$t_{10}(h(x,y))=\int_1^x\frac{V_{10}(\Delta_{10}h(u,y))}{u}du, \tag{25}$$

$$t_{01}(h(x,y))=\int_1^y\frac{V_{01}(\Delta_{01}h(x,v))}{v}dv. \tag{26}$$

Indeed, using (12), we get

$$\frac{xy\Delta_{11}(t_{11}(h(x,y)))}{xy}=\frac{V_{11}(\Delta_{11}h(x,y))}{xy}$$

and then we have

$$\int_1^x\int_1^y\frac{\partial^2 t_{11}(h(u,v))}{\partial u\partial v}dudv=\int_1^x\int_1^y\frac{V_{11}(\Delta_{11}(h(u,v)))}{uv}dudv.$$

From the last equation, we obtain (24).

The other identities can be obtained similarly. From Kronecker identities, we get

$$h(x,y)-t_{11}(h(x,y))=V_{10}(\Delta_{10}h(x,y))+V_{01}(\Delta_{01}h(x,y))-V_{11}(\Delta_{11}h(x,y)). \tag{27}$$

From the previous identity and (24), we obtain

$$h(x,y) = V_{10}(\Delta_{10}h(x,y)) + V_{01}(\Delta_{01}h(x,y)) - V_{11}(\Delta_{11}h(x,y)) + \int_1^x \int_1^y \frac{V_{11}(\Delta h(u,v))}{uv} dudv.$$

Hence $V_{10}(\Delta_{10}h(x,y))$, $V_{01}(\Delta_{01}h(x,y))$ and $V_{11}(\Delta_{11}h(x,y))$ are generators of $h(x,y)$. In addition, from (10) and (25), we get

$$h(x,y) = V_{10}(\Delta_{10}h(x,y)) + \int_1^x \frac{V_{10}(\Delta_{10}h(u,y))}{u} du.$$

Hence, $V_{10}(\Delta_{10}h(x,y))$ is the generator of $h(x,y)$. Similarly, from (11) and taking (26) into account, we get

$$h(x,y) = V_{01}(\Delta_{01}h(x,y)) + \int_1^y \frac{V_{01}(\Delta_{01}h(x,v))}{v} dv.$$

Hence, $V_{01}(\Delta_{01}h(x,y))$ is the generator of $h(x,y)$.

Lemma 3.1. *Let $h(x,y) \in \mathfrak{L}$ and $\mathfrak{D}, \mathfrak{E} \subset \mathfrak{L}$.*
(i) *If $h(x,y) \in C_{10}(\mathfrak{D})$, then $V_{10}(\Delta_{10}h(x,y)) \in \mathfrak{D}$.*
(ii) *If $h(x,y) \in C_{01}(\mathfrak{E})$, then $V_{01}(\Delta_{01}h(x,y)) \in \mathfrak{E}$.*

Proof. (i) Since $h(x,y) \in C_{10}(\mathfrak{D})$, then

$$h(x,y) = \xi(x,y) + \int_1^x \frac{\xi(u,y)}{u} du \tag{28}$$

for some $\xi(x,y) \in \mathfrak{D}$. Taking backward difference in sense $(1,0)$ of both sides of (28) and then multiplying both sides by x, we obtain

$$x\Delta_{10}h(x,y) = x\Delta_{10}\xi(x,y) + x\frac{\xi(x,y)}{x}. \tag{29}$$

Taking $(C,1,0)$ means of both sides of (29), we have

$$V_{10}(\Delta_{10}h(x,y)) = V_{10}(\Delta_{10}\xi(x,y)) + t_{10}(\xi(x,y)). \tag{30}$$

Replacing $h(x,y)$ by $\xi(x,y)$ in Kronecker identity (10), we obtain

$$\xi(x,y) - t_{10}(\xi(x,y)) = V_{10}(\Delta_{10}\xi(x,y)). \tag{31}$$

From (30) and (31), we see that $V_{10}(\Delta_{10}h(x,y)) = \xi(x,y)$. Therefore, we conclude that $V_{10}(\Delta_{10}h(x,y)) \in \mathfrak{D}$.
(ii) The proof of (ii) can be done in a similar way. □

In proving our main results, we use the following theorems and lemmas given by Móricz [9].

Theorem 3.2. *Let $h(x,y) = O(1)$.*
(i) *If (2) is $(C,1,0)$ summable to s and and there exist constants $H > 0$ and $u_0 \geq 0$ such that $x\Delta_{10}h(x,y) \geq -H$ is satisfied for all $(x,y) \in \mathbb{R}_+^2$ with $x, y > u_0$, then $h(x,y)$ is convergent to s.*
(ii) *If (2) is $(C,0,1)$ summable to s and and there exist constants $H > 0$ and $u_0 \geq 0$ such that $y\Delta_{01}h(x,y) \geq -H$ is satisfied for all $(x,y) \in \mathbb{R}_+^2$ with $x, y > u_0$, then $h(x,y)$ is convergent to s.*

Theorem 3.3. *Let $h(x,y) = O(1)$.*
(i) *If (2) is $(C,1,0)$ summable to s and $h(x,y) \in S_{10}$, then $h(x,y)$ is convergent to s.*
(ii) *If (2) is $(C,0,1)$ summable to s and $h(x,y) \in S_{01}$, then $h(x,y)$ is convergent to s.*

Lemma 3.4. *Let $h(x,y)$ be a double integral over $[1,x]\times[1,y]$. For sufficiently large x and y:*
(i) *If $\mu>1$,*

$$\begin{aligned} h(x,y)-t_{11}(h(x,y)) &= \left(\frac{\mu}{\mu-1}\right)^2(t_{11}(h(\mu x,\mu y))-t_{11}(h(x,y)))\\ &+ \frac{\mu}{(\mu-1)^2}(t_{11}(h(x,y))-t_{11}(h(\mu x,y)))\\ &+ \frac{\mu}{(\mu-1)^2}(t_{11}(h(x,y))-t_{11}(h(x,\mu y)))\\ &- \frac{1}{(\mu x-x)(\mu y-y)}\int_x^{\mu x}\int_y^{\mu y}(h(u,v)-h(x,y))\,dudv. \end{aligned}$$

(ii) *If $0<\mu<1$,*

$$\begin{aligned} h(x,y)-t_{11}(h(x,y)) &= \left(\frac{\mu}{1-\mu}\right)^2(t_{11}(h(\mu x,\mu y))-t_{11}(h(x,y)))\\ &+ \frac{\mu}{(1-\mu)^2}(t_{11}(h(x,y))-t_{11}(h(\mu x,y)))\\ &+ \frac{\mu}{(1-\mu)^2}(t_{11}(h(x,y))-t_{11}(h(x,\mu y)))\\ &+ \frac{1}{(x-\mu x)(y-\mu y)}\int_{\mu x}^{x}\int_{\mu y}^{y}(h(x,y)-h(u,v))\,dudv. \end{aligned}$$

Lemma 3.5. *Let $h(x,y)$ be a double integral over $[1,x]\times[1,y]$. For sufficiently large x and y:*
(i) *If $\mu>1$,*

$$h(x,y)-t_{10}(h(x,y)) = \frac{\mu}{\mu-1}(t_{10}(h(\mu x,y))-t_{10}(h(x,y)))-\frac{1}{\mu x-x}\int_x^{\mu x}(h(u,y)-h(x,y))\,du.$$

(ii) *If $0<\mu<1$,*

$$h(x,y)-t_{10}(h(x,y)) = \frac{\mu}{1-\mu}(t_{10}(h(x,y))-t_{10}(h(\mu x,y)))+\frac{1}{x-\mu x}\int_{\mu x}^{x}(h(x,y)-h(u,y))\,du.$$

4. Main results

Theorem 4.1. *If (2) is $(C,1,1)$ summable to s and $h(x,y)\in C_{11}(\mathfrak{RB},\mathfrak{RB},\mathfrak{RB})$, $h(x,y)\in C_{10}(\mathfrak{R})$ and $h(x,y)\in C_{01}(\mathfrak{R})$, then $h(x,y)$ is convergent to s.*

Proof. Assume that (2) is $(C,1,1)$ summable to s. Since $h(x,y)\in C_{10}(\mathfrak{R})$, we have

$$V_{10}(\Delta_{10}h(x,y))\in\mathfrak{R} \tag{32}$$

by Lemma 3.1 (i). Since $h(x,y)\in C_{01}(\mathfrak{R})$, we have

$$V_{01}(\Delta_{01}h(x,y))\in\mathfrak{R} \tag{33}$$

by Lemma 3.1 (ii). It follows by the assumption $h(x,y)\in C_{11}(\mathfrak{RB},\mathfrak{RB},\mathfrak{RB})$ that

$$h(x,y)=\xi(x,y)+\nu(x,y)+\int_1^x\int_1^y\frac{\eta(u,v)}{uv}dudv-\eta(x,y) \tag{34}$$

for some $\xi(x,y) \in \mathfrak{RB}$, $\nu(x,y) \in \mathfrak{RB}$, $\eta(x,y) \in \mathfrak{RB}$. Taking the backward difference in sense $(1,1)$ of both sides of (34), we obtain

$$\Delta_{11}h(x,y) = \Delta_{11}\xi(x,y) + \Delta_{11}\nu(x,y) + \frac{\eta(x,y)}{xy} - \Delta_{11}\eta(x,y). \tag{35}$$

Multiplying both sides of (35) by xy and then taking $(C,1,1)$ means of both sides, we get

$$V_{11}(\Delta_{11}h(x,y)) \quad = \quad V_{11}(\Delta_{11}\xi(x,y)) + V_{11}(\Delta_{11}\nu(x,y)) + \sigma_{11}(\eta(x,y)) - V_{11}(\Delta_{11}\eta(x,y)). \tag{36}$$

Replacing $h(x,y)$ by $\xi(x,y)$, $\nu(x,y)$ and $\eta(x,y)$ in Kronecker identities (9), (10) and (11), respectively, we have

$$\xi(x,y) - t_{10}(\xi(x,y)) - t_{01}(\xi(x,y)) + t_{11}(\xi(x,y)) = V_{11}(\Delta_{11}\xi(x,y)), \tag{37}$$

$$\nu(x,y) - t_{10}(\nu(x,y)) = V_{10}(\Delta_{10}\nu(x,y)), \tag{38}$$

and

$$\eta(x,y) - t_{01}(\eta(x,y)) = V_{01}(\Delta_{01}\eta(x,y)). \tag{39}$$

Since $\xi(x,y)$, $\nu(x,y)$ and $\eta(x,y)$ are assumed to be bounded, $(C,1,1)$, $(C,1,0)$ and $(C,0,1)$ means of $\xi(x,y)$, $\nu(x,y)$ and $\eta(x,y)$ is convergent to 0. Hence using (37), we conclude that $V_{11}(\Delta_{11}\xi(x,y)) \in \mathfrak{R}$. If we replace $\nu(x,y)$ by $V_{01}(\Delta_{01}\nu(x,y))$ and $\eta(x,y)$ by $V_{10}(\Delta_{10}\eta(x,y))$ in (38) and (39), respectively, we obtain that $V_{11}(\Delta_{11}\nu(x,y)) \in \mathfrak{R}$ and $V_{11}(\Delta_{11}\eta(x,y)) \in \mathfrak{R}$. Thus from (36), we get

$$V_{11}(\Delta_{11}h(x,y)) \in \mathfrak{R}. \tag{40}$$

Therefore we conclude that $h(x,y)$ is convergent to s by using (27) from (32),(33) and (40). □

Remark 4.2. *If $h(x,y) \in \mathfrak{B}$ as in Theorem 4.1, then condition $h(x,y) \in C_{11}(\mathfrak{RB},\mathfrak{RB},\mathfrak{RB})$ is omitted. Indeed, if we replace $h(x,y)$ by $V_{10}(\Delta_{10}h(x,y))$ in (11), we get*

$$V_{10}(\Delta_{10}h(x,y)) - t_{01}(V_{10}(\Delta_{10}h(x,y))) = V_{11}(\Delta_{11}h(x,y)). \tag{41}$$

Because of $h(x,y) \in \mathfrak{B}$ and $V_{10}(\Delta_{10}h(x,y)) \in \mathfrak{R}$, we conclude that $t_{01}(V_{10}(\Delta_{10}h(x,y))) \in \mathfrak{R}$ by regularity. Thus, we obtain $V_{11}(\Delta_{11}h(x,y)) \in \mathfrak{R}$ by (41).

Theorem 4.3. *Let $h(x,y) = O(1)$. If (2) is $(C,1,1)$ summable to s and $V_{01}(\Delta_{01}h(x,y)) \in C_{10}(S_{10})$ and $V_{10}(\Delta_{10}h(x,y)) \in C_{01}(SS_{01})$ (or $V_{10}(\Delta_{10}h(x,y)) \in C_{01}(S_{01})$ and $V_{01}(\Delta_{01}h(x,y)) \in C_{10}(SS_{10})$), then $h(x,y)$ is convergent to s.*

Proof. Assume that (2) is $(C,1,1)$ summable to s. Replacing $h(x,y)$ by $V_{11}(\Delta_{11}h(x,y))$ in Lemma 3.4 (i), we get

$$\begin{aligned}
&V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\\
&= \left(\frac{\mu}{\mu-1}\right)^2 (t_{11}(V_{11}(\Delta_{11}h(\mu u,\mu y))) - t_{11}(V_{11}(\Delta_{11}h(x,y))))\\
&+ \frac{\mu}{(\mu-1)^2} (t_{11}(V_{11}(\Delta_{11}h(x,y))) - t_{11}(V_{11}(\Delta_{11}h(\mu x,y))))\\
&+ \frac{\mu}{(\mu-1)^2} (t_{11}(V_{11}(\Delta_{11}h(x,y))) - t_{11}(V_{11}(\Delta_{11}h(x,\mu y))))\\
&\qquad - \frac{1}{(\mu x - x)(\mu y - y)} \int_x^{\mu x}\int_y^{\mu y} (V_{11}(\Delta_{11}h(u,v)) - V_{11}(\Delta_{11}h(x,y)))\,dudv
\end{aligned} \tag{42}$$

for $\mu > 1$. From the above equality, we have

$$\left|V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\right| \leq \left(\frac{\mu}{\mu-1}\right)^2 \left|t_{11}(V_{11}(\Delta_{11}h(\mu x,\mu y))) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\right|$$
$$+\frac{\mu}{(\mu-1)^2}\left|t_{11}(V_{11}(\Delta_{11}h(x,y))) - t_{11}(V_{11}(\Delta_{11}h(\mu x,y)))\right| + \frac{\mu}{(\mu-1)^2}\left|t_{11}(V_{11}(\Delta_{11}h(x,y))) - t_{11}(V_{11}(\Delta_{11}h(x,\mu y)))\right|$$
$$+\max_{\substack{x\leq u\leq \mu x\\ y\leq v\leq \mu y}}\left|V_{11}(\Delta_{11}h(u,v)) - V_{11}(\Delta_{11}h(x,y))\right|.$$

Since the $(C,1,1)$, $(C,1,0)$ and $(C,0,1)$ summability methods are regular under the boundedness condition of $h(x,y)$, we obtain that $t_{11}(h(x,y))$, $t_{10}(h(x,y))$ and $t_{01}(h(x,y))$ are $(C,1,1)$ summable to s by using Lemma 2.2. Taking $(C,1,1)$ means of Kronecker equality (9), we conclude that $V_{11}(\Delta_{11}h(x,y))$ is $(C,1,1)$ summable to 0. The first three terms on the right-hand side of the previous inequality are vanished and then we obtain

$$\left|V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\right| \leq \max_{\substack{x\leq u\leq \mu x\\ y\leq v\leq \mu y}}\left|V_{11}(\Delta_{11}h(u,v)) - V_{11}(\Delta_{11}h(u,y))\right|$$
$$+\max_{x\leq u\leq \mu x}\left|V_{11}(\Delta_{11}h(u,y)) - V_{11}(\Delta_{11}h(x,y))\right|. \quad (43)$$

Taking the limit superior of (43) as $x,y\to\infty$ and then taking the limit of the resulting inequality as $\mu\to 1^+$, we have

$$\lim_{\mu\to 1^+}\limsup_{x,y\to\infty}\left|V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\right| \leq \lim_{\mu\to 1^+}\limsup_{x,y\to\infty}\max_{\substack{x\leq u\leq \mu x\\ y\leq v\leq \mu y}}\left|V_{11}(\Delta_{11}h(u,v)) - V_{11}(\Delta_{11}h(u,y))\right|$$
$$+\lim_{\mu\to 1^+}\limsup_{x,y\to\infty}\max_{x\leq u\leq \mu x}\left|V_{11}(\Delta_{11}h(u,y)) - V_{11}(\Delta_{11}h(x,y))\right|.$$

Since $V_{01}(\Delta_{01}h(x,y)) \in C_{10}(S_{10})$ and $V_{10}(\Delta_{10}h(x,y)) \in C_{01}(SS_{01})$, then we have $V_{11}(\Delta_{11}h(x,y)) \in S_{10}$ and $V_{11}(\Delta_{11}h(x,y)) \in SS_{01}$ by Lemma 3.1. We then have

$$\limsup_{x,y\to\infty}\left|V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\right| \leq 0.$$

So we conclude

$$\lim_{x,y\to\infty} V_{11}(\Delta_{11}h(x,y)) = 0. \quad (44)$$

Now we prove that $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ converges to zero. Because of this, we use Theorem 3.3 (i). Firstly, we indicate that $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ is $(C,1,0)$ summable to zero and then $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ is slowly oscillating in sense $(1,0)$.

Taking $(C,0,1)$ means of Kronecker identity (10), we get

$$t_{01}(h(x,y)) - t_{01}(t_{10}(h(x,y))) = t_{01}(V_{10}(\Delta_{10}h(x,y))).$$

If we take $(C,1,0)$ means of the previous equality, we get

$$t_{11}(h(x,y)) - t_{10}(t_{11}(h(x,y))) = t_{10}(t_{01}(V_{10}(\Delta_{10}h(x,y))))$$

by Lemma 2.2. By regularity and $(C,1,1)$ summability, we conclude that $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ is $(C,1,0)$ summable to zero.

Replacing $h(x,y)$ by $V_{10}(\Delta_{10}h(x,y))$ in Kronecker identity (11), we get

$$\begin{aligned} V_{10}(\Delta_{10}h(x,y)) - t_{01}(V_{10}(\Delta_{10}h(x,y))) &= V_{01}(\Delta_{01}V_{10}(\Delta_{10}h(x,y))) \\ &= V_{11}(\Delta_{11}h(x,y)). \end{aligned} \quad (45)$$

Since $V_{11}(\Delta_{11}h(x,y))$ converges to zero, $V_{11}(\Delta_{11}h(x,y))$ is bounded and slowly oscillating in sense $(1,0)$, $(0,1)$ and $(1,1)$. In the light of this information , we obtain $V_{10}(\Delta_{10}h(x,y))$ is slowly oscillating in sense $(1,0)$, $(0,1)$ and $(1,1)$ by (45). Therefore, we conclude that $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ is slowly oscillating in sense $(1,0)$. Since $t_{01}(V_{10}\Delta_{10}h(x,y))$ and $V_{11}(\Delta_{11}h(x,y))$ converge to zero, we obtain

$$\lim_{x,y\to\infty} V_{10}(\Delta_{10}h(x,y)) = 0 \tag{46}$$

by (45). Similarly, it can be obtained that

$$\lim_{x,y\to\infty} V_{01}(\Delta_{01}h(x,y)) = 0 \tag{47}$$

by Theorem 3.3 (ii). Considering (27), we conclude that $h(x,y)$ converges to s by (44), (46) and (47). □

Theorem 4.4. *Let $h(x,y) = O(1)$. If (2) is $(C,1,1)$ summable to s, $x\Delta_{10}h(x,y) \in C_{10}(\mathfrak{B}^{>})$, $y\Delta_{01}h(x,y) \in C_{01}(\mathfrak{B}^{>})$, $x\Delta_{10}V_{10}(\Delta_{10}h(x,y)) \in C_{10}(\mathfrak{B}^{>})$ and $y\Delta_{01}V_{01}(\Delta_{01}h(x,y)) \in C_{01}(\mathfrak{B}^{>})$, then $h(x,y)$ is convergent to s.*

Proof. Assume that (2) is $(C,1,1)$ summable to s. Taking the limit superior of both sides of (42) as $x,y\to\infty$, we have

$$\begin{aligned}
&\limsup_{x,y\to\infty} \left(V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\right) \leq \\
&\left(\frac{\mu}{\mu-1}\right)^2 \limsup_{x,y\to\infty} \left(t_{11}(V_{11}(\Delta_{11}h(\mu u,\mu v))) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\right) \\
&+ \frac{\mu}{(\mu-1)^2} \limsup_{x,y\to\infty} \left(t_{11}(V_{11}(\Delta_{11}h(x,y))) - t_{11}(V_{11}(\Delta_{11}h(\mu x,y)))\right) \\
&+ \frac{\mu}{(\mu-1)^2} \limsup_{x,y\to\infty} \left(t_{11}(V_{11}(\Delta_{11}h(x,y))) - t_{11}(V_{11}(\Delta_{11}h(x,\mu y)))\right) \\
&\qquad + \limsup_{x,y\to\infty} \left(-\frac{1}{(\mu x - x)(\mu y - y)} \int_x^{\mu x}\int_y^{\mu y} \left(V_{11}(\Delta_{11}h(u,v)) - V_{11}(\Delta_{11}h(x,y))\right) du dv\right).
\end{aligned}$$

As in the proof of Theorem 4.3, we have that $V_{11}(\Delta_{11}h(x,y))$ is $(C,1,1)$ summable to 0. The first three terms on the right-hand side of the previous inequality vanish and we obtain

$$\begin{aligned}
&\limsup_{x,y\to\infty} \left(V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))\right) \leq \\
&\qquad + \limsup_{x,y\to\infty} \left(-\frac{1}{(\mu x - x)(\mu y - y)} \int_x^{\mu x}\int_y^{\mu y} \left(V_{11}(\Delta_{11}h(u,v)) - V_{11}(\Delta_{11}h(x,y))\right) du dv\right).
\end{aligned} \tag{48}$$

Since $x\Delta_{10}V_{10}(\Delta_{10}h(x,y)) \in C_{10}(\mathfrak{B}^{>})$ and $y\Delta_{01}V_{01}(\Delta_{01}h(x,y)) \in C_{01}(\mathfrak{B}^{>})$, then we have $x\Delta_{10}V_{11}(\Delta_{11}h(x,y)) \in \mathfrak{B}^{>}$ and $y\Delta_{01}V_{11}(\Delta_{11}h(x,y)) \in \mathfrak{B}^{>}$ by Lemma 3.1, respectively. Therefore, we get

$$\begin{aligned}
V_{11}(\Delta_{11}h(x,y)) - V_{11}(\Delta_{11}h(u,v)) &= \int_u^x \frac{\partial V_{11}(\Delta_{11}h(r,v))}{\partial r} dr \\
&+ \int_v^y \frac{\partial V_{11}(\Delta_{11}h(x,t))}{\partial t} dt \\
&\geq -H\left(\int_u^x \frac{dr}{r} + \int_v^y \frac{dt}{t}\right) \\
&= -H\left(\ln\left(\frac{x}{u}\right) + \ln\left(\frac{y}{v}\right)\right)
\end{aligned} \tag{49}$$

for some $H > 0$. From (48) and (49), we have

$$\limsup_{x,y\to\infty} (V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))) \leq 2H\left(\ln\left(\frac{1}{\mu}\right)\right).$$

From the last inequality, we have

$$\limsup_{x,y\to\infty} (V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))) \leq 0. \tag{50}$$

For $0 < \mu < 1$, in a similar way by using Lemma 3.4 (ii) we have

$$\liminf_{x,y\to\infty} (V_{11}(\Delta_{11}h(x,y)) - t_{11}(V_{11}(\Delta_{11}h(x,y)))) \geq 0. \tag{51}$$

From (50) and (51), we conclude

$$\liminf_{x,y\to\infty} V_{11}(\Delta_{11}h(x,y)) = 0. \tag{52}$$

Now we show that $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ converges to zero. Because of this, we use Theorem 3.2 (i). Firstly we indicate that $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ is $(C,1,0)$ summable to zero and then $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ is one-sided bounded.

As in the proof of Theorem 4.3, we obtain that $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ is $(C,1,0)$ summable to zero.

Moreover, by hypothesis, since $x\Delta_{10}h(x,y) \in C_{10}(\mathfrak{B}^{>})$, $y\Delta_{01}h(x,y) \in C_{01}(\mathfrak{B}^{>})$, then we have $x\Delta_{10}V_{10}(\Delta_{10}h(x,y)) \in \mathfrak{B}^{>}$ and $y\Delta_{01}V_{01}(\Delta_{01}h(x,y)) \in \mathfrak{B}^{>}$ by Lemma 3.1, respectively. If we replace $h(x,y)$ by $V_{10}(\Delta_{10}h(x,y))$ in Kronecker identity (11), we get

$$V_{10}(\Delta_{10}h(x,y)) - t_{01}(V_{10}(\Delta_{10}h(x,y))) = V_{11}(\Delta_{11}h(x,y)). \tag{53}$$

Taking the backward difference in sense $(1,0)$ of both sides of above equality and after by multiplying both sides of its by x, we have

$$x\Delta_{10}V_{10}(\Delta_{10}h(x,y)) - x\Delta_{10}t_{01}(V_{10}(\Delta_{10}h(x,y))) = x\Delta_{10}V_{11}(\Delta_{11}h(x,y)).$$

Hence, we obtain that $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ converges to zero. Since $x\Delta_{10}V_{10}(\Delta_{10}h(x,y)) \in \mathfrak{B}^{>}$ and $x\Delta_{10}V_{11}(\Delta_{11}h(x,y)) \in \mathfrak{B}^{>}$, we conclude $x\Delta_{10}t_{01}(V_{10}(\Delta_{10}h(x,y))) \in \mathfrak{B}^{>}$. Since $t_{01}(V_{10}(\Delta_{10}h(x,y)))$ and $V_{11}(\Delta_{11}h(x,y))$ converge to zero, we obtain

$$\lim_{x,y\to\infty} V_{10}(\Delta_{10}h(x,y)) = 0 \tag{54}$$

by (53) . Similarly, it can be obtained that

$$\lim_{x,y\to\infty} V_{01}(\Delta_{01}h(x,y)) = 0 \tag{55}$$

by Theorem 3.2 (ii). Considering (27), we conclude that $h(x,y)$ converges to s by (52), (54) and (55). □

5. Conclusion

In this paper, we define the concept of regularly generated integrals in different senses and study Tauberian theorems for Cesàro summability method of double integrals by means of this newly defined concept. For future research, we plan to extend the Tauberian results for Cesàro summability method of double integrals to the weighted mean method of double integrals.

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