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An algorithmic approach for a system of extended multi-valued variational-like inclusions in Banach spaces

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Abstract. In this paper, we pursue two purposes. Our first goal is to study a new system of extended multi-valued nonlinear variational inclusions in Banach spaces and to establish its equivalence with a system of fixed point problems with the help of the concept of (H, η)-proximal mapping. The obtained alternative equivalent formulation is used and a new iterative algorithm for finding its approximate solution is proposed. Under some appropriate assumptions imposed on the mappings and parameters involved in the system of extended multi-valued nonlinear variational inclusions, the existence of solution for the system mentioned above is proved and the convergence analysis of the sequences generated by our suggested iterative algorithm is discussed. The second objective of this paper is to investigate and analyze the notion of C_n-η-monotone mapping, which is an extension of the concept of C_n-monotone mapping, and to point out some remarks relating to C_n-η-monotone mapping and the results concerning it appeared in the literature. The results presented in this paper are new, and improve and generalize many known corresponding results.

1. Introduction

The theory of variational inequalities which its history can be traced back to 1959 with the work of Signorini [24], in which the author posed the first problem involving a variational inequality the so-called Signorini contact problem, plays an important role in many different areas of mathematics such as optimization theory, economic equilibrium, partial differential equations, mechanics, management, engineering, etc. In fact, variational inequalities and their generalizations have been recognized as suitable mathematical models for dealing with many problems arising in the fields mentioned above. Later, because of its wide applications in different branches of sciences, the theory of variational inequality has been extended and generalized in many different directions. There is no doubt that among the generalizations, variational inclusion is one of the most interesting and well-known ones and this is the reason why in the last two decades

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many researchers have shown interest in studying various classes of variational inclusion problems. For additional references among with more details, the reader is referred to [1–13, 15–20, 22, 25–28, 32] and the references therein.

It is worthwhile to emphasize that the existence of solutions and approximation of solutions by the iterative algorithms are two important problems in the theory of variational inequalities and their generalizations. For this reason, in recent decades, several numerical methods have been devised for solving variational inequalities and related optimization problems in Euclidean and Hilbert spaces, such as the projection methods and its variant forms, linear approximation, descent method, Newton's method and the method based on auxiliary principle technique. In particular, the method based on the resolvent operator technique is a generalization of the projection method and has been widely used for solving variational inclusions. Over the last few decades, the study of problems and equations with monotone and accretive operators have been one of the most active research areas of optimization theory and nonlinear functional analysis. The study of the notion of invexity as an important and significant generalization of convexity was first made by Hanson [14] in 1981. By replacing the linear term $y - x$ appearing in the formulation of variational inequalities by a vector-valued term $\eta(y, x)$, where η is a vector-valued bifunction, Parida et al. [23] and Yang and Chen [31] introduced, independently, the notion of variational-like inequality or pre-variational inequality. Here it is to be noted that due to the nature of variational-like inequalities, that is the involvement of the vector-valued term $\eta(y, x)$ in the formulations of variational-like inequalities, among numerical techniques available in the literature, only few of them can be used to compute approximate solutions of variational-like inequality problems. The auxiliary principle technique and the proximal method are the most studied methods for solving variational-like inequalities. The introduction of the notions of η-subdifferential and η-proximal point mappings of a proper functional was first made by Ding and Luo [8] and Lee et al. [18], independently. In 2002, Ding and Xia [9] succeeded to introduce the notion of J-proximal mapping for a lower semicontinuous subdifferentiable proper (may not be convex) functional on reflexive Banach spaces, and proved its existence and Lipschitz continuity under some suitable conditions. Attempts in this direction have been continued and further resolvent operators have been introduced. For example, in 2005, Ahmad et al. [1] and Kazmi and Bhat [16] succeeded, independently, to introduce the notion of J^η-proximal (also referred to as P-η-proximal) mapping for a nonconvex lower semicontinuous η-subdifferentiable proper functional on reflexive Banach spaces as an extension of the concept of J-proximal mapping appeared in [9]. The existence and Lipschitz continuity of such proximal mappings have been proved under some appropriate conditions in [1, 16]. They also proposed some iterative algorithms for solving some classes of generalized multivalued nonlinear variational-like inequalities in the framework of Banach spaces.

Inspired by their wide applications in modern optimization and variational analysis, during the last two decades, much attention has given to develop and generalize the notion of maximal monotone operator. One of the first efforts in this direction was carried out by Fang and Huang [10] in 2003, who introduced the concept of H-monotone operator and defined the resolvent operator associated with it. Subsequently, Xia and Huang [28] introduced the concept of general H-monotone mapping as a generalization of the notions of J-proximal mapping [9] and H-monotone operator [10]. They defined the proximal mapping associated with general H-monotone operator, which is different from the resolvent operator associated with the H-accretive operator considered and studied by Fang and Huang [11]. They also introduced a new class of variational inclusions with general H-monotone operator and constructed an iterative algorithm for solving this class of variational inclusions by using the proximal mapping. The efforts in this direction have been continued and in 2010, Luo and Huang [20] introduced the concept of B-monotone operator as a generalization of general H-monotone mapping and by using the notion of the proximal mapping, they constructed an iterative algorithm for solving a class of variational inclusions involving B-monotone operators in Banach spaces. Two years later, Nazemi [22] introduced and studied the notion of C_n-monotone mappings as a generalization of general H-monotone and B-monotone operators. She considered a class of variational inclusions involving C_n-monotone mappings in Banach spaces and suggested an iterative algorithm for solving this class of variational inclusions by using the technique of proximal mapping. In the meanwhile, she discussed the convergence of the sequences generated by the proposed iterative algorithm under some suitable conditions. One year later, Guan and Hu [13] introduced and studied the notion C_n-η-monotone mapping as an extension of C_n-monotone mapping. They considered a class of variational inclusions involving C_n-η-monotone mapping which is a generalized form of the class of variational inclusions involving C_n-monotone mapping considered in [22]. They proposed a new proximal mapping and proved its Lipschitz continuity and suggested an iterative algorithm by using the new proximal mapping. They also studied the convergence analysis of the sequences generated by the suggested iterative algorithm under some appropriate conditions.

The rest of the paper is organized as follows. Section 2 recalls the basic definitions and preliminaries concerning general (H,η)-monotone operator and its associated proximal mappings in a q-uniformly smooth Banach space setting that are broadly used throughout the whole paper. This section is ended with a new conclusion, in which the Lipschitz continuity of the proximal-point mapping associated with a general (H,η)-monotone operator is proved and a new estimate of its Lipschitz constant is computed. In Sect. 3, a new system of extended multi-valued nonlinear variational inclusions (for short, SEMNVI) is considered and its equivalence with a system of fixed point problems is demonstrated. By using the obtained equivalence, an iterative algorithm for finding an approximate solution of the SEMNVI is constructed. As an application of the proposed algorithm, at the end of Sect. 3, under some suitable assumptions imposed on the parameters and operators, the strong convergence of the sequences generated by our suggested iterative algorithm to the solution of the SEMNVI is proved. Section 4 is devoted to the investigation and analysis of the notion of C_n-η-monotone mapping introduced and studied in [13]. We point out that under the conditions imposed on C_n-η-monotone mapping in [13], every C_n-η-monotone mapping is actually a general (H,η)-monotone operator and is not a new one. Moreover, we review and investigate the results appeared in [13] and by pointing out some comments regarding them, we show that one can deduce all the conclusions existing in [13] with the aid of the results given in the previous sections.

2. Preliminaries and Basic Results

Let E be a real Banach space with the topological dual space E^*. Suppose that $CB(E)$ denote the family of all the nonempty closed and bounded subsets of E. Furthermore, let $\widehat{H}(.,.)$ be the Hausdorff metric on $CB(E)$ defined by

$$\widehat{H}(A,B)=\max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\sup_{y\in B}\inf_{x\in A}\|x-y\|\},\quad \forall A,B\in CB(E).$$

For a real constant $q>1$, the generalized duality mapping $J_q: E\rightrightarrows E^*$ is defined by

$$J_q(x)=\{f^*\in E^*:\langle x,f^*\rangle=\|x\|^q,\|f^*\|=\|x\|^{q-1}\},\quad \forall x\in E.$$

In particular, $J_2=J$ is the usual normalized duality mapping. It is known that, in general, $J_q(x)=\|x\|^{q-2}J_2(x)$, for all $x\neq 0$ and J_q is single-valued if E^* is strictly convex. We recall that a Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2}<1$ for all $x,y\in U=\{z\in E:\|z\|=1\}$ with $x\neq y$. If E is a Hilbert space, then J_2 becomes the identity mapping on E.

The modulus of smoothness of E is the function $\rho_E:[0,+\infty)\to[0,+\infty)$ defined by

$$\rho_E(t)=\sup\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1:\|x\|\leq 1,\|y\|\leq t\}.$$

A Banach space E is called *uniformly smooth* if $\lim\limits_{t\to 0}\frac{\rho_E(t)}{t}=0$.

For a real constant $q>1$, a Banach space E is called q-uniformly smooth if there exists a constant $C>0$ such that $\rho_E(t)\leq Ct^q$ for all $t\in[0,+\infty)$. It is well known that (see e.g. [29]) L_q (or l_q) is q-uniformly smooth for $1<q\leq 2$ and is 2-uniformly smooth if $q\geq 2$. Note that J_q is single-valued if E is uniformly smooth.

In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [29] proved the following result.

Lemma 2.1. *Let E be a real uniformly smooth Banach space. For a real constant $q > 1$, E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

We also recall the following concepts and some known results which shall be used in the sequel.

Definition 2.2. *Let E be a real Banach space with the dual space E^* and let $\eta : E\times E \to E$ be a vector-valued mapping. A single-valued mapping $T : E \to E^*$ is said to be*

(i) *monotone if*

$$\langle T(x) - T(y), x - y\rangle \geq 0, \quad \forall x, y \in E;$$

(ii) *η-monotone if*

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq 0, \quad \forall x, y \in E;$$

(iii) *k-strongly monotone if there exists a constant $k > 0$ such that*

$$\langle T(x) - T(y), x - y\rangle \geq k\|x - y\|^2, \quad \forall x, y \in E;$$

(iv) *γ-strongly η-monotone if there exists a constant $\gamma > 0$ such that*

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq \gamma\|x - y\|^2, \quad \forall x, y \in E;$$

(v) *ρ-Lipschitz continuous if there exists a constant $\rho > 0$ such that*

$$\|T(x) - T(y)\| \leq \rho\|x - y\|, \quad \forall x, y \in E.$$

Definition 2.3. *Let E be a real Banach space with the dual space E^* and let $\eta : E\times E \to E$ be a vector-valued mapping. A multi-valued mapping $\widehat{M} : E \rightrightarrows E$ is said to be*

(i) *monotone if*

$$\langle u - v, x - y\rangle \geq 0, \quad \forall x, y \in E, u \in \widehat{M}(x), v \in \widehat{M}(y);$$

(ii) *η-monotone if*

$$\langle u - v, \eta(x, y)\rangle \geq 0, \quad \forall x, y \in E, u \in \widehat{M}(x), v \in \widehat{M}(y);$$

(iii) *strongly monotone with constant r (or r-strongly monotone) if there exists a constant $r > 0$ such that*

$$\langle u - v, \eta(x, y)\rangle \geq r\|x - y\|^2, \quad \forall x, y \in E, u \in \widehat{M}(x), v \in \widehat{M}(y);$$

(iv) *strongly η-monotone with constant θ (or θ-strongly η-monotone) if there exists a constant $\theta > 0$ such that*

$$\langle u - v, \eta(x, y)\rangle \geq \theta\|x - y\|^2, \quad \forall x, y \in E, u \in \widehat{M}(x), v \in \widehat{M}(y).$$

Definition 2.4. [15] *A multi-valued operator $\widehat{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ is said to be*

(i) *maximal monotone if $\widehat{M}$ is monotone and $(I + \lambda\widehat{M})(\mathcal{H}) = \mathcal{H}$ holds for all $\lambda > 0$, where I stands for the identity mapping on $\mathcal{H}$;*

(ii) *maximal η-monotone if $\widehat{M}$ is η-monotone and $(I + \lambda\widehat{M})(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$.*

Here it is to be noted that $\widehat{M}$ is a maximal η-monotone operator if and only if $\widehat{M}$ is η-monotone and there is no other η-monotone operator whose graph contains strictly Graph($\widehat{M}$), where Graph($\widehat{M}$) $= \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in \widehat{M}(x)\}$.

Xia and Huang [28] introduced a class of generalized monotone operators the so-called general H-monotone operators as follows.

Definition 2.5. [28, Theorem 3.1] *Let E be a Banach space with the dual space E^* and let $H : E \to E$ be a single-valued mapping, and $\widehat{M} : E \rightrightarrows E^*$ be a multi-valued mapping. $\widehat{M}$ is said to be general H-monotone if $\widehat{M}$ is monotone and $(H + \lambda\widehat{M})(E) = E^*$ holds for every $\lambda > 0$.*

Lou et al. [19] and Alimohammady and Roohi [3] introduced and studied the class of general (H, η)-monotone operators (also referred to as (H, η)-monotone operators in literature, see for example, [19, Definition 1.2(7)]) as a generalization of the class of general H-monotone operators as follows.

Definition 2.6. *A multi-valued operator $\widehat{M} : E \rightrightarrows E^*$ is said to be general (H, η)-monotone operator if $\widehat{M}$ is η-monotone and $(H + \lambda\widehat{M})(E) = E^*$ holds for every $\lambda > 0$.*

Remark 2.7. When $E = \mathcal{H}$ is a Hilbert space, the general (H, η)-monotone operator reduces to the (H, η)-monotone operator introduced in [12]. If $\eta(x, y) = x - y$ for all $x, y \in E$, then Definition 2.6 reduces to Definition 2.5, that is, the general (H, η)-monotone operator coincides with the general H-monotone operator. For the case where $E = \mathcal{H}$ and $\eta(x, y) = x - y$, for all $x, y \in E$, then Definition 2.6 reduces to Definition 2.1 in [10], that is, the definition H-monotone operator. If $E = \mathcal{H}$ and $H = g$, then Definition 2.6 reduces to the definition g-η-monotone operator introduced in [32]. If $E = \mathcal{H}$ and $H = I$, the identity mapping, then Definition 2.6 reduces to Definition 2.4(ii), that is, the definition maximal η-monotone operator considered in [15].

The following two examples illustrate that for the vector-valued mappings $\eta : E \times E \to E$ and $H : E \to E$, a general (H, η)-monotone operator may be neither general H-monotone nor maximal η-monotone.

Example 2.8. *Let $E = \mathbb{R}$ and let the operators $\widehat{M} : E \rightrightarrows E$ and $\eta : E \times E \to E$ be defined by*

$$\widehat{M}(x) = \begin{cases} x + \beta, & \text{if } x < \gamma, \\ \{-\beta - \gamma, \beta - \gamma\}, & \text{if } x = \gamma, \\ -x - \beta, & \text{if } x > \gamma, \end{cases}$$

and $\eta(x, y) = \alpha x^q y^q (y^n - x^n)$, for all $x, y \in E$, respectively, where $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$ are arbitrary but fixed, and q and n are arbitrary but fixed even and odd natural numbers, respectively. For all $x, y > \gamma$, $x \neq y$, we have

$$\langle \widehat{M}(x) - \widehat{M}(y), x - y \rangle = -(x - y)^2 < 0,$$

that is, $\widehat{M}$ is not monotone. Since $y^n - x^n = (y - x)\sum_{j=1}^{n} y^{n-j}x^{j-1}$, for all $x, y \in E$, $x \neq y$, we have $\sum_{j=1}^{n} y^{n-j}x^{j-1} = \frac{y^n - x^n}{y - x}$. If $x = y = 0$, then $\sum_{j=1}^{n} y^{n-j}x^{j-1} = 0$. If $x, y > 0$ or $x, y < 0$, obviously, $\sum_{j=1}^{n} y^{n-j}x^{j-1} > 0$. For the case where $x > 0$ and $y < 0$, in view of the fact that n is an odd natural number, we have $y - x < 0$ and $y^n - x^n < 0$. If $x < 0$ and $y > 0$, since n is an odd natural number, it follows that $y - x > 0$ and $y^n - x^n > 0$. Hence, in both cases, we conclude that $\sum_{j=1}^{n} y^{n-j}x^{j-1} > 0$. In the case when $x = 0$ and $y \neq 0$, or $x \neq 0$ and $y = 0$, clearly $\sum_{j=1}^{n} y^{n-j}x^{j-1} > 0$. Therefore, $\sum_{j=1}^{n} y^{n-j}x^{j-1} \geq 0$, for all $x, y \in E$. If $x > \gamma$ and $y < \gamma$, then $\widehat{M}(x) = x - \beta$, $\widehat{M}(y) = y + \beta$ and we have

$$\begin{aligned} \langle \widehat{M}(x) - \widehat{M}(y), \eta(x, y) \rangle &= (x - \beta - y - \beta)\alpha x^q y^q (y^n - x^n) \\ &= [-(y - x) - 2\beta]\alpha x^q y^q (y - x) \sum_{j=1}^{n} y^{n-j}x^{j-1} \\ &= -\alpha(y - x)^2 x^q y^q \sum_{j=1}^{n} y^{n-j}x^{j-1} \\ &\quad - 2\alpha\beta x^q y^q (y - x) \sum_{j=1}^{n} y^{n-j}x^{j-1}. \end{aligned}$$

Taking into account that $\beta > 0$, $y < x$, q is an even natural number and $\sum_{j=1}^{n} y^{n-j}x^{j-1} \geq 0$, the above equality implies that $\langle \widehat{M}(x) - \widehat{M}(y), \eta(x,y)\rangle \geq 0$. For the case where $x < \gamma$ and $y > \gamma$, then $\widehat{M}(x) = x + \beta$, $\widehat{M}(y) = y - \beta$ and we have

$$\langle \widehat{M}(x) - \widehat{M}(y), \eta(x,y)\rangle = -\alpha(y-x)^2x^qy^q\sum_{j=1}^{n} y^{n-j}x^{j-1} + 2\alpha\beta x^qy^q(y-x)\sum_{j=1}^{n} y^{n-j}x^{j-1}.$$

Considering the facts that $\beta > 0$, $y > x$, q is an even natural number, and $\sum_{j=1}^{n} y^{n-j}x^{j-1} \geq 0$, from the preceding equality it follows that $\langle \widehat{M}(x) - \widehat{M}(y), \eta(x,y)\rangle \geq 0$. Similarly, one can deduce that

$$\langle u - v, \eta(x,y)\rangle \geq 0, \quad \forall x,y \in E, u \in \widehat{M}(x), v \in \widehat{M}(y),$$

that is, $\widehat{M}$ is an η-monotone operator. Since

$$(I + \lambda\widehat{M})(x) = \begin{cases} (1-\lambda)x + \lambda\beta, & \text{if } x < \gamma, \\ \{(1-\lambda)\gamma - \lambda\beta, (1-\lambda)\gamma + \lambda\beta\}, & \text{if } x = \gamma, \\ (1-\lambda)x - \lambda\beta, & \text{if } x > \gamma, \end{cases}$$

for all $x, y \in E$, it is easy to see that $(I + \lambda\widehat{M})(E) \neq E$, for all $\lambda \geq 1$, that is, $\widehat{M}$ is not a maximal η-monotone operator. Now, let us define the operator $H : E \to E$ as follows:

$$H(x) = \begin{cases} -(x-\gamma)^2, & \text{if } x < \gamma, \\ (x-\gamma)^2, & \text{if } x \geq \gamma. \end{cases}$$

In virtue of the fact that $\widehat{M}$ is not monotone, it follows that $\widehat{M}$ is not a general H-monotone operator. It can be easily observed that $(H + \lambda\widehat{M})(E) = E$, for every $\lambda > 0$. This fact implies that $\widehat{M}$ is a general (H,η)-monotone operator.

Example 2.9. Let $E = \mathbb{R}$ and let the operators $\widehat{M} : E \rightrightarrows E$ and $\eta : E \times E \to E$ be defined by

$$\widehat{M}(x) = \begin{cases} \alpha x + \beta, & \text{if } x < 0, \\ \{-\beta, \beta\}, & \text{if } x = 0, \\ \alpha x - \beta, & \text{if } x > 0, \end{cases}$$

and $\eta(x,y) = \gamma x^qy^q(y^n - x^n)$, for all $x, y \in E$, respectively, where $\alpha < 0$ and $\beta, \gamma > 0$ are arbitrary but fixed real numbers, and q and n are arbitrary but fixed even and odd natural numbers, respectively. Relying on the fact that $\alpha < 0$, for all $x, y > 0$, $x \neq y$, we have

$$\langle \widehat{M}(x) - \widehat{M}(y), x - y\rangle = \alpha(x-y)^2 < 0,$$

that is, $\widehat{M}$ is not monotone. Let $x > \gamma$ and $y < \gamma$. Then $\widehat{M}(x) = \alpha x - \beta$ and $\widehat{M}(y) = \alpha y + \beta$. Since $\alpha < 0 < \beta$, $y < x$, q is an even natural number and $\sum_{j=1}^{n} y^{n-j}x^{j-1} \geq 0$, we have

$$\langle \widehat{M}(x) - \widehat{M}(y), \eta(x,y)\rangle = -\alpha\gamma(y-x)^2x^qy^q\sum_{j=1}^{n} y^{n-j}x^{j-1} - 2\beta\gamma x^qy^q(y-x)\sum_{j=1}^{n} y^{n-j}x^{j-1} \geq 0.$$

If $x < \gamma$ and $y > \gamma$, then $\widehat{M}(x) = \alpha x + \beta$, $\widehat{M}(y) = \alpha y + \beta$ and we have

$$\langle \widehat{M}(x) - \widehat{M}(y), \eta(x,y)\rangle = -\alpha\gamma(y-x)^2x^qy^q\sum_{j=1}^{n} y^{n-j}x^{j-1} + 2\beta\gamma x^qy^q(y-x)\sum_{j=1}^{n} y^{n-j}x^{j-1} \geq 0.$$

In a similar fashion to the preceding analysis, one can show that

$$\langle u - v, \eta(x, y)\rangle \geq 0, \quad \forall x, y \in E, u \in \widehat{M}(x), v \in \widehat{M}(y),$$

that is, $\widehat{M}$ is an η-monotone operator. Thanks to the fact that

$$(I + \lambda\widehat{M})(x) = \begin{cases} (1 + \lambda\alpha)x + \lambda\beta, & \text{if } x < 0, \\ \{-\lambda\beta, \lambda\beta\}, & \text{if } x = 0, \\ (1 + \lambda\alpha)x - \lambda\beta, & \text{if } x > 0, \end{cases}$$

for all $x, y \in E$, it is easy to check that $(I + \lambda\widehat{M})(E) \neq E$, for all $\lambda \geq -\frac{1}{\alpha}$, that is, $\widehat{M}$ is not a maximal η-monotone operator. We now define the operator $H : E \to E$ as follows:

$$H(x) = \begin{cases} -x^2, & \text{if } x < 0, \\ x^2, & \text{if } x \geq 0. \end{cases}$$

Since $\widehat{M}$ is not monotone, we conclude that $\widehat{M}$ is not a general H-monotone operator. It is easy to see that $(H + \lambda\widehat{M})(E) = E$, for every $\lambda > 0$. This fact ensures that $\widehat{M}$ is a general (H, η)-monotone operator.

The following example shows that for given vector-valued mappings $H : E \to E$ and $\eta : E \times E \to E$, a maximal η-monotone operator need not be general (H, η)-monotone operator.

Example 2.10. Let $E = \mathbb{R}$ and let the operators $\widehat{M} : E \rightrightarrows E$ and $\eta : E \times E \to E$ be defined by $\widehat{M}(x) = x^k$ and $\eta(x, y) = \alpha x^\beta y^\beta (x^n - y^n)$, where α is a positive real number, k and n are two arbitrary but fixed odd natural numbers, and β is an arbitrary but fixed even natural number. Since k is an odd natural number, for all $x, y \in E$, we have

$$\langle \widehat{M}(x) - \widehat{M}(y), x - y\rangle = (x^k - y^k)(x - y) = (x - y)^2 \sum_{j=1}^{k} x^{k-j} y^{j-1} \geq 0,$$

that is, $\widehat{M}$ is monotone. In the meanwhile, for all $x, y \in E$, we have

$$\begin{aligned} \langle \widehat{M}(x) - \widehat{M}(y), \eta(x, y)\rangle &= \alpha(x^k - y^k)x^\beta y^\beta (x^n - y^n) \\ &= \alpha x^\beta y^\beta (x - y)^2 (\sum_{j=1}^{k} x^{k-j} y^{j-1})(\sum_{j=1}^{n} x^{n-j} y^{j-1}). \end{aligned}$$

Taking into account that k and n are two odd natural numbers, it follows that $\sum_{j=1}^{k} x^{k-j} y^{j-1} \geq 0$ and $\sum_{j=1}^{n} x^{n-j} y^{j-1} \geq 0$, for all $x, y \in E$. This fact and the facts that q is an even natural number and $\alpha > 0$ guarantee that $\langle \widehat{M}(x) - \widehat{M}(y), \eta(x, y)\rangle \geq 0$, for all $x, y \in E$, that is, $\widehat{M}$ is an η-monotone operator. On the other hand, for every $\lambda > 0$ and $x \in E$, we have $(I + \lambda\widehat{M})(x) = x + \lambda x^k$. Owing to the fact that k is an odd natural number, it follows that $(I + \lambda\widehat{M})(E) = E$, for every $\lambda > 0$. This fact implies that $\widehat{M}$ is a maximal η-monotone operator. Now, let us define the operator $H : E \to E$ by $H(x) = x^{2k}$, for all $x \in E$. Then, by taking $\lambda = 1$, for all $x \in E$, we have

$$(H + \lambda\widehat{M})(x) = H(x) + \widehat{M}(x) = x^{2k} + x^k = x^k(x^k + 1).$$

Since k is an odd natural number, it follows that $(H + \widehat{M})(E) \neq E$, that is, $(H + \lambda\widehat{M})(E) = E$ does not hold for all $\lambda > 0$. Hence, $\widehat{M}$ is not a general H-monotone and general (H, η)-monotone operator.

Theorem 2.11. *Let E be a reflexive Banach space with the dual space E^* and $\eta : E \times E \to E$ be a vector-valued operator. Suppose that $H : E \to E^*$ is an η-monotone operator and $\widehat{M} : E \rightrightarrows E^*$ is a θ-strongly η-monotone operator. Then, the mapping $(H + \lambda\widehat{M})^{-1} : \mathrm{Range}(H + \lambda\widehat{M}) \to E$ is single-valued for every real constant $\lambda > 0$.*

Proof. Suppose, by contradiction, that there exists some $z \in \text{Range}(H + \lambda\widehat{M})$ such that $x, y \in (H + \lambda\widehat{M})^{-1}(z)$ and $x \neq y$. Then, we have $z \in (H + \lambda\widehat{M})(x)$ and $z \in (H + \lambda\widehat{M})(y)$, and so there exists $u \in \widehat{M}(x)$ and $v \in \widehat{M}(y)$ such that

$$H(x) + \lambda u = H(y) + \lambda v. \tag{1}$$

Since H is η-monotone and $\widehat{M}$ is θ-strongly η-monotone, by (1), yields

$$\lambda\theta\|x - y\|^2 \leq \langle H(x) - H(y), \eta(x, y)\rangle + \lambda\langle u - v, \eta(x, y)\rangle = 0,$$

which implies that $x = y$. Obviously, this is in contradiction to our assumption. $\square$

We obtain the following corollary as a direct consequent of the previous theorem immediately.

Corollary 2.12. *Let E be a reflexive Banach space with the dual space E^* and $\eta : E \times E \to E$ be a vector-valued operator. Let $H : E \to E^*$ be an η-monotone operator and $\widehat{M} : E \rightrightarrows E^*$ be a general (H, η)-strongly monotone operator with constant θ. Then, the mapping $(H + \lambda\widehat{M})^{-1} : E^* \to E$ is single-valued for every constant $\lambda > 0$.*

By utilizing Corollary 2.12, we can define the proximal mapping $R^{H,\eta}_{\widehat{M},\lambda}$ associated with $H, \eta, \widehat{M}$ and constant $\lambda > 0$ as follows.

Definition 2.13. *Let E be a reflexive Banach space with the dual space E^* and let $\eta : E \times E \to E$ be a vector-valued operator. Suppose that $H : E \to E^*$ is an η-monotone operator and $\widehat{M} : E \rightrightarrows E^*$ is a general strongly (H, η)-monotone mapping with constant γ. For every real constant $\lambda > 0$, the proximal mapping $R^{H,\eta}_{\widehat{M},\lambda} : E^* \to E$ associated with $H, \eta, \widehat{M}$ and constant $\lambda > 0$ is defined by $R^{H,\eta}_{\widehat{M},\lambda}(x^*) = (H + \lambda\widehat{M})^{-1}(x^*)$ for all $x^* \in E^*$.*

Definition 2.14. *A vector-valued mapping $\eta : E \times E \to E$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \leq \tau\|x - y\|$, for all $x, y \in E$.*

Theorem 2.15. *Let E be a reflexive Banach space with the dual space E^* and $\eta : E \times E \to E$ be a κ-Lipschitz continuous operator. Suppose that $H : E \to E^*$ is an η-monotone operator and $\widehat{M} : E \rightrightarrows E^*$ is a general (H, η)-strongly monotone with constant θ. Then, the proximal mapping $R^{H,\eta}_{\widehat{M},\lambda} : E^* \to E$ is $\frac{\kappa}{\lambda\theta}$-Lipschitz continuous, i.e.,*

$$\|R^{H,\eta}_{\widehat{M},\lambda}(x^*) - R^{H,\eta}_{\widehat{M},\lambda}(y^*)\| \leq \frac{\kappa}{\lambda\theta}\|x^* - y^*\|, \quad \forall x^*, y^* \in E^*.$$

Proof. In view of the fact that $\widehat{M}$ is a general (H, η)-monotone operator, for any given $x^*, y^* \in E^*$ with $\|R^{H,\eta}_{\widehat{M},\lambda}(x^*) - R^{H,\eta}_{\widehat{M},\lambda}(y^*)\| \neq 0$, we have

$$R^{H,\eta}_{\widehat{M},\lambda}(x^*) = (H + \lambda\widehat{M})^{-1}(x^*) \text{ and } R^{H,\eta}_{\widehat{M},\lambda}(y^*) = (H + \lambda\widehat{M})^{-1}(y^*),$$

from which we deduce that

$$\lambda^{-1}(x^* - H(R^{H,\eta}_{\widehat{M},\lambda}(x^*))) \in \widehat{M}(R^{H,\eta}_{\widehat{M},\lambda}(x^*)) \text{ and } \lambda^{-1}(y^* - H(R^{H,\eta}_{\widehat{M},\lambda}(y^*))) \in \widehat{M}(R^{H,\eta}_{\widehat{M},\lambda}(y^*)).$$

Since $\widehat{M}$ is θ-strongly η-monotone, it follows that

$$\lambda^{-1}\langle x^* - H(R^{H,\eta}_{\widehat{M},\lambda}(x^*)) - (y^* - H(R^{H,\eta}_{\widehat{M},\lambda}(y^*))), \eta(R^{H,\eta}_{\widehat{M},\lambda}(x^*), R^{H,\eta}_{\widehat{M},\lambda}(y^*))\rangle \geq \theta\|R^{H,\eta}_{\widehat{M},\lambda}(x^*) - R^{H,\eta}_{\widehat{M},\lambda}(y^*)\|^2.$$

Taking into account that $\lambda^{-1} > 0$, from the above inequality, we obtain

$$\langle x^* - y^*, \eta(R^{H,\eta}_{\widehat{M},\lambda}(x^*), R^{H,\eta}_{\widehat{M},\lambda}(y^*))\rangle \geq \langle H(R^{H,\eta}_{\widehat{M},\lambda}(x^*)) - H(R^{H,\eta}_{\widehat{M},\lambda}(y^*)), \eta(R^{H,\eta}_{\widehat{M},\lambda}(x^*), R^{H,\eta}_{\widehat{M},\lambda}(y^*))\rangle + \lambda\theta\|R^{H,\eta}_{\widehat{M},\lambda}(x^*) - R^{H,\eta}_{\widehat{M},\lambda}(y^*)\|^2.$$

The preceding inequality and the facts that η is κ-Lipschitz continuous and H is η-monotone imply that

$$\begin{aligned}\kappa\|x^*-y^*\|\|R^{H,\eta}_{\widehat{M},\lambda}(x^*)-R^{H,\eta}_{\widehat{M},\lambda}(y^*)\| &\geq \|x^*-y^*\|\|\eta(R^{H,\eta}_{\widehat{M},\lambda}(x^*),R^{H,\eta}_{\widehat{M},\lambda}(y^*))\|\\ &\geq \langle x^*-y^*,\eta(R^{H,\eta}_{\widehat{M},\lambda}(x^*),R^{H,\eta}_{\widehat{M},\lambda}(y^*))\rangle\\ &\geq \langle H(R^{H,\eta}_{\widehat{M},\lambda}(x^*))-H(R^{H,\eta}_{\widehat{M},\lambda}(y^*)),\eta(R^{H,\eta}_{\widehat{M},\lambda}(x^*),\\ &\quad R^{H,\eta}_{\widehat{M},\lambda}(y^*))\rangle+\lambda\theta\|R^{H,\eta}_{\widehat{M},\lambda}(x^*)-R^{H}_{\widehat{M},\lambda}(y^*)\|^2\\ &\geq \lambda\theta\|R^{H,\eta}_{\widehat{M},\lambda}(x^*)-R^{H,\eta}_{\widehat{M},\lambda}(y^*)\|^2.\end{aligned}$$

In view of the fact that $\|R^{H,\eta}_{\widehat{M},\lambda}(x^*)-R^{H,\eta}_{\widehat{M},\lambda}(y^*)\|\neq 0$, dividing both sides of the last inequality by $\|R^{H,\eta}_{\widehat{M},\lambda}(x^*)-R^{H,\eta}_{\widehat{M},\lambda}(y^*)\|\neq 0$, we yield

$$\|R^{H,\eta}_{\widehat{M},\lambda}(x^*)-R^{H,\eta}_{\widehat{M},\lambda}(y^*)\|\leq\frac{\kappa}{\lambda\theta}\|x^*-y^*\|.$$

This completes the proof. □

Remark 2.16. By a careful reading, we found that there is a small mistake in Theorem 3.9 of [28]. In fact, the space E must be assumed as a reflexive Banach space with the dual space E^*, as we have added the mentioned assumption to Theorem 2.15. At the same time, it is worthwhile to stress that Theorem 2.15 improves Theorem 3.9 in [28]. Indeed, in Theorem 3.9 of [28], the strict η-monotonicity condition of the operator H has been reduced to the η-monotonicity condition in Theorem 2.15.

Corollary 2.17. [28, Theorem 3.2] *Suppose that E is a reflexive Banach space with the dual space E^*. Let $H:E\to E^*$ be a mapping, and $M:E\rightrightarrows E^*$ be a general H-monotone mapping. Then the following conclusions hold.*

(i) *If H is a strongly monotone mapping with constant γ, then the proximal mapping $R^H_{M,\lambda}:E^*\to E$ is Lipschitz continuous with constant $\frac{1}{\gamma}$;*

(ii) *If H is a strictly monotone mapping and M is a strongly monotone mapping with constant β, then the proximal mapping $R^H_{M,\lambda}:E^*\to E$ is Lipschitz continuous with constant $\frac{1}{\lambda\beta}$*

Remark 2.18. It is significant to mention that Theorem 2.15 generalizes Theorem 3.2 in [28]. In fact, if $\eta(x,y)=x-y$, for all $x,y\in E$, then the conclusion of Theorem 2.15 reduces to the conclusion (b) of Theorem 3.2 in [28]. By comparing Theorem 3.2(b) in [28] and Theorem 2.15, and due to the fact that the strict η-monotonicity condition of the operator H in [28, Theorem 3.2(b)] has been reduced to the η-monotonicity condition in Theorem 2.15, we note that Theorem 2.15 is an improvement version of Theorem 3.2(b) in [28]. Note, in particular, that Theorem 2.15 extends and improves Theorem 2.2 in [10], Lemma 2.2 in [12] and Theorem 2.2 in [32].

3. Formulations, Existence Results of Solution and Convergence Analysis

Let for each $i\in\{1,2,\ldots,n\}$ and $j\in\{1,2\}$, $E_{i,j}$ be a real Banach space equipped with the dual space $E^*_{i,j}$ and norm $\|.\|_{i,j}$, $\langle .,.\rangle_{i,j}$ be the dual pair between $E_{i,j}$ and $E^*_{i,j}$. Suppose that for $i=1,2,\ldots,n$, $A_i:E_{i,1}\to E^*_{i,1}$, $p_i:E_{i,1}\to E_{i,1}$, $g_i:E_{n-(i-1),1}\to E_{n-(i-1),1}$, $P_i:\prod\limits_{k=1}^{n}E_{n-(k-1),1}\to E^*_{i,1}$ and $F_i:\prod\limits_{k=1}^{n}E_{k,2}\to E^*_{i,1}$ are single-valued operators. Assume further that for $i=1,2\ldots,n$, $T_i:E_{i,1}\rightrightarrows CB(E_{i,2})$, $S_i:E_{n-(i-1),1}\to CB(E_{n-(i-1),1})$ and $M_i:E_{i,1}\rightrightarrows E^*_{i,1}$ are multi-valued operators. For each $i\in\{1,2,\ldots,n\}$ and any given $a_i\in E^*_{i,1}$, we consider the following system of extended multi-valued nonlinear variational inclusions (SEMNVI): Find $(x_1, x_2, \dots, x_n) \in \prod_{i=1}^{n} E_{i,1}$, $(s_1, s_2, \dots, s_n) \in \prod_{i=1}^{n} S_i(x_{n-(i-1)})$ and $(t_1, t_2, \dots, t_n) \in \prod_{i=1}^{n} T_i(x_i)$ such that for each $i \in \{1, 2, \dots, n\}$,

$$a_i \in A_i(x_i - p_i(x_i)) + P_i(g_1(s_1), g_2(s_2), \dots, g_n(s_n)) + M_i(x_i) - F_i(t_1, t_2, \dots, t_n). \tag{2}$$

If for $i = 1, 2, \dots, n$, $E_{i,1} = E$, $E^*_{i,1} = E^*$, $E_{i,2} = E_i$, $S_i = P_i \equiv 0$, $p_i = p$, $A_i = A$, $M_i = \widehat{M}$, $F_i = F$ and $a_i = a$, then the SEMNVI (2) collapses to the following multi-valued nonlinear variational inclusion problem (NMVIP): Find $x \in E$ and $(t_1, t_2, \dots, t_n) \in \prod_{i=1}^{n} T_i(x)$ such that

$$a \in A(x - p(x)) + \widehat{M}(x) - F(t_1, t_2, \dots, t_n). \tag{3}$$

We remark that for suitable and appropriate choices of the operators $A_i, F_i, T_i, S_i, P_i, M_i, g_i, p_i$ and the underlying spaces $E_{i,j}$ for $i = 1, 2, \dots, n$ and $j = 1, 2$, the SEMNVI (2) reduces to various classes of variational inclusions and variational inequalities, see for example, [12, 17, 22, 27, 30] and the references therein.

With the goal of finding a characterization of a solution of the SEMNVI (2), we present the following result in which the equivalence between the SEMNVI (2) and a fixed point problem is stated.

Theorem 3.1. *Let for each $i \in \{1, 2, \dots, n\}$, $E_{i,1}$ be a reflexive Banach space with the dual space $E^*_{i,1}$ and $E_{i,2}$ be a real Banach space with the dual space $E^*_{i,2}$. Let for $i = 1, 2, \dots, n$, $A_i, S_i, T_i, F_i, P_i, g_i, p_i$ and a_i be the same as in the* SEMNVI (2). *Suppose further that for $i = 1, 2, \dots, n$, $\eta_i : E_{i,1} \times E_{i,1} \to E_{i,1}$ is a vector-valued operator, $H_i : E_{i,1} \to E^*_{i,1}$ is an η_i-monotone operator, and $M_i : E_{i,1} \rightrightarrows E^*_{i,1}$ is a general (H_i, η_i)-strongly monotone operator. Then, $(x_1, x_2, \dots, x_n) \in \prod_{i=1}^{n} E_{i,1}$, $(s_1, s_2, \dots, s_n) \in \prod_{i=1}^{n} S_i(x_{n-(i-1)})$ and $(t_1, t_2, \dots, t_n) \in \prod_{i=1}^{n} T_i(x_i)$ are the solution of the* SEMNVI (2), *if and only if for $i = 1, 2, \dots, n$,*

$$x_i = R^{H_i,\eta_i}_{M_i,\lambda_i}[H_i(x_i) - \lambda_i(A_i(x_i - p_i(x_i)) + P_i(g_1(s_1), g_2(s_2), \dots, g_n(s_n)) - a_i - F_i(t_1, t_2, \dots, t_n))] \tag{4}$$

where $\lambda_i > 0$ $(i = 1, 2, \dots, n)$ are constants.

Proof. By using Definition 2.13, it follows that $(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_n, t_1, \dots, t_n)$ is a solution of the SEMNVI (2) if and only if for each $i \in \{1, 2, \dots, n\}$,

$$\begin{aligned} & a_i \in A_i(x_i - p_i(x_i)) + P_i(g_1(s_1), g_2(s_2), \dots, g_n(s_n)) + M_i(x_i) - F_i(t_1, t_2, \dots, t_n) \\ & \Leftrightarrow \lambda_i[a_i + F_i(t_1, t_2, \dots, t_n) - A_i(x_i - p_i(x_i)) - P_i(g_1(s_1), g_2(s_2), \dots, g_n(s_n))] \in \lambda_i M_i(x_i) \\ & \Leftrightarrow H_i(x_i) - \lambda_i[A_i(x_i - p_i(x_i)) + P_i(g_1(s_1), g_2(s_2), \dots, g_n(s_n)) - a_i - F_i(t_1, t_2, \dots, t_n)] \in H_i(x_i) + \lambda_i M_i(x_i) \\ & \quad = (H_i + \lambda_i M_i)(x_i) \\ & \Leftrightarrow x_i = R^{H_i,\eta_i}_{M_i,\lambda_i}[H_i(x_i) - \lambda_i(A_i(x_i - p_i(x_i)) + P_i(g_1(s_1), g_2(s_2), \dots, g_n(s_n)) - a_i - F_i(t_1, t_2, \dots, t_n))] \end{aligned}$$

where $\lambda_i > 0$ $(i = 1, 2, \dots, n)$ are constants. This gives the desired result. □

Corollary 3.2. *Let E be a reflexive Banach space with the dual space E^*, and for each $i \in \{1, 2, \dots, n\}$, E_i be a real Banach space with the dual space E^*_i. Let A, F, T_i $(i = 1, 2, \dots, n)$, p and a be the same as in the* NMVIP (3). *Suppose further that $\eta : E \times E \to E$ is a vector-valued operator, $H : E \to E^*$ is an η-monotone operator, and $\widehat{M} : E \rightrightarrows E^*$ is a general (H, η)-strongly monotone operator. Then, $(x, t_1, t_2, \dots, t_n) \in E \times \prod_{i=1}^{n} T_i(x) \subseteq E \times \prod_{i=1}^{n} E_i$ is a solution of the* NMVIP (3), *if and only if*

$$x = R^{H,\eta}_{\widehat{M},\lambda}[H(x) - \lambda(A(x - p(x)) - a - F(t_1, t_2, \dots, t_n))], \tag{5}$$

where $\lambda > 0$ is a constant.

Lemma 3.3. [21] *Let (E, d) be a complete metric space and $T : E \to CB(E)$ be a multi-valued mapping. Then for any $\epsilon > 0$ and for any given $x, y \in E$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u, v) \leq (1 + \epsilon)\widehat{H}(T(x), T(y)),$$

where $\widehat{H}(.,.)$ is the Hausdorff metric on $CB(E)$.

We now apply the fixed point formulation (4) and Nadler's technique [21] to construct the following iterative algorithm for solving the SEMNVI (2).

Algorithm 3.4. *Let for each $i \in \{1, 2, \ldots, n\}$, $E_{i,1}$ be a real reflexive Banach space with the dual space $E^*_{i,1}$ and $E_{i,2}$ be a real Banach space with the dual space $E^*_{i,2}$. Suppose that $A_i, S_i, T_i, F_i, P_i, g_i, p_i$ and a_i $(i = 1, 2, \ldots, n)$ are the same as in the* SEMNVI (2). *Assume further that for $i = 1, 2, \ldots, n$, $\eta_i : E_{i,1} \times E_{i,1} \to E_{i,1}$ is a vector-valued operator, $H_i : E_{i,1} \to E^*_{i,1}$ is an η_i-monotone operator, and $M_i : E_{i,1} \rightrightarrows E^*_{i,1}$ is a general (H_i, η_i)-strongly monotone operator. For any given $(x_{1,0}, x_{2,0}, \ldots, x_{n,0}) \in \prod_{i=1}^{n} E_{i,1}$, $(s_{1,0}, s_{2,0}, \ldots, s_{n,0}) \in \prod_{i=1}^{n} S_i(x_{n-(i-1),0})$ and $(t_{1,0}, t_{2,0}, \ldots, t_{n,0}) \in \prod_{i=1}^{n} T_i(x_{i,0})$, compute the iterative sequences $\{(x_{1,m}, x_{2,m}, \ldots, x_{n,m})\}_{m=0}^{\infty}$, $\{(s_{1,m}, s_{2,m}, \ldots, s_{n,m})\}_{m=0}^{\infty}$ and $\{(t_{1,m}, t_{2,m}, \ldots, t_{n,m})\}_{m=0}^{\infty}$ in $\prod_{i=1}^{n} E_{i,1}$, $\prod_{i=1}^{n} E_{n-(i-1),1}$ and $\prod_{i=1}^{n} E_{i,2}$, respectively, by the iterative schemes*

$$\begin{cases} x_{i,m+1} = (1-\alpha)x_{i,m} + \alpha R^{H_i,\eta_i}_{M_i,\lambda_i}[H_i(x_{i,m}) - \lambda_i(A_i(x_{i,m} - p_i(x_{i,m})) + P_i(g_1(s_{1,m}), g_2(s_{2,m}), \ldots, g_n(s_{n,m})) - a_i \\ \qquad\qquad -F_i(t_{1,m}, t_{2,m}, \ldots, t_{n,m}))] + \alpha e_{i,m} + r_{i,m}, \\ s_{i,m} \in S_i(x_{n-(i-1),m}) : \|s_{i,m+1} - s_{i,m}\|_{n-(i-1),1} \leq (1 + (1+m)^{-1})\widehat{H}_{n-(i-1),1}(S_i(x_{n-(i-1),m+1}), S_i(x_{n-(i-1),m})), \\ t_{i,m} \in T_i(x_{i,m}) : \|t_{i,m+1} - t_{i,m}\|_{i,2} \leq (1 + (1+m)^{-1})\widehat{H}_{i,2}(T_i(x_{i,m+1}), T_i(x_{i,m})), \end{cases} \tag{6}$$

where $i = 1, 2, \ldots, n$; $m = 0, 1, 2, \ldots$; $\lambda_{m,i} > 0$ are constants; $\alpha \in (0, 1]$ is a relaxation parameter, and $\{(e_{1,m}, e_{2,m}, \ldots, e_{n,m})\}_{m=0}^{\infty}$ and $\{(r_{1,m}, r_{2,m}, \ldots, r_{n,m})\}_{m=0}^{\infty}$ are two sequences in $\prod_{i=1}^{n} E_{i,1}$ to take into account a possible inexact computation of the resolvent mapping point satisfying the following conditions:

$$\begin{cases} \lim\limits_{m\to\infty} \|e_{i,m}\|_{i,1} = \lim\limits_{m\to\infty} \|r_{i,m}\|_{i,1} = 0, \quad i = 1, 2, \ldots, n; \\ \sum\limits_{m=0}^{\infty} \|e_{i,m} - e_{i,m-1}\|_{i,1} < \infty, \quad i = 1, 2, \ldots, n; \\ \sum\limits_{m=0}^{\infty} \|r_{i,m} - r_{i,m-1}\|_{i,1} < \infty, \quad i = 1, 2, \ldots, n. \end{cases} \tag{7}$$

By using the fixed point formulation (6) and Nadler's technique [21], we are able to suggest the following iterative algorithm for solving the NMVIP (3).

Algorithm 3.5. *Assume that E is a real reflexive Banach space with the dual space E^*, and for each $i \in \{1, 2, \ldots, n\}$, E_i is a real Banach space with the dual space E^*_i. Let A, F, T_i $(i = 1, 2, \ldots, n)$, p and a be the same as in the* NMVIP (3). *Suppose further that $\eta : E \times E \to E$ is a vector-valued operator, $H : E \to E^*$ is an η-monotone operator, and $\widehat{M} : E \rightrightarrows E^*$ is a general (H, η)-strongly monotone operator. For any given $x_0 \in E$ and $(t_{1,0}, t_{2,0}, \ldots, t_{n,0}) \in \prod_{i=1}^{n} T_i(x_0)$, define the iterative sequences $\{x_m\}_{n=0}^{\infty}$ in E and $\{(t_{1,m}, t_{2,m}, \ldots, t_{n,m})\}_{m=0}^{\infty} \subseteq \prod_{i=1}^{n} T_i(x_m)$ by the iterative scmemes*

$$\begin{cases} x_{m+1} = R^{H,\eta}_{\widehat{M},\lambda}[H(x_m) - \lambda(A(x_m) - p(x_m)) - a - F(t_{1,m}, t_{2,m}, \ldots, t_{n,m}))], \\ t_{i,m} \in T_i(x_m) : \|t_{i,m+1} - t_{i,m}\|_i \leq (1 + (1+m)^{-1})\widehat{H}_i(T_i(x_{m+1}), T_i(x_m)), \end{cases}$$

where $i = 1, 2, \ldots, n$; $m = 0, 1, 2, \ldots$, and $\lambda > 0$ is a constant.

Before turning to the main result of this paper, we need to recall the definitions mentioned below which shall be used in the sequel.

Definition 3.6. *A set-valued mapping $T : E \rightrightarrows CB(E)$ is said to be $\widehat{H}$-Lipschitz continuous with constant ϱ (or ϱ-$\widehat{H}$-Lipschitz continuous) if there exists a constant $\varrho > 0$ such that*

$$\widehat{H}(T(x), T(y)) \leq \varrho\|x - y\|, \quad \forall x, y \in E,$$

where $\widehat{H}(.,.)$ is the Hausdorff metric on $CB(E)$.

Definition 3.7. *Let E be a real q-uniformly smooth Banach space. A mapping $g : E \to E$ is said to be (γ, μ)-relaxed cocoercive if there exist two constants $\gamma, \mu > 0$ such that*

$$\langle g(x) - g(y), J_q(x - y)\rangle \geq -\gamma\|g(x) - g(y)\|^q + \mu\|x - y\|^q, \quad \forall x, y \in E.$$

Definition 3.8. *Let E be a real Banach space with the dual space E^*; for $i = 1, , 2, \ldots, n$, E_i be real Banach spaces and $T_i : E \rightrightarrows CB(E_i)$ be multi-valued mappings. A mapping $F : \prod_{i=1}^{n} E_i \to E^*$ is said to be λ_{F_i}-Lipschitz continuous in the ith argument with respect to T_i $(i = 1, 2, \ldots, n)$ if there exists a constant $\lambda_{F_i} > 0$ such that*

$$\begin{aligned}&\|F(x_1, x_2, \ldots, x_{i-1}, u_{i,1}, x_{i+1}, \ldots, x_n) - F(x_1, x_2, \ldots, x_{i-1}, u_{i,2}, x_{i+1}, \ldots, x_n)\|\\ &\leq \lambda_{F_i}\|u_{i,1} - u_{i,2}\|, \forall y_1, y_2 \in E, x_1 \in E_1, \ldots, x_{i-1} \in E_{i-1}, x_{i+1} \in E_{i+1}, \ldots, x_n \in E_n, u_{i,1} \in T_i(y_1), u_{i,2} \in T_i(y_2).\end{aligned}$$

Theorem 3.9. *Let for each $i \in \{1, 2, \ldots, n\}$, $E_{i,1}$ be a $q_{i,1}$-uniformly smooth Banach space with $q_{i,1} > 1$ and the dual space $E^*_{i,1}$, and $E_{i,2}$ be a real Banach space with the dual space $E^*_{i,2}$. Assume that for each $i \in \{1, 2, \ldots, n\}$, $A_i : E_{i,1} \to E^*_{i,1}$ is a τ_i-Lipschitz continuous mapping, $p_i : E_{i,1} \to E_{i,1}$ is a (γ_i, μ_i)-relaxed cocoercive and λ_{p_i}-Lipschitz continuous mapping and $P_i : \prod_{k=1}^{n} E_{n-(k-1),1} \to E^*_{i,1}$ is a $\lambda_{P_{i,j}}$-Lipschitz continuous mapping in the jth argument $(j = 1, 2, \ldots, n)$. Let for each $i \in \{1, 2, \ldots, n\}$, the mapping $g_i : E_{n-(i-1),1} \to E_{n-(i-1),1}$ be λ_{g_i}-Lipschitz continuous, $S_i : E_{n-(i-1),1} \rightrightarrows CB(E_{n-(i-1),1})$ and $T_i : E_{i,1} \rightrightarrows CB(E_{i,2})$ be λ_{S_i}-$\widehat{H}_{n-(i-1),1}$-Lipschitz continuous and λ_{T_i}-$\widehat{H}_{i,2}$-Lipschitz continuous mappings, respectively, and $F_i : \prod_{k=1}^{n} E_{k,2} \to E^*_{i,1}$ be a $\lambda_{F_{i,j}}$-Lipschitz continuous mapping in the jth argument $(j = 1, 2, \ldots, n)$. Suppose further that for each $i \in \{1, 2, \ldots, n\}$, $\eta_i : E_{i,1} \times E_{i,1} \to E_{i,1}$ is a k_i-Lipschitz continuous mapping, $H_i : E_{i,1} \to E_{i,1}$ is a δ_i-Lipschitz continuous mapping and $M_i : E_{i,1} \rightrightarrows E^*_{i,1}$ is a θ_i-strongly monotone mapping. If there exists constants $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ such that for each $i \in \{1, 2, \ldots, n\}$,*

$$\frac{k_i(\delta_i + \lambda_i\tau_i \sqrt[q_{i,1}]{1 - q_{i,1}\mu_i + (q_{i,1}\gamma_i + c_{q_{i,1}})\lambda_{p_i}^{q_{i,1}}})}{\lambda_i\theta_i} + \sum_{j=1}^{n} \frac{k_j P_{j,n-(i-1)}\lambda_{\theta_j\lambda_{S_{n-(i-1)}}}}{\theta_j} + \sum_{j=1}^{n} \frac{k_j\lambda_{F_{j,i}}\lambda_{T_i}}{\theta_j} < 1, \tag{8}$$

and for the case when $q_{i,1}$ $(i \in \{1, 2, \ldots, n\})$ is an even natural number, in addition to (8), we have $q_{i,1}\mu_i < 1 + (q_{i,1}\gamma_i + c_{q_{i,1}}\lambda_{p_i}^{q_{i,1}}$, where $c_{q_{i,1}}$ $(i = 1, 2, \ldots, n)$ are constants guaranteed by Lemma 2.1, then the iterative sequences $\{(x_{1,m}, x_{2,m}, \ldots, x_{n,m})\}_{m=0}^{\infty}$, $\{(s_{1,m}, s_{2,m}, \ldots, s_{n,m})\}_{m=0}^{\infty}$ and $\{(t_{1,m}, t_{2,m}, \ldots, t_{n,m})\}_{m=0}^{\infty}$ generated by Algorithm 3.4 converge strongly to $(x_1, x_2, \ldots, x_n)$, $(s_1, s_2, \ldots, s_n)$ and $(t_1, t_2, \ldots, t_n)$, respectively, and $(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n)$ is a solution of the SEMNVI (2).

Proof. It follows from (6), Theorem 3.1 and the assumptions that for each $i \in \{1, 2, \ldots, n\}$ and $m \in \mathbb{N}$,

$$
\begin{aligned}
\|x_{i,m+1} - x_{i,m}\|_{i,1} = {} & \|(1-\alpha)x_{i,m} + \alpha R^{H_i,\eta_i}_{M_i,\lambda_i}[H_i(x_{i,m}) - \lambda_i(A_i(x_{i,m} - p_i(x_{i,m}))\\
& + P_i(g_1(s_{1,m}), g_2(s_{2,m}), \ldots, g_n(s_{n,m})) - a_i - F_i(t_{1,m}, t_{2,m}, \ldots, t_{n,m}))]\\
& + \alpha e_{i,m} + r_{i,m} - ((1-\alpha)x_{i,m-1} + \alpha R^{H_i,\eta_i}_{M_i,\lambda_i}[H_i(x_{i,m-1})\\
& - \lambda_i(A_i(x_{i,m-1} - p_i(x_{i,m-1})) + P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), \ldots, g_n(s_{n,m-1}))\\
& - a_i - F_i(t_{1,m-1}, t_{2,m-1}, \ldots, t_{n,m-1}))] + \alpha e_{i,m-1} + r_{i,m-1})\|_{i,1}\\
\leq {} & (1-\alpha)\|x_{i,m} - x_{i,m-1}\|_{i,1} + \alpha\|R^{H_i,\eta_i}_{M_i,\lambda_i}[H_i(x_{i,m}) - \lambda_i(A_i(x_{i,m} - p_i(x_{i,m}))\\
& + P_i(g_1(s_{1,m}), g_2(s_{2,m}), \ldots, g_n(s_{n,m})) - a_i - F_i(t_{1,m}, t_{2,m}, \ldots, t_{n,m}))]\\
& - R^{H_i,\eta_i}_{M_i,\lambda_i}[H_i(x_{i,m-1}) - \lambda_i(A_i(x_{i,m-1} - p_i(x_{i,m-1}))\\
& + P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), \ldots, g_n(s_{n,m-1}))\\
& - a_i - F_i(t_{1,m-1}, t_{2,m-1}, \ldots, t_{n,m-1}))]\|_{i,1}\\
& + \alpha\|e_{i,m} - e_{i,m-1}\|_{i,1} + \|r_{i,m} - r_{i,m-1}\|_{i,1}\\
\leq {} & (1-\alpha)\|x_{i,m} - x_{i,m-1}\|_{i,1} + \alpha\Big(\frac{k_i}{\lambda_i\theta_i}\|H_i(x_{i,m}) - \lambda_i(A_i(x_{i,m} - p_i(x_{i,m}))\\
& + P_i(g_1(s_{1,m}), g_2(s_{2,m}), \ldots, g_n(s_{n,m})) - a_i - F_i(t_{1,m}, t_{2,m}, \ldots, t_{n,m})\\
& - (H_i(x_{i,m-1}) - \lambda_i(A_i(x_{i,m-1} - p_i(x_{i,m-1}))\\
& + P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), \ldots, g_n(s_{n,m-1}))\\
& - a_i - F_i(t_{1,m-1}, t_{2,m-1}, \ldots, t_{n,m-1})))\|_{i,1})\\
& + \alpha\|e_{i,m} - e_{i,m-1}\|_{i,1} + \|r_{i,m} - r_{i,m-1}\|_{i,1}\\
\leq {} & (1-\alpha)\|x_{i,m} - x_{i,m-1}\|_{i,1} + \alpha\Big(\frac{k_i}{\lambda_i\theta_i}(\|H_i(x_{i,m}) - H_i(x_{i,m-1})\|_{i,1}\\
& + \lambda_i\|A_i(x_{i,m} - p_i(x_{i,m})) - A_i(x_{i,m-1} - p_i(x_{i,m-1}))\|_{i,1}\\
& + \lambda_i\|P_i(g_1(s_{1,m}), g_2(s_{2,m}), \ldots, g_n(s_{n,m}))\\
& - P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), \ldots, g_n(s_{n,m-1}))\|_{i,1}\\
& + \lambda_i\|F_i(t_{1,m}, t_{2,m}, \ldots, t_{n,m}) - F_i(t_{1,m-1}, t_{2,m-1}, \ldots, t_{n,m-1})\|_{i,1}\Big)\\
& + \alpha\|e_{i,m} - e_{i,m-1}\|_{i,1} + \|r_{i,m} - r_{i,m-1}\|_{i,1}.
\end{aligned}
\tag{9}
$$

Since for each $i \in \{1, 2, \ldots, n\}$ and $m \in \mathbb{N}$, the mappings H_i and A_i are δ_i-Lipschitz continuous and τ_i-Lipschitz continuous, respectively, we yield

$$
\|H_i(x_{i,m}) - H_i(x_{i,m-1})\|_{i,1} \leq \delta_i\|x_{i,m} - x_{i,m-1}\|_{i,1} \tag{10}
$$

and

$$
\|A_i(x_{i,m} - p_i(x_{i,m})) - A_i(x_{i,m-1} - p_i(x_{i,m-1}))\|_{i,1} \leq \tau_i\|x_{i,m} - x_{i,m-1} - (p_i(x_{i,m}) - p_i(x_{i,m-1}))\|_{i,1}. \tag{11}
$$

Owing to the facts that for each $i \in \{1, 2, \ldots, n\}$, p_i is a (γ_i, μ_i)-relaxed cocoercive and λ_{p_i}-Lipschitz continuous mapping, $E_{i,1}$ is a real $q_{i,1}$-uniformly smooth Banach space with $q_{i,1} > 1$, invoking Lemma 2.1, for each

$i \in \{1,2,\ldots,n\}$ there exists $c_{q_{i,1}} > 0$ such that for each $m \in \mathbb{N}$, we have

$$
\begin{aligned}
\|x_{i,m} - x_{i,m-1} - (p_i(x_{i,m}) - p_i(x_{i,m-1}))\|_{i,1}^{q_{i,1}} &\le \|x_{i,m} - x_{i,m-1}\|_{i,1}^{q_{i,1}} + c_{q_{i,1}}\|p_i(x_{i,m}) - p_i(x_{i,m-1})\|_{i,1}^{q_{i,1}} \\
&\quad - q_{i,1}\langle p_i(x_{i,m}) - p_i(x_{i,m-1}), J_{q_{i,1}}(x_{i,m} - x_{i,m-1})\rangle_{i,1} \\
&\le \|x_{i,m} - x_{i,m-1}\|_{i,1}^{q_{i,1}} + c_{q_{i,1}}\|p_i(x_{i,m}) - p_i(x_{i,m-1})\|_{i,1}^{q_{i,1}} \\
&\quad - q_{i,1}\Big(\gamma_i\|p_i(x_{i,m}) - p_i(x_{i,m-1})\|_{i,1}^{q_{i,1}} + \mu_i\|x_{i,m} - x_{i,m-1}\|_{i,1}^{q_{i,1}}\Big) \\
&= (1 - q_{i,1}\mu_i)\|x_{i,m} - x_{i,m-1}\|_{i,1}^{q_{i,1}} + (q_{i,1}\gamma_i + c_{q_{i,1}})\|p_i(x_{i,m}) - p_i(x_{i,m-1})\|_{i,1}^{q_{i,1}} \\
&\le \Big(1 - q_{i,1}\mu_i + (q_{i,1}\gamma_i + c_{q_{i,1}})\lambda_{p_i}^{q_{i,1}}\Big)\|x_{i,m} - x_{i,m-1}\|_{i,1}^{q_{i,1}},
\end{aligned}
$$

which implies that

$$
\|x_{i,m} - x_{i,m-1} - (p_i(x_{i,m}) - p_i(x_{i,m-1}))\|_{i,1} \le \sqrt[q_{i,1}]{1 - q_{i,1}\mu_i + (q_{i,1}\gamma_i + c_{q_{i,1}})\lambda_{p_i}^{q_{i,1}}}\,\|x_{i,m} - x_{i,m-1}\|_{i,1}. \tag{12}
$$

Since for each $i \in \{1,2,\ldots,n\}$, the mapping P_i is $\lambda_{i,j}$-Lipschitz continuous in the jth argument ($j = 1,2,\ldots,n$), the mapping g_i is λ_{g_i}-Lipschitz continuous and the mapping S_i is λ_{S_i}-$\widehat{H}_{n-(i-1),1}$-Lipschitz continuous, by using (6), for each $i \in \{1,2,\ldots,n\}$ and $m \in \mathbb{N}$, we get

$$
\begin{aligned}
&\|P_i(g_1(s_{1,m}), g_2(s_{2,m}), g_3(s_{3,m}), \ldots, g_n(s_{n,m})) - P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), g_3(s_{3,m}), \ldots, g_n(s_{n,m-1}))\|_{i,1} \\
&\le \|P_i(g_1(s_{1,m}), g_2(s_{2,m}), g_3(s_{3,m}), \ldots, g_n(s_{n,m})) - P_i(g_1(s_{1,m-1}), g_2(s_{2,m}), g_3(s_{3,m}), \ldots, g_n(s_{n,m}))\|_{i,1} \\
&\quad + \|P_i(g_1(s_{1,m-1}), g_2(s_{2,m}), g_3(s_{3,m}), \ldots, g_n(s_{n,m})) - P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), g_3(s_{3,m}), \ldots, g_n(s_{n,m}))\|_{i,1} + \ldots \\
&\quad + \|P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), \ldots, g_{n-1}(s_{n-1,m-1}), g_n(s_{n,m})) \\
&- P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), \ldots, g_{n-1}(s_{n-1,m}), g_n(s_{n,m-1}))\|_{i,1} \\
&= \sum_{i=1}^{n} \|P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), \ldots, g_{j-1}(s_{j-1,m-1}), g_j(s_{j,m}), g_{j+1}(s_{j+1,m}), \ldots, g_n(s_{n,m})) \\
&- P_i(g_1(s_{1,m-1}), g_2(s_{2,m-1}), \ldots, g_{j-1}(s_{j-1,m-1}), g_j(s_{j,m-1}), g_{j+1}(s_{j+1,m}), \ldots, g_n(s_{n,m}))\|_{i,1} \\
&\le \sum_{j=1}^{n} \lambda_{P_{i,j}}\|g_j(s_{j,m}) - g_j(s_{j,m-1})\|_{n-(j-1),1} \\
&\le \sum_{j=1}^{n} \lambda_{P_{i,j}}\lambda_{g_j}\|s_{j,m} - s_{j,m-1}\|_{n-(j-1),1} \\
&\le \sum_{j=1}^{n} \lambda_{P_{i,j}}\lambda_{g_j}(1 + m^{-1})\widehat{H}_{n-(j-1),1}(S_j(x_{n-(j-1),m}), S_j(x_{n-(j-1),m-1})) \\
&\le \sum_{j=1}^{n} \lambda_{P_{i,j}}\lambda_{g_j}\lambda_{S_j}(1 + m^{-1})\|x_{n-(j-1),m} - x_{n-(j-1),m-1}\|_{n-(j-1),1}.
\end{aligned} \tag{13}
$$

Taking into account that for each $i \in \{1,2,\ldots,n\}$, the mapping F_i is $\lambda_{F_{i,j}}$-Lipschitz continuous in the jth argument ($j = 1,2,\ldots,n$), the mapping T_i is λ_{T_i}-$\widehat{H}_{i,2}$-Lipschitz continuous, by using (6), for each $i \in$

$\{1, 2, \ldots, n\}$ and $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\|F_i(t_{1,m}, t_{2,m}, t_{3,m}, \ldots, t_{n,m}) - F_i(t_{1,m-1}, t_{2,m-1}, t_{3,m-1}, \ldots, t_{n,m-1})\|_{i,1} \\
&\qquad \le \|F_i(t_{1,m}, t_{2,m}, \ldots, t_{n,m}) - F_i(t_{1,m-1}, t_{2,m}, \ldots, t_{n,m})\|_{i,1} \\
&\qquad\quad + \|F_i(t_{1,m-1}, t_{2,m}, t_{3,m}, \ldots, t_{n,m}) - F_i(t_{1,m-1}, t_{2,m-1}, t_{3,m}, \ldots, t_{n,m})\| + \ldots \\
&\qquad\quad + \|F_i(t_{1,m-1}, t_{2,m-1}, \ldots, t_{n,m-1}, t_{n,m}) \\
&\qquad\quad - F_i(t_{1,m-1}, t_{2,m-1}, \ldots, t_{n,m-1}, t_{n,m-1})\|_{i,1} \\
&\qquad = \sum_{j=1}^{n} \|F_i(t_{1,m-1}, t_{2,m-1}, \ldots, t_{j-1,m-1}, t_{j,m}, t_{j+1,m}, \ldots, t_{n,m}) \\
&\qquad\quad - F_i(t_{1,m-1}, t_{2,m-1}, \ldots, t_{j-1,m-1}, t_{j,m-1}, t_{j+1,m}, \ldots, t_{n,m})\|_{i,1} \\
&\qquad \le \sum_{j=1}^{n} \lambda_{F_{i,j}} \|t_{j,m} - t_{j,m-1}\|_{j,2} \\
&\qquad \le \sum_{j=1}^{n} \lambda_{F_{i,j}} (1 + m^{-1}) \widehat{H}_{j,2}(T_j(x_{j,m}), T_j(x_{j,m-1})) \\
&\qquad \le \sum_{j=1}^{n} \lambda_{F_{i,j}} \lambda_{T_j} (1 + m^{-1}) \|x_{j,m} - x_{j,m-1}\|_{j,1}.
\end{aligned} \tag{14}$$

Combining (9)–(14), for each $i \in \{1, 2, \ldots, n\}$ and $m \in \mathbb{N}$, yields

$$\begin{aligned}
\|x_{i,m+1} - x_{i,m}\|_{i,1} &\le (1-\alpha)\|x_{i,m} - x_{i,m-1}\|_{i,1} \\
&\quad + \alpha\Big(\frac{k_i(\delta_i + \lambda_i \tau_i \sqrt[q_{i,1}]{1 - q_{i,1}\mu_i + (q_{i,1}\gamma_i + c_{q_{i,1}})\lambda_{p_i}^{q_{i,1}}}\Big)}{\lambda_i \theta_i} \|x_{i,m} - x_{i,m-1}\|_{i,1} \\
&\quad + \frac{k_i}{\theta_i} \sum_{j=1}^{n} \lambda_{P_{i,j}} \lambda_{g_i} \lambda_{S_j} (1 + m^{-1}) \|x_{n-(j-1),m} - x_{n-(j-1),m-1}\|_{n-(j-1),1} \\
&\quad + \frac{k_i}{\theta_i} \sum_{j=1}^{n} \lambda_{F_{i,j}} \lambda_{T_j} (1 + m^{-1}) \|x_{j,m} - x_{j,m-1}\|_{j,1}\Big) \\
&\quad + \alpha\|e_{i,m} - e_{i,m-1}\|_{i,1} + \|r_{i,m} - r_{i,m-1}\|_{i,1} \\
&= (1-\alpha)\|x_{i,m} - x_{i,m-1}\|_{i,1} + \alpha\Big(\omega_{i,m}\|x_{i,m} - x_{i,m-1}\|_{i,1} \\
&\quad + \frac{k_i}{\theta_i} \sum_{j=1}^{n} \lambda_{P_{i,j}} \lambda_{g_i} \lambda_{S_j} (1 + m^{-1}) \|x_{n-(j-1),m} - x_{n-(j-1),m-1}\|_{n-(j-1),1} \\
&\quad + \frac{k_i}{\theta_i} \sum_{j=1}^{n} \lambda_{F_{i,j}} \lambda_{T_j} (1 + m^{-1}) \|x_{j,m} - x_{j,m-1}\|_{j,1}\Big) \\
&\quad + \alpha\|e_{i,m} - e_{i,m-1}\|_{i,1} + \|r_{i,m} - r_{i,m-1}\|_{i,1},
\end{aligned} \tag{15}$$

where for each $i \in \{1, 2, \ldots, n\}$,

$$\omega_i = \frac{k_i\Big(\delta_i + \lambda_i \tau_i \sqrt[q_{i,1}]{1 - q_{i,1}\mu_i + (q_{i,1}\gamma_i + c_{q_{i,1}})\lambda_{p_i}^{q_{i,1}}}\Big)}{\lambda_i \theta_i}.$$

Setting $\vartheta_{j,m} = \sum_{i=1}^{n} \frac{k_i \lambda_{P_{i,j}} \lambda_{g_i} \lambda_{S_j}(1+m^{-1})}{\theta_i}$ and $Q_{j,m} = \sum_{i=1}^{n} \frac{k_i \lambda_{F_{i,j}} \lambda_{T_j}(1+m^{-1})}{\theta_i}$ for each $i \in \{1, 2, \ldots, n\}$, and $\Lambda(m) = \max\{\omega_{j,m} +$

$\vartheta_{n-(j-1),m} + Q_{j,m} : j = 1, 2, \dots, n\}$, it follows from (15) that

$$\begin{aligned}\sum_{i=1}^{n} \|x_{i,m+1} - x_{i,m}\|_{i,1} \leq (1-\alpha) \sum_{i=1}^{n} \|x_{i,m} - x_{i,m-1}\|_{i,1} + \alpha\Lambda(n) \sum_{i=1}^{n} \|x_{i,m} - x_{i,m-1}\|_{i,1} \\ + \alpha \sum_{i=1}^{n} \|e_{i,m} - e_{i,m-1}\|_{i,1} + \sum_{i=1}^{n} \|r_{i,m} - r_{i,m-1}\|_{i,1}.\end{aligned} \tag{16}$$

In virtue of the facts that for each $j \in \{1, 2, \dots, n\}$, $\vartheta_{j,m} \to \vartheta_j$ and $Q_{j,m} \to Q_j$, as $m \to \infty$, we deduce that $\Lambda(m) \to \Lambda$, as $m \to \infty$, where $\Lambda = \max\{\omega_j + \vartheta_{n-(j-1)} + Q_j : j = 1, 2, \dots, n\}$. Now, letting $\varphi(m) = 1 - \alpha + \alpha\Lambda(m)$, for each $m \geq 0$, we know that $\varphi(m) \to \varphi$, as $m \to \infty$, where $\varphi = 1 - \alpha + \alpha\Lambda$. Clearly, (8) implies that $\Lambda < 1$ and so $\varphi \in (0, 1)$. Hence, there exists $\hat{\varphi} \in (0, 1)$ (take $\hat{\varphi} = \frac{\varphi+1}{2} \in (\varphi, 1)$) and $n_0 \in \mathbb{N}$ such that $\varphi(n) \leq \hat{\varphi}$, for all $n \geq n_0$. Then, for all $n > n_0$, by (16), it follows that

$$\begin{aligned}\sum_{i=1}^{n} \|x_{i,m+1} - x_{i,m}\|_{i,1} &\leq \hat{\varphi} \sum_{i=1}^{n} \|x_{i,m} - x_{i,m-1}\|_{i,1} + \alpha \sum_{i=1}^{n} \|e_{i,m} - e_{i,m-1}\|_{i,1} + \sum_{i=1}^{n} \|r_{i,m} - r_{i,m-1}\|_{i,1} \\ &\leq \hat{\varphi}[\hat{\varphi} \sum_{i=1}^{n} \|x_{i,m-1} - x_{i,m-2}\|_{i,1} + \alpha \sum_{i=1}^{n} \|e_{i,m-1} - e_{i,m-2}\|_{i,1} + \sum_{i=1}^{n} \|r_{i,m-1} - r_{i,m-2}\|_{i,1}] \\ &\quad + \alpha \sum_{i=1}^{n} \|e_{i,m} - e_{i,m-1}\|_{i,1} + \sum_{i=1}^{n} \|r_{i,m} - r_{i,m-1}\|_{i,1} \\ &= \hat{\varphi}^2 \sum_{i=1}^{n} \|x_{i,m-1} - x_{i,m-2}\|_{i,1} + \alpha\Big(\hat{\varphi} \sum_{i=1}^{n} \|e_{i,m-1} - e_{i,m-2}\|_{i,1} + \sum_{i=1}^{n} \|e_{i,m} - e_{i,m-1}\|_{i,1}\Big) \\ &\quad + \hat{\varphi} \sum_{i=1}^{n} \|r_{i,m-1} - r_{i,m-2}\|_{i,1} + \sum_{i=1}^{n} \|r_{i,m} - r_{i,m-1}\|_{i,1} \\ &\leq \cdots \leq \hat{\varphi}^{m-n_0} \sum_{i=1}^{n} \|x_{i,n_0+1} - x_{i,n_0}\|_{i,1} + \alpha \sum_{j=1}^{m-n_0} \sum_{i=1}^{n} \hat{\varphi}^{j-1} \|e_{i,m-(j-1)} - e_{i,m-j}\|_{i,1} \\ &\quad + \sum_{j=1}^{m-n_0} \sum_{i=1}^{n} \hat{\varphi}^{j-1} \|r_{i,m-(j-1)} - r_{i,m-j}\|_{i,1}.\end{aligned} \tag{17}$$

The preceding relation (17) implies that for any $k \geq l > n_0$,

$$\begin{aligned}\sum_{i=1}^{n} \|x_{i,k} - x_{i,l}\|_{i,1} &\leq \sum_{i=1}^{n} \sum_{m=l}^{k-1} \|x_{i,m+1} - x_{i,m}\|_{i,1} = \sum_{m=l}^{k-1} \sum_{i=1}^{n} \|x_{i,m+1} - x_{i,m}\|_{i,1} \\ &\leq \sum_{m=l}^{k-1} \sum_{i=1}^{n} \hat{\varphi}^{m-n_0} \|x_{i,n_0+1} - x_{i,n_0}\|_{i,1} + \alpha \sum_{m=l}^{k-1} \sum_{j=1}^{m-n_0} \sum_{i=1}^{n} \hat{\varphi}^{j-1} \|e_{i,m-(j-1)} - e_{i,m-j}\|_{i,1} \\ &\quad + \sum_{m=l}^{k-1} \sum_{j=1}^{m-n_0} \sum_{i=1}^{n} \hat{\varphi}^{j-1} \|r_{i,m-(j-1)} - r_{i,m-j}\|_{i,1}.\end{aligned} \tag{18}$$

In the light of the fact $\hat{\varphi} < 1$, (7) and (18) guarantee that

$$\sum_{i=1}^{n} \|x_{i,k} - x_{i,l}\|_{i,1} \to 0, \text{ as } l \to \infty. \tag{19}$$

Now, let us define a norm $\|.\|_*$ on $\prod_{i=1}^{n} E_{i,1}$ by

$$\|(x_1, x_2, \dots, x_n)\|_* = \sum_{i=1}^{n} \|x_i\|_{i,1}, \quad \forall (x_1, x_2, \dots, x_n) \in \prod_{i=1}^{n} E_{i,1}.$$

It is easy to see that $(\prod_{i=1}^{n} E_{i,1}, \|.\|_*)$ is a Banach space. Then (19) ensures that

$$\|(x_{1,k}, x_{2,k}, \dots, x_{n,k}) - (x_{1,l}, x_{2,l}, \dots, x_{n,l})\|_* \to 0, \quad \text{as } l \to \infty,$$

that is, the sequence $\{(x_{1,l}, x_{2,l}, \dots, x_{n,l})\}_{l=0}^{\infty}$ is a Cauchy sequence in $\prod_{i=1}^{n} E_{i,1}$. The completeness of $\prod_{i=1}^{n} E_{i,1}$ implies the existence of $(x_1, x_2, \dots, x_l) \in \prod_{i=1}^{n} E_{i,1}$ such that

$$(x_{1,m}, x_{2,m}, \dots, x_{n,m}) \to (x_1, x_2, \dots, x_n), \quad \text{as } m \to \infty.$$

Considering the fact that for each $i \in \{1, 2, \dots, n\}$, the mapping S_i is λ_{S_i}-$\widehat{H}_{i,1}$-Lipschitz continuous, by using (6), for each $i \in \{1, 2, \dots, n\}$ and $m \geq 0$, we get

$$\begin{aligned} \|s_{i,m+1} - s_{i,m}\|_{n-(i-1),1} &\leq (1 + (1 + m)^{-1})\widehat{H}_{n-(i-1),1}(S_i(x_{n-(i-1),m+1}), S_i(x_{n-(i-1),m})) \\ &\leq (1 + (1 + m)^{-1})\lambda_{S_i}\|x_{n-(i-1),m+1} - x_{n-(i-1),m}\|_{n-(i-1)}. \end{aligned} \tag{20}$$

Since for each $i \in \{1, 2, \dots, n\}$, $\|x_{i,m+1} - x_{i,m}\|_{i,1} \to 0$, as $m \to \infty$, from (20), we conclude that $\|s_{i,m+1} - s_{i,m}\|_{n-(i-1),1} \to 0$, as $n \to \infty$. Hence,

$$\lim_{m\to\infty} \sum_{i=1}^{n} \|s_{i,m+1} - s_{i,m}\|_{n-(i-1),1} = \|(s_{1,m+1}, s_{2,m+1}, \dots, s_{n,m+1}) - (s_{1,m}, s_{2,m}, \dots, s_{n,m})\|_* = 0,$$

that is, the sequence $\{(s_{1,m}, s_{2,m}, \dots, s_{n,m})\}_{n=0}^{\infty}$ is also a Cauchy sequence in $\prod_{i=1}^{n} S_i(x_{n-(i-1)}) \subseteq \prod_{i=1}^{n} E_{i,1}$. Consequently, there exists $(s_1, s_2, \dots, s_n) \in \prod_{i=1}^{n} E_{i,1}$ such that

$$(s_{1,m}, s_{2,m}, \dots, s_{n,m}) \to (s_1, s_2, \dots, s_n), \quad \text{as } m \to \infty.$$

In the meanwhile, due to the fact that for each $i \in \{1, 2, \dots, n\}$, the mapping S_i is λ_{S_i}-$\widehat{H}_{n-(i-1),1}$-Lipschitz continuous, we infer that

$$\begin{aligned} d(s_i, S_i(x_{n-(i-1)})) &= \inf\{\|s_i - t\|_{n-(i-1),1} : t \in S_i(x_{n-(i-1)})\} \\ &\leq \|s_i - s_{i,m}\|_{n-(i-1),1} + d(s_{i,m}, S_i(x_{n-(i-1)})) \\ &\leq \|s_i - s_{i,m}\|_{n-(i-1),1} + \widehat{H}_{n-(i-1),1}(S_i(x_{n-(i-1),m}), S_i(x_{n-(i-1)})) \\ &\leq \|s_i - s_{i,m}\|_{n-(i-1),1} + \lambda_{S_i}\|x_{n-(i-1),m} - x_{n-(i-1)}\|_{n-(i-1)}. \end{aligned}$$

The right-hand side of the above inequality approaches zero, as $m \to \infty$. Accordingly, for $i = 1, 2, \dots, n$ we derive that $s_i \in S_i(x_{n-(i-1)})$. In a similar way, on can show that the sequence $\{(t_{1,m}, t_{2,m}, \dots, t_{n,m})\}_{m=0}^{\infty}$ is a Cauchy sequence in $\prod_{i=1}^{n} T_i(x_{i,m}) \subseteq \prod_{i=1}^{n} E_{i,2}$,

$$(t_{1,m}, t_{2,m}, \dots, t_{n,m}) \to (t_1, t_2, \dots, t_n), \quad \text{as } m \to \infty,$$

for some $(t_1, t_2, \dots, t_m) \in \prod_{i=1}^{n} E_{i,2}$ and $t_i \in T_i(x_i)$ for $i = 1, 2, \dots, n$. Since the mappings $R^{H_i,\eta_i}_{M_i,\lambda_i}$, H_i, η_i, A_i, P_i, S_i, T_i, F_i, g_i and p_i ($i = 1, 2, \dots, n$) are continuous, it follows from (5) and (7) that for each $i \in \{1, 2, \dots, n\}$,

$$x_i = R^{H_i,\eta_i}_{M_i,\lambda_i}[H_i(x_i) - \lambda_i(A_i(x_i - p_i(x_i)) + P_i(g_1(s_1), g_2(s_2), \dots, g_n(s_n)) - a_i - F_i(t_1, t_2, \dots, t_n))].$$

Now, this fact and Theorem 3.1 guarantee that $(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, x_n, t_1, t_2, \ldots, t_n) \in \prod_{i=1}^{n} E_{i,1} \times \prod_{i=1}^{n} S_i(x_{n-(i-1)}) \times \prod_{i=1}^{n} T_i(x_i)$ is a solution of the SEMNVI (2). This completes the proof. □

The following corollary is an immediately consequence of the above theorem.

Corollary 3.10. *Let E be a real q-uniformly smooth Banach space with the dual space E^*, and let for $i = 1, 2, \ldots, n$, E_i be real Banach spaces. Let the vector-valued mapping $\eta : E \times E \to E$ be k-Lipschitz continuous, $H : E \to E^*$ an η-monotone and δ-Lipschitz continuous mapping, $p : E \to E$ a (γ, μ)-relaxed cocoercive and λ_p-Lipschitz continuous mapping, and $\widehat{M} : E \rightrightarrows E^*$ a general (H, η)-strongly monotone mapping with constant θ. Let $A : E \to E^*$ be a τ-Lipslchitz continuous mapping and for each $i \in \{1, 2, \ldots, n\}$, $T_i : E \rightrightarrows CB(E_i)$ be a λ_{t_i}-$\widehat{H}_i$-Lipschitz continuous mapping. Suppose further that $F : \prod_{i=1}^{n} E_i \to E^*$ is λ_{F_i}-Lipschitz continuous in the ith argument $(i = 1, 2, \ldots, n)$ and there exists a constant $\lambda > 0$ such that*

$$\frac{k}{\lambda\theta}(\delta + \lambda\tau \sqrt[q]{1 - q\mu + (q\gamma + c_q)\lambda_p^q} + \lambda \sum_{i=1}^{n} \lambda_{F_i}\lambda_{t_i}) < 1, \tag{21}$$

and for the case when q is an even natural number, in addition to (21)*, the condition $q\mu < 1 + (q\gamma + c_q)\lambda_p^q$ holds, where c_q is a constant guaranteed by Lemma 2.1. Then, the iterative sequences $\{x_m\}_{m=0}^{\infty}$ and $\{(t_{1,m}, t_{2,m}, \ldots, t_{n,m})\}_{m=0}^{\infty}$ generated by Algorithm 3.5 converges strongly to $x \in E$ and $(t_1, t_2, \ldots, t_n) \in \prod_{i=1}^{n} T_i(x) \subseteq \prod_{i=1}^{n} E_i$, respectively, and $(x, t_1, t_2, \ldots, t_n)$ is a solution of the* NMVIP (3).

4. Remarks on C_n-η-monotone mappings

In this section, the C_n-η-monotone mapping introduced in [13] is investigated and analyzed and some important facts concerning it are pointed out. At the same time, we show that one can derive the results in [13] by using the conclusions of the previous sections.

Definition 4.1. *[13, Definition 6] Let $n \geq 3$ and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ be a multi-valued mapping, $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, and $\eta : E \times E \to E$ single-valued mappings.*

(i) *For each $1 \leq i \leq n$, $M(\ldots, f_i, \ldots)$ is said to be α_i-strongly η-monotone with respect to f_i (in the ith argument) if there exists a constant $\alpha_i > 0$ such that*

$$\langle w_i - w_i', \eta(x, y)\rangle \geq \alpha_i\|x - y\|^2, \forall x, y, u_1 \in E_1, u_2 \in E_2, \ldots, u_{i-1} \in E_{i-1}, u_{i+1} \in E_{i+1}, \ldots, u_n \in E_n,$$
$$w_i \in M(u_1, \ldots, u_{i-1}, f_i(x), u_{i+1}, \ldots, u_n), w_i' \in M(u_1, \ldots, u_{i-1}, f_i(y), u_{i+1}, \ldots, u_n).$$

(ii) *For each $1 \leq i \leq n$, $M(\ldots, f_i, \ldots)$ is said to be β_i-relaxed η-monotone with respect to f_i (in the ith argument) if there exists a constant $\beta_i > 0$ such that*

$$\langle w_i - w_i', \eta(x, y)\rangle \geq -\beta_i\|x - y\|^2, \forall x, y \in E, u_1 \in E_1, u_2 \in E_2, \ldots, u_{i-1} \in E_{i-1}, u_{i+1} \in E_{i+1}, \ldots, u_n \in E_n,$$
$$w_i \in M(u_1, \ldots, u_{i-1}, f_i(x), u_{i+1}, \ldots, u_n), w_i' \in M(u_1, \ldots, u_{i-1}, f_i(y), u_{i+1}, \ldots, u_n).$$

(iii) *By assumption that n be an even natural number, M is said to be $\alpha_1\beta_2\alpha_3\beta_4 \ldots \alpha_{n-1}\beta_n$-symmetric η-monotone with respect to $f_1, f_2, \ldots, f_n$ if, for each $i \in \{1, 3, \ldots, n-1\}$, $M(\ldots, f_i, \ldots)$ is α_i-strongly η-monotone with respect to f_i (in the ith argument) and for each $j \in \{2, 4, \ldots, n\}$, $M(\ldots, f_j, \ldots)$ is β_j-relaxed η-monotone with respect to f_j (in the jth argument) with $\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} > \beta_2 + \beta_4 + \cdots + \beta_n$.*

(iv) *By assumption that n be an odd natural number, M is said to be $\alpha_1\beta_2\alpha_3\beta_4 \ldots \beta_{n-1}\alpha_n$-symmetric η-monotone with respect to $f_1, f_2, \ldots, f_n$ if, for each $i \in \{1, 3, \ldots, n\}$, $M(\ldots, f_i, \ldots)$ is α_i-strongly η-monotone with respect to f_i (in the ith argument) and for each $j \in \{2, 4, \ldots, n-1\}$, $M(\ldots, f_j, \ldots)$ is β_j-relaxed η-monotone with respect to f_j (in the jth argument) with $\alpha_1 + \alpha_3 + \cdots + \alpha_n > \beta_2 + \beta_4 + \cdots + \beta_{n-1}$.*

Proposition 4.2. *Let E be a real Banach space with the dual space E^*, and let for $i = 1,2,\dots,n$, E_i be real Banach spaces. Suppose that $n \geq 3$ and $f_i : E \to E_i$ ($i = 1,2,\dots,n$) and $\eta : E \times E \to E$ are vector-valued mappings and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ a multi-valued mapping. Furthermore, let $\widehat{M} : E \rightrightarrows E^*$ be defined by $\widehat{M}(x) = M(f_1(x), f_2(x),\dots, f_n(x))$, for all $x \in E$. Then, the following conclusions hold:*

(i) *If n is an even natural number and M is an $\alpha_1\beta_2\alpha_3\beta_4\dots\alpha_{n-1}\beta_n$-symmetric η-monotone mapping with respect to $f_1, f_2,\dots, f_n$, then $\widehat{M}$ is a $\sum_{i=1}^{\frac{n}{2}}(\alpha_{2i-1} - \beta_{2i})$-strongly η-monotone mapping.*

(ii) *If n is an odd natural number and M is an $\alpha_1\beta_2\alpha_3\beta_4\dots\beta_{n-1}\alpha_n$-symmetric η-monotone mapping with respect to $f_1, f_2,\dots, f_n$, then $\widehat{M}$ is a $\Big(\sum_{i=1}^{\frac{n+1}{2}}\alpha_{2i-1} - \sum_{i=1}^{\frac{n-1}{2}}\beta_{2i}\Big)$-strongly η-monotone mapping.*

Proof. We first let n be an even natural number. Owing to the fact that M is an $\alpha_1\beta_2\alpha_3\beta_4\dots\alpha_{n-1}\beta_n$-symmetric η-monotone mapping with respect to $f_1, f_2,\dots, f_n$, for all $x, y \in E$, $u \in \widehat{M}(x) = M(f_1(x), f_2(x),\dots, f_n(x))$ and $v \in \widehat{M}(y) = M(f_1(y), f_2(y),\dots, f_n(y))$, we yield

$$\begin{aligned}
\langle u - v, \eta(x,y)\rangle &= \langle u + \sum_{i=1}^{n-1}(-w_i + w_i) - v, \eta(x,y)\rangle \\
&= \langle u - w_1, \eta(x,y)\rangle + \sum_{i=1}^{n-2}\langle w_i - w_{i+1}, \eta(x,y)\rangle + \langle w_{n-1} - v, \eta(x,y)\rangle \\
&= \langle u - w_1, \eta(x,y)\rangle + \sum_{i=1}^{\frac{n-2}{2}}\langle w_{2i-1} - w_{2i}, \eta(x,y)\rangle + \sum_{i=1}^{\frac{n-2}{2}}\langle w_{2i} - w_{2i+1}, \eta(x,y)\rangle + \langle w_{n-1} - v, \eta(x,y)\rangle \qquad (22)\\
&\geq \alpha_1\|x-y\|^2 - \sum_{i=1}^{\frac{n-2}{2}}\beta_{2i}\|x-y\|^2 + \sum_{i=1}^{\frac{n-2}{2}}\alpha_{2i+1}\|x-y\|^2 - \beta_n\|x-y\|^2 \\
&= \sum_{i=1}^{\frac{n}{2}}\alpha_{2i-1}\|x-y\|^2 - \sum_{i=1}^{\frac{n}{2}}\beta_{2i}\|x-y\|^2 = \sum_{i=1}^{\frac{n}{2}}(\alpha_{2i-1} - \beta_{2i})\|x-y\|^2,
\end{aligned}$$

where for each $1 \leq i \leq n-1$, $w_i \in M(f_1(y), f_2(y),\dots, f_i(y), f_{i+1}(x),\dots, f_n(x))$. Taking into account that $\alpha_1+\alpha_3+\dots+\alpha_{n-1} = \sum_{i=1}^{\frac{n}{2}}\alpha_{2i-1} > \beta_2+\beta_4+\dots+\beta_n = \sum_{i=1}^{\frac{n}{2}}\beta_{2i}$, it follows from (22) that $\widehat{M}$ is a $\sum_{i=1}^{\frac{n}{2}}(\alpha_{2i-1}-\beta_{2i})$-strongly η-monotone mapping.

We now prove conclusion (ii). Assume that n is an odd natural number. Since M is an $\alpha_1\beta_2\alpha_3\beta_4\dots\beta_{n-1}\alpha_n$-symmetric η-monotone mapping with respect to $f_1, f_2,\dots, f_n$, for all $x, y \in E$, $u \in \widehat{M}(x) = M(f_1(x), f_2(x),\dots, f_n(x))$ and $v \in \widehat{M}(y) = M(f_1(y), f_2(y),\dots, f_n(y))$, we obtain

$$\begin{aligned}
\langle u - v, \eta(x,y)\rangle &= \langle u + \sum_{i=1}^{n-1}(-w_i + w_i) - v, \eta(x,y)\rangle \\
&= \langle u - w_1, \eta(x,y)\rangle + \sum_{i=1}^{n-2}\langle w_i - w_{i+1}, \eta(x,y)\rangle + \langle w_{n-1} - v, \eta(x,y)\rangle \\
&= \langle u - w_1, \eta(x,y)\rangle + \sum_{i=1}^{\frac{n-1}{2}}\langle w_{2i-1} - w_{2i}, \eta(x,y)\rangle + \sum_{i=1}^{\frac{n-3}{2}}\langle w_{2i} - w_{2i+1}, \eta(x,y)\rangle + \langle w_{n-1} - v, \eta(x,y)\rangle
\end{aligned}$$

$$\begin{aligned}&\geq \alpha_1\|x-y\|^2-\sum_{i=1}^{\frac{n-1}{2}}\beta_{2i}\|x-y\|^2+\sum_{i=1}^{\frac{n-3}{2}}\alpha_{2i+1}\|x-y\|^2+\alpha_n\|x-y\|^2\\&=\sum_{i=1}^{\frac{n+1}{2}}\alpha_{2i-1}\|x-y\|^2-\sum_{i=1}^{\frac{n-1}{2}}\beta_{2i}\|x-y\|^2\Big(\sum_{i=1}^{\frac{n+1}{2}}\alpha_{2i-1}-\sum_{i=1}^{\frac{n-1}{2}}\beta_{2i})\|x-y\|^2,\end{aligned} \tag{23}$$

where for each $1 \leq i \leq n-1$, $w_i \in M(f_1(y), f_2(y), \dots, f_i(y), f_{i+1}(x), \dots, f_n(x))$. Thanks to the facts that $\alpha_1 + \alpha_3 + \cdots + \alpha_n = \sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} > \beta_2 + \beta_4 + \cdots + \beta_{n-1} = \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i}$, making use of (23) we conclude that $\widehat{M}$ is a $\big(\sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} - \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i}\big)$-strongly η-monotone mapping. This completes the proof. □

Remark 4.3. *In virtue of Proposition 4.2 and the arguments mentioned above, we found that the notions of $\alpha_1\beta_2\alpha_3\beta_4\dots\alpha_{n-1}\beta_n$-symmetric η-monotonicity and $\alpha_1\beta_2\alpha_3\beta_4\dots\beta_{n-1}\alpha_n$-symmetric η-monotonicity of the mapping $M : E^n \rightrightarrows E^*$ with respect to the mappings $f_1, f_2, \dots, f_n : E \to E$, given in parts* (iii) *and* (iv) *of Definition 4.1 are* actually the same notion of θ-strong monotonicity of the mapping $\widehat{M} = M(f_1, f_2, \dots, f_n)$ presented in Definition 2.3(iv), where $\theta = \sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})$ for the case when n is an even natural number, and $\theta = \sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} - \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i}$ when n is an odd natural number.

Guan and Hu [13] introduced and studied a class of generalized monotone mappings the so-called C_n-η-monotone mappings as follows.

Definition 4.4. *[13, Definition 10] Let E be a real Banach space with the dual space E^*. Let $n \geq 3$; $f_i : E \to E_i$ ($i = 1, 2, \dots, n$) and $C_n : E \to E^*$ be single-valued mappings and let $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ a multi-valued mapping.*

- (i) *In case n is an even natural number, M is said to be a C_n-η-monotone mapping if M is $\alpha_1\beta_2\alpha_3\beta_4\dots\alpha_{n-1}\beta_n$-symmetric η-monotone with respect to $f_1, f_2, \dots, f_n$ and $(C_n + \lambda M(f_1, f_2, \dots, f_n))(E) = E^*$ holds, for every $\lambda > 0$.*
- (ii) *In case n is an odd natural number, M is said to be a C_n-η-monotone mapping if M is $\alpha_1\beta_2\alpha_3\beta_4\dots\beta_{n-1}\alpha_n$-symmetric η-monotone with respect to $f_1, f_2, \dots, f_n$ and $(C_n + \lambda M(f_1, f_2, \dots, f_n))(E) = E^*$ holds, for every $\lambda > 0$.*

With the purpose of showing the existence of the class of C_n-η-monotone mappings, the authors [13] gave [13, Example 12]. But, we show that contrary to the claim in [13], the C_n-η-monotone mapping presented in [13, Example 12] is actually a general (H, η)-strongly monotone mapping and is not a new one.

Example 4.5. Let $E = l^2$ denote the space of all square-summable sequences, i.e., the space of all sequences $\{x_m\}_{m=1}^{\infty}$ for which $\sum_{m=1}^{\infty} |x_m|^2$ converges, and $\|.\|_2$ be a norm defined on l^2 by $\|x\|_2 = (\sum_{m=1}^{\infty} |x_m|^2)^{\frac{1}{2}}$, for all $x = \{x_m\}_{m=1}^{\infty} \in l^2$. It is well known that l^2 together with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i\bar{y}_i, \quad \forall x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in l^2,$$

is a Hilbert space and so $E^* = l^2$. In the meanwhile, $\{e_m\}_{m=1}^{\infty}$, where for each $m \in \mathbb{N}$, $e_m = (0, 0, \dots, 1, 0, 0, \dots)$, 1 at the nth coordinate and all other coordinates are zero, is a Schauder basis of $E = l^2$.

Let n be an even natural number and let for each $i \in \{1, 2, \dots, n\}$, $E_i = (E, \|.\|_i)$, where $\|.\|_i$ ($i = 1, 2, \dots, n$) are the equivalent norms on l^2 space. Suppose that for $i = 1, 2, \dots, n$, the mappings $f_i : E \to E_i$ are defined by

$$f_i(x) = \begin{cases} \alpha_i x + e_i, & \text{if } i = 1, 3, \dots, n-1,\\ -\beta_i x + e_i, & \text{if } i = 2, 4, \dots, n,\end{cases}$$

for all $x = (x_1, x_2, \dots, x_n, \dots) \in l^2$, where $\alpha_{2i-1}, \beta_{2i} > 0$ $(i = 1, 2, \dots, \frac{n}{2})$ are constants such that $\sum\limits_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i}) = \gamma > 0$.

Let $M : \prod\limits_{i=1}^{n} E_i \rightrightarrows E^*$ be defined by $M(u_1, u_2, \dots, u_n) = \sum\limits_{i=1}^{n} (u_i - e_i)$, for all $(u_1, u_2, \dots, u_n) \in \prod\limits_{i=1}^{n} E_i$, and define the mappings $C_n : E \to E^*$ and $\eta : E \times E \to E$ by $C_n(x) = x + e_{n+1}$ and $\eta(x, y) = x - y$, for all $x, y \in E$. Since for each $i \in \{1, 2, \dots, \frac{n}{2}\}$, we have $f_{2i-1}(x) = \alpha_{2i-1} x + e_{2i-1}$ and $f_{2i}(x) = -\beta_{2i} x + e_{2i}$, for all $x, y \in E$, $u_j \in E_j$, $j = 1, 2, \dots, 2(i-1), 2i, \dots, n$ and $i = 1, 2, \dots, \frac{n}{2}$, we obtain

$$\begin{aligned}&\langle M(u_1, u_2, \dots, u_{2i-2}, f_{2i-1}(x), u_{2i}, \dots, u_n) - M(u_1, u_2, \dots, u_{2i-2}, f_{2i-1}(y), u_{2i}, \dots, u_n), \eta(x, y)\rangle \\ &\qquad = \langle f_{2i-1}(x) - f_{2i-1}(y), x - y\rangle = \langle \alpha_{2i-1} x - \alpha_{2i-1} y, x - y\rangle = \alpha_{2i-1} \|x - y\|_2^2,\end{aligned}$$

and

$$\begin{aligned}&\langle M(u_1, u_2, \dots, u_{2i-1}, f_{2i}(x), u_{2i+1}, \dots, u_n) - M(u_1, u_2, \dots, u_{2i-1}, f_{2i}(y), u_{2i+1}, \dots, u_n), \eta(x, y)\rangle \\ &\qquad = \langle f_{2i}(x) - f_{2i}(y), x - y\rangle = \langle -\beta_{2i} x + \beta_{2i} y, x - y\rangle = -\beta_{2i} \|x - y\|_2^2,\end{aligned}$$

that is, for each $i \in \{1, 2, \dots, \frac{n}{2}\}$, $M(\dots, f_{2i-1}, \dots)$ and $M(\dots, f_{2i}, \dots)$ are α_{2i-1}-strongly η-monotone with respect to f_{2i-1} in the $(2i-1)$th argument, and β_{2i}-relaxed η-monotone with respect to f_{2i} in the $(2i)$th argument. Taking into account that for each $i \in \{1, 2, \dots, n-1\}$, $M(\dots, f_i, \dots)$ is α_i-strongly η-monotone with respect to f_i in the ith argument and for each $j \in \{2, 4, \dots, n\}$, $M(\dots, f_j, \dots)$ is β_j-relaxed η-monotone with respect to f_j (in the jth argument) and $\sum\limits_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i}) = \gamma > 0$, it follows that M is an $\alpha_1\beta_2\alpha_3\beta_4 \dots \alpha_{n-1}\beta_n$-symmetric η-monotone mapping with respect to the mappings $f_1, f_2, \dots, f_n$.

At the same time, for all $x = \{x_m\}_{m=1}^{\infty} \in l^2$, we yield

$$\begin{aligned}M(f_1(x), f_2(x), \dots, f_n(x)) &= \sum_{i=1}^{n} (f_i(x) - e_i) = \sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{n} e_i \\ &= \sum_{i=1}^{\frac{n}{2}} (f_{2i-1}(x) + f_{2i}(x)) - \sum_{i=1}^{n} e_i \\ &= \sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i}) x - \hat{e} = \gamma x - \hat{e},\end{aligned}$$

where $\hat{e} = \sum\limits_{i=1}^{n} e_i = (1, 1, \dots, 1, 0, 0, \dots)$, is an element of $E = l^2$ having the first n entries 1 and the rest 0. Owing to the fact that for each $x \in l^2$, there is $\frac{x + \lambda\hat{e} - e_{n+1}}{1 + \lambda\gamma} \in l^2$ such that we have $(C_n + \lambda M)(\frac{x + \lambda\hat{e} - e_{n+1}}{1 + \lambda\gamma}) = x$, where $\gamma = \sum\limits_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})$, we deduce that $(C_n + \lambda M)(E) = E^*$ for every $\lambda > 0$. Thanks to the above-mentioned arguments, the author [13] concluded that M is a C_n-η-monotone mapping.

Let us now define the mapping $\widehat{M} : E \rightrightarrows E^*$ as $\widehat{M}(x) = M(f_1(x), f_2(x), \dots, f_n(x))$, for all $x \in E = l^2$. Then, for all $x = \{x_m\}_{m=1}^{\infty} \in l^2$, we have $\widehat{M}(x) = \sum\limits_{i=1}^{n} (f_i(x) - e_i) = \sum\limits_{i=1}^{n} f_i(x) - \sum\limits_{i=1}^{n} e_i = \gamma x - \hat{e}$. Moreover, for all $x = \{x_m\}_{m=1}^{\infty}, y = \{y_m\}_{m=1}^{\infty} \in l^2$, we get

$$\langle \widehat{M}(x) - \widehat{M}(y), \eta(x, y)\rangle = \langle \gamma x - \hat{e} - (\gamma y - \hat{e}), \eta(x, y)\rangle = \gamma \|x - y\|_2^2,$$

that is, $\widehat{M}$ is a γ-strongly η-monotone mapping. By taking $H = C_n$, it can be easily observed that $(H + \lambda\widehat{M})(E) = E^*$ holds, for every real constant $\lambda > 0$. Hence, according to Definition 2.6, $\widehat{M}$ is a general (H, η)-monotone mapping.

Here it is to be noted that in the light of the arguments mentioned above, in contrary to the claim in [13], the C_n-η-monotone mapping given in [13, Example 12] is actually a general (H,η)-strongly monotone mapping with constant γ, and is not a new one. In general, if E is a real Banach space with the dual space E^*, E_i $(i = 1, 2, \dots, n)$ are real Banach spaces, $f_i : E \to E_i$, $\eta : E \times E \to E$ and $C_n : E \to E^*$ $(n \geq 3)$ are single-valued mappings and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ is a C_n-η-monotone mapping, then in view of Definition 4.4, for the case when n is an even natural number, M is an $\alpha_1\beta_2\alpha_3\beta_4\dots\alpha_{n-1}\beta_n$-symmetric η-monotone mapping with respect to $f_1, f_2, \dots, f_n$, and in the case where n is an odd natural number, M is an $\alpha_1\beta_2\alpha_3\beta_4\dots\beta_{n-1}\alpha_n$-symmetric η-monotone mapping with respect to $f_1, f_2, \dots, f_n$. In the meanwhile, in both the cases, we have $(C_n + \lambda M(f_1, f_2, \dots, f_n))(E) = E^*$ for every real constant $\lambda > 0$. Then, by defining $\widehat{M} : E \rightrightarrows E^*$ as $\widehat{M}(x) := M(f_1(x), f_2(x), \dots, f_n(x))$ for all $x \in E$, and by taking $H = C_n$, invoking Proposition 4.2, $\widehat{M}$ is a strongly η-monotone mapping. Consequently, in accordance with Definition 2.6, $\widehat{M}$ is a general (H,η)-monotone mapping and so Definition 4.4 reduces to the definition of a general (H,η)-monotone mapping which has been introduced in [3, 19]. In other words, the class of C_n-η-monotone mappings presented in Definition 4.4 is exactly the same class of general (H,η)-strongly monotone mappings and is not a new one.

Theorem 4.6. [13, Theorem 14] *Let E be a real reflexive Banach space with the dual space E^*. Let $n \geq 3$ and $f_i : E \to E_i$ $(i = 1, 2, \dots, n)$ and $\eta : E \times E \to E$ be single-valued mappings, $C_n : E \to E^*$ be an η-monotone mapping and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ be a C_n-η-monotone mapping. Then the mapping $(C_n + \lambda M(f_1, f_2, \dots, f_n))^{-1}$ is single-valued for every $\lambda > 0$.*

Proof. Let $\widehat{M} : E \rightrightarrows E^*$ be defined by $\widehat{M}(x) := M(f_1(x), f_2(x), \dots, f_n(x))$ for all $x \in E$. Proposition 4.2 implies that $\widehat{M}$ is a strongly η-monotone mapping. By taking $H = C_n$, it follows that $\widehat{M}$ is a general (H,η)-strongly monotone mapping with constant θ, where $\theta = \sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})$ for the case when n is an even natural number, and $\theta = \sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} - \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i}$, in the case where n is an odd natural number. Then, all the conditions of Corollary 2.12 hold and so Corollary 2.12 guarantees that $(H + \lambda\widehat{M})^{-1} = (C_n + \lambda M(f_1, f_2, \dots, f_n))^{-1}$ is single-valued for every $\lambda > 0$. The proof is finished. □

Based on Theorem 4.6, Guan and Hu [13] defined the proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)}$ associated with C_n, λ and the C_n-η-monotone mapping $M(f_1, f_2, \dots, f_n)$ as follows.

Definition 4.7. *[13, Definition 15] Let E be a real reflexive Banach space with the dual space E^*. Let $n \geq 3$ and $f_i : E \to E_i$ $(i = 1, 2, \dots, n)$ be single-valued mappings, $C_n : E \to E^*$ be an η-monotone mapping and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ be a C_n-η-monotone mapping. A proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)} : E^* \to E$ is defined by*

$$R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)}(x^*) = (C_n + \lambda M(f_1, f_2, \dots, f_n))^{-1}(x^*), \quad \forall x^* \in E^*.$$

By defining $\widehat{M} : E \rightrightarrows E^*$ as $\widehat{M}(x) := M(f_1(x), f_2(x), \dots, f_n(x))$ for all $x \in E$, and by taking $H = C_n$, from Proposition 4.2 we deduce that $\widehat{M}$ is a general (H,η)-strongly monotone mapping. In accordance with Definition 2.13, for any real constant $\lambda > 0$, the mapping $R^{H,\eta}_{\widehat{M},\lambda}$, that is, the proximal mapping associated with H, λ and $\widehat{M}$ is defined for any $x^* \in E^*$ as follows:

$$R^{H,\eta}_{\widehat{M},\lambda}(x^*) = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)}(x^*) = (H + \lambda\widehat{M})^{-1}(x^*) = (C_n + \lambda M(f_1, f_2, \dots, f_n))^{-1}(x^*).$$

In fact, the notion of the proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)}$ associated with an η-monotone mapping C_n, an arbitrary real constant $\lambda > 0$, and a C_n-η-monotone mapping $M(f_1, f_2, \dots, f_n)$ is actually the same notion of the proximal mapping $R^{C_n,\eta}_{\widehat{M},\lambda}$ associated with C_n, λ and the general $(H = C_n, \eta)$-strongly monotone mapping $\widehat{M} = M(f_1, f_2, \ldots, f_n)$, and is not a new one.

By assumption that $n > 3$ is an even natural number, and under some appropriate assumptions, Guan and Hu [13] proved the Lipschitz continuity of the proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}$ as follows.

Theorem 4.8. [13, Theorem 16] *Let E be a real reflexive Banach space with the dual space E^*, and let $\eta : E \times E \to E$ be a k-Lipschitz continuous mapping. Let $n \geq 3$ and $f_i : E \to E_i$ $(i = 1, 2, \ldots, n)$ be single-valued mappings, $C_n : E \to E^*$ be an η-monotone mapping, and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ be a C_n-η-monotone mapping. Then, the proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)} : E^* \to E$ is $\frac{k}{\lambda K_n}$-Lipschitz continuous, where $K_n = \alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} - (\beta_2 + \beta_4 + \cdots + \beta_n)$.*

Proof. Let $\widehat{M} : E \rightrightarrows E^*$ be defined by $\widehat{M}(x) := M(f_1(x), f_2(x), \ldots, f_n(x))$ for all $x \in E$. Since n is an even natural number, M is a C_n-η-monotone mapping, in the light of Definition 4.4(i), we deduce that M is an $\alpha_1\beta_2\alpha_3\beta_4 \ldots \alpha_{n-1}\beta_n$-symmetric monotone mapping with respect to $f_1, f_2, \ldots, f_n$. By using Proposition 4.2(i), it follows that $\widehat{M}$ is a $\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})$-strongly η-monotone mapping. At the same time, by taking $H = C_n$, we note that $\widehat{M}$ is a general (H, η)-strongly monotone with constant $\theta = \sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})$. Now, Theorem 2.15 implies that $R^{H,\eta}_{\widehat{M},\lambda} = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)} : E^* \to E$ is $\frac{k}{\lambda \sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})} = \frac{k}{\lambda K_n}$-Lipschitz continuous. This completes the proof. □

Remark 4.9. It should be pointed out that in a similar way to that in the proof of Theorem 4.8, for the case when n is an odd natural number, the proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}$ is $\frac{k}{\lambda(\sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} - \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i})}$-Lipschitz continuous. At the same time, by a careful reading the proofs of Theorems 14 and 15 in [13], we note that there is a small mistake in the contexts of Theorem 14, Definition 15 and Theorem 16. In fact, the Banach space E must be assumed reflexive, as we have added the assumption reflexivity of E to Theorem 4.6, Definition 4.7 and Theorem 4.8.

Let $n \geq 3$ and $A : E \to E^*$, $p : E \to E$, $f_i : E \to E_i$ $(i = 1, 2, \ldots, n)$, $F : \prod_{i=1}^{n} E_i \to E^*$ be single-valued mappings and let $T_i : E \rightrightarrows CB(E_i)$ $(i = 1, 2, \ldots, n)$ and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ be multi-valued mappings. Recently, for any given $a \in E^*$, Guan and Hu [13] considered and studied the variational inclusion problem of finding $x \in E$, $t_1 \in T_1(x), t_2 \in T_2(x), \ldots, t_n \in T_n(x)$ such that

$$a \in A(x - p(x)) + M(f_1(x), f_2(x), \ldots, f_n(x)) - F(t_1, t_2, \ldots, t_n). \tag{24}$$

In order to find a solution of the problem (24), they gave a characterization of the solution of the problem (24) by utilizing the proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}$ as follows.

Theorem 4.10. [13, Theorem 17] *Let $n \geq 3$ and $A : E \to E^*$, $p : E \to E$, $f_i : E \to E_i$ $(i = 1, 2, \ldots, n)$, $F : \prod_{i=1}^{n} E_i \to E^*$ be single-valued mappings and let $T_i : E \rightrightarrows CB(E_i)$ $(i = 1, 2, \ldots, n)$ be multi-valued mappings. Let $C_n : E \to E^*$ be an η-monotone mapping and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ be a C_n-η-monotone mapping with respect to $f_1, f_2, \ldots, f_n$. Then, $(x, t_1, t_2, \ldots, t_n)$ is a solution of the problem (24) if and only if*

$$x = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}[C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n)],$$

where $t_1 \in T_1(x), t_2 \in T_2(x), \ldots, t_n \in T_n(x)$ and $\lambda > 0$ is a real constant.

Proof. Define $\widehat{M} : E \rightrightarrows E^*$ by $\widehat{M}(x) := M(f_1(x), f_2(x), \ldots, f_x(x))$ for all $x \in E$. From the assumptions and Proposition 4.2 it follows that $\widehat{M}$ is a strongly η-monotone mapping. By taking $H = C_n$, the assumption implies that $\widehat{M}$ is a general strongly (H, η)-strongly monotone mapping. Now, the conclusion follows from Corollary 3.2 immediately. □

In view of the above-mentioned argument, it is worthwhile to stress that contrary to the claim in [13], the characterization of the solution for the problem (24), presented in Theorem 4.10, is exactly the same characterization of the solution for the problem (3) given in Corollary 3.2 and is not a new one.

In order to find an approximate solution of the problem (24), Guan and Hu [13] proposed the following iterative algorithm based on Theorem 4.10.

Algorithm 4.11. [13, Iterative Algorithm 1] *Let E be a real reflexive Banach space with the dual space E^*. For any given $x_0 \in E$, we choose $t_{1,0} \in T_1(x_0), t_{2,0} \in T_2(x_0), \ldots, t_{n,0} \in T_n(x_0)$ and compute $\{x_m\}_{m=0}^{\infty}, \{t_{1,m}\}_{m=0}^{\infty}, \{t_{2,m}\}_{m=0}^{\infty}, \ldots, \{t_{n,m}\}_{m=0}^{\infty}$ by iterative schemes*

$$
\begin{aligned}
&x_{n+1} = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}[C_n(x_m) - \lambda A(x_m - p(x_m)) + \lambda a + \lambda F(t_{1,m}, t_{2,m}, \ldots, t_{n,m})];\\
&t_{1,m} \in T_1(x_m); \|t_{1,m+1} - t_{1,m}\| \leq (1 + \frac{1}{m+1})\widehat{H}(T_1(x_{m+1}), T_1(x_m));\\
&t_{2,m} \in T_2(x_m); \|t_{2,m+1} - t_{2,m}\| \leq (1 + \frac{1}{m+1})\widehat{H}(T_2(x_{m+1}), T_2(x_m));\\
&\vdots\\
&t_{n,m} \in T_n(x_m); \|t_{n,m+1} - t_{n,m}\| \leq (1 + \frac{1}{m+1})\widehat{H}(T_n(x_{m+1}), T_n(x_m)),
\end{aligned}
$$

for all $m = 0, 1, 2, \ldots$.

It is significant to mention that by defining the multi-valued mapping $\widehat{M} : E \rightrightarrows E^*$ as $\widehat{M}(x) = M(f_1(x), f_2(x), \ldots, f_n(x))$, for all $x \in E$, and by taking $H = C_n$, it follows that $\widehat{M}$ is a general (H, η)-strongly monotone mapping and we note that Algorithm 4.11 is actually the same Algorithm 3.5 and is not a new one.

Under the assumption that $n > 3$ is an even natural number and some appropriate assumptions, they proved the strong convergence of the sequences generated by iterative Algorithm 3.4 to a solution of the problem (24) as follows.

Theorem 4.12. [13, Theorem 18] *Let E be a real q-uniformly smooth Banach space and E^* the dual space of E. Let $\eta : E \times E \to E$ be k-Lipschitz continuous. Let $n \geq 3$ and $f_i : E \to E_i$ $(i = 1, 2, \ldots, n)$ be single-valued mappings, $C_n : E \to E^*$ an η-monotone and δ-Lipschitz continuous mapping, $p : E \to E$ a (γ, μ)-relaxed cocoercive and λ_p-Lipschitz continuous mapping, and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ a C_n-η-monotone mapping. Let $A : E \to E^*$ be a τ-Lipschitz continuous mapping and for each $i \in \{1, 2, \ldots, n\}$, let $T_i : E \rightrightarrows CB(E_i)$ be $\widehat{H}_i$-Lipschitz continuous with constant λ_{t_i}. Suppose that $F : \prod_{i=1}^{n} E_i \to E^*$ is λ_{F_i}-Lipschitz continuous in the ith argument with respect to T_i $(i = 1, 2, \ldots, n)$ and there exists a constant $\lambda > 0$ such that the following condition is satisfied:*

$$
\frac{k}{\lambda K_n}(\delta + \lambda\tau(1 + q\gamma\lambda_p^q - q\mu + c_q\lambda_p^q)^{\frac{1}{q}} + \lambda\sum_{i=1}^{n}\lambda_{F_i}\lambda_{t_i}) < 1, \tag{25}
$$

where $K_n = \alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} - (\beta_2 + \beta_4 + \cdots + \beta_n)$ and c_q is a constant guaranteed by Lemma 2.1, and for the case where q is an even natural number, in addition to (25)*, the condition $q\mu < 1 + (q\gamma + c_q)\lambda_p^q$ holds. Then, the iterative sequences $\{x_m\}_{m=0}^{\infty}, \{t_{1,m}\}_{m=0}^{\infty}, \{t_{2,m}\}_{m=0}^{\infty}, \ldots, \{t_{n,m}\}_{m=0}^{\infty}$ generated by Algorithm 4.11 converge strongly to $x, t_1, t_2, \ldots, t_n$, respectively, and $(x, t_1, t_2, \ldots, t_n)$ is a solution of the problem* (24).

Proof. Let us define $\widehat{M} : E \rightrightarrows E^*$ as $\widehat{M}(x) := M(f_1(x), f_2(x), \ldots, f_n(x))$ for all $x \in E$. Relying on the fact that M is a C_n-η-monotone mapping, and $n > 3$ is an even natural number, we conclude that M is an $\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{n-1}\beta_n$-symmetric η-monotone mapping with respect to $f_1, f_2, \ldots, f_n$. In accordance with Proposition 4.2(i), $\widehat{M}$ is a $\sum\limits_{i=1}^{\frac{n}{2}}(\alpha_{2i-1} - \beta_{2i})$-strongly η-monotone mapping. Furthermore, by taking $H = C_n$, it follows that $\widehat{M}$ is a general (H, η)-strongly monotone mapping with constant $\sum\limits_{i=1}^{\frac{n}{2}}(\alpha_{2i-1} - \beta_{2i})$ and Algorithm 4.11 coincides with Algorithm 3.5. Taking $\theta = \sum\limits_{i=1}^{\frac{n}{2}}(\alpha_{2i-1} - \beta_{2i})$, (25) reduces to (21) in Corollary 3.10. Now, all the conditions of Corollary 3.10 hold and so Corollary 3.10 guarantees that the iterative sequences $\{x_m\}_{m=0}^{\infty}$ and $\{t_{i,m}\}_{m=0}^{\infty}$ $(i = 1, 2, \ldots, n)$ generated by Algorithm 4.11 converge strongly to x and t_i $(i = 1, 2, \ldots, n)$, respectively, and $(x, t_1, t_2, \ldots, t_n)$ is a solution of the problem (24). This completes the proof. □

Remark 4.13. *It is worthwhile to emphasize that*

(i) *owing to the facts that $n > 3$ is an even natural number and M is a C_n-η-monotone mapping, Definition 4.1*(iii) implies that the constants α_i $(i = 1, 2, \ldots, n-1)$ and β_i $(i = 2, 4, \ldots, n)$ must be satisfied the condition $\sum\limits_{i=1}^{\frac{n}{2}}(\alpha_{2i-1} - \beta_{2i}) > 0$, that is, $\sum\limits_{i=1}^{\frac{n}{2}}\alpha_{2i-1} > \sum\limits_{i=1}^{\frac{n}{2}}\beta_{2i}$;

(ii) by a careful reading Theorem 18 in [13], we found that there are two small mistakes in its context. Firstly, since the constants $k, \lambda, \delta, \tau, \gamma, \lambda_p, \mu, c_q, q$ and λ_{t_i} $(i = 1, 2, \ldots, n)$ are all positive and $\sum\limits_{i=1}^{\frac{n}{2}}\alpha_{2i-1} > \sum\limits_{i=1}^{\frac{n}{2}}\beta_{2i}$, it follows that

$$\frac{k}{\lambda K_n}\Big(\delta + \lambda\tau(1 + q\gamma\lambda_p^q - q\mu + c_q\lambda_p^q)^{\frac{1}{q}} + \lambda\sum_{i=1}^{n}\lambda_{F_i}\lambda_{t_i}\Big) > 0.$$

Hence, (39) in [13], that is, the condition

$$0 < \frac{k}{\lambda K_n}\Big(\delta + \lambda\tau(1 + q\gamma\lambda_p^q - q\mu + c_q\lambda_p^q)^{\frac{1}{q}} + \lambda\sum_{i=1}^{n}\lambda_{F_i}\lambda_{t_i}\Big) < 1$$

must be replaced by (25) in Theorem 4.12. Secondly, in the case where q is an even natural number, then the constants μ, γ, λ_p and c_q, in addition to (25), must be also satisfied the condition $q\mu < 1 + (q\gamma + c_q)\lambda_p^q$, as we have added the mentioned condition to the assumptions of Theorem 4.12.

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