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Asymptotic normality of the Stirling-Whitney-Riordan triangle

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Abstract. Recently, Zhu [34] introduced a Stirling-Whitney-Riordan triangle $[T_{n,k}]_{n,k\geq 0}$ satisfying the recurrence

$$T_{n,k} = (b_1k + b_2)T_{n-1,k-1} + [(2\lambda b_1 + a_1)k + a_2 + \lambda(b_1 + b_2)]T_{n-1,k} + \lambda(a_1 + \lambda b_1)(k + 1)T_{n-1,k+1},$$

where initial conditions $T_{n,k} = 0$ unless $0 \leq k \leq n$ and $T_{0,0} = 1$. Denote by $T_n = \sum_{k=0}^{n} T_{n,k}$. In this paper, we show the asymptotic normality of $T_{n,k}$ and give an asymptotic formula of T_n. As applications, we show the asymptotic normality of many famous combinatorial numbers, such as the Stirling numbers of the second kind, the Whitney numbers of the second kind, the r-Stirling numbers and the r-Whitney numbers of the second kind.

1. Introduction

Let $a(n,k)$ be a double-index sequence of nonnegative numbers and let

$$p(n,k) = \frac{a(n,k)}{\sum_{j=0}^{n} a(n,j)} \tag{1}$$

denote the normalized probabilities. Following Bender [3], we say that $a(n,k)$ is *asymptotically normal by a central limit theorem*, if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k\leq \mu_n + x\delta_n} p(n,k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0, \tag{2}$$

where μ_n and σ_n^2 are the mean and the variance of $a(n,k)$ respectively. We say that $a(n,k)$ is *asymptotically normal by a local limit theorem* on $\mathbb{R}$, if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sigma_n p(n, \lfloor \mu_n + x\sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0. \tag{3}$$

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In this case,

$$a(n,k) \sim \frac{e^{-x^2/2\sum_{j=0}^{n} a(n,j)}}{\sigma_n\sqrt{2\pi}}, \text{ as } n \to \infty,$$

where $k = \mu_n + x\sigma_n$ and $x = O(1)$. Clearly, the validity of (3) implies that of (2).

Many combinatorial numbers satisfy the central and local limit theorems, including the binomial coefficients $\binom{n}{k}$, the signless Stirling numbers of the first kind [17], the Stirling numbers of the second kind [21], the Eulerian numbers [15], the q-derangement numbers [11], the coefficients of q-Catalan numbers [10], and the Laplacian coefficients of graphs [31]. We refer the reader to the survey of Canfield [9]. Recently, Hwang et al. [22] investigated the asymptotic distributions of recurrence sequence of Eulerian type, which includes hundreds of examples. Liu et al. [25] proved the asymptotic normality of combinatorial numbers related to Dowling lattices.

Let $\mathbb{R}$ (resp. $\mathbb{R}^{\geq 0}, \mathbb{R}^{>0}$) denote the set of all (resp. nonnegative, positive) real numbers. For $\{\lambda, a_1, a_2, b_1, b_2\} \subseteq \mathbb{R}$, Zhu [34] defined a *Stirling-Whitney-Riordan triangle* $[T_{n,k}]_{n,k\geq 0}$, which satisfies the recurrence relation

$$T_{n,k} = (b_1k + b_2)T_{n-1,k-1} + [(2\lambda b_1 + a_1)k + a_2 + \lambda(b_1 + b_2)]T_{n-1,k} + \lambda(a_1 + \lambda b_1)(k + 1)T_{n-1,k+1}, \tag{4}$$

where initial conditions $T_{n,k} = 0$ unless $0 \leq k \leq n$ and $T_{0,0} = 1$. Let its row generating function $T_n(q) = \sum_{k\geq 0} T_{n,k}q^k$ for $n \geq 0$. In [34], it was proved that under some inequalities of the coefficients,

(i) $[T_{n,k}]_{n,k}$ is coefficientwise totally positive in all the indeterminates;

(ii) $T_n(q)$ has only real zeros;

(iii) the Turán-type polynomial $T_{n+1}(q)T_{n-1}(q) - T_n^2(q)$ is stable;

(iv) $T_n(q)$ is q-Stieltjes moment and 3-q-log-convex.

These properties can be applied to many famous combinatorial numbers.

Example 1.1. (1) *For $a_1 = b_2 = 1$ and $a_2 = b_1 = \lambda = 0$, $T_{n,k}$ is the Stirling number of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, which enumerates the number of partitions of a set with n elements consisting of k disjoint nonempty subsets. Its row generating function, i.e., the Bell polynomial, is*

$$B_n(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k.$$

See [2, 7, 12, 14, 18, 19, 23, 24, 30, 32, 33] for many nice properties of the Stirling number of the second kind and the Bell polynomial.

(2) *For $a_1 = b_1 = 1$, $a_2 = b_2 = \lambda = 0$, $T_{n,k} = k!\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$. Let $G_{n,k} = k!\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, which counts the number of distinct ordered partitions of a set with n elements. The row generating function $G_n(x) = \sum_{k=1}^{n} G_{n,k}x^k$ is called the geometric polynomial and was studied by Tanny in [29].*

(3) *For $\lambda = 0$, $T_{n,k}$ are the coefficients of a generalized ordered Bell polynomial, which were studied by Barbero et al. [1], Guo and Zhu [20].*

(4) *For $a_1 = m$, $b_1 = \lambda = 0$, $a_2 = b_2 = 1$, $T_{n,k}$ is the Whitney number of the second kind, denote by $W_m(n, k)$. In 1973, Dowling [16] introduced a class of geometric lattices based on finite group G of order $m \geq 1$, called Dowling lattices. Let $Q_n(G)$ be Dowling lattices of rank n associated to G. When $m = 1$, that is, G is the trivial group, $Q_n(G)$ is isomorphic to the lattice Π_{n+1} of partition of an $(n + 1)$ set. Its row generatinng function $D_n(m, x) = \sum_{k=0}^{n} W_m(n, k)x^k$ is called the Dowling polynomial by Benoumhani [5].*

(5) *For $a_1 = m$, $b_2 = \lambda = 0$, $a_2 = b_1 = 1$, $T_{n,k} = k!W_m(n,k)$, where $W_m(n,k)$ is the Whitney number of the second kind defined as (4). Its row generating function $F_m(n,x) = \sum_{k=0}^{n} k!W_m(n,k)x^k$ is called the Tanny-geometric polynomial in [5]. See [5, 6, 12, 23, 30, 32, 33] for some properties of the Whitney number of the second kind and the Dowling polynomial.*

(6) *For $\lambda = b_2 = 0$, $a_2 = 1$ and $a_1 = b_1 = m$, $T_{n,k} = m^k k!W_m(n,k)$, which are the coefficients of a generalized Dowling polynomial*

$$F_{n,m,2}(x) = \sum_{k=0}^{n} k!W_m(n,k)m^k x^k$$

introduced by Benoumhani [5]. See Benoumhani [4–6] for the recurrence relations, the exponential generating functions and the reality of zeros of these Dowling polynomials.

(7) *For $\lambda = b_1 = 0$, $a_1 = b_2 = 1$, $a_2 = r$, $T_{n,k}$ is the r-Stirling number $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r$ defined by Broder [8], which enumerates the number of partitions of the set $[n]$ having k non-empty disjoint subset, such that the numbers $1, 2, \ldots, r$ are in distinct subsets [8]. The row generating function*

$$B_{n,r}(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r x^k,$$

is called the r-Bell polynomial by Mezö [8].

(8) *For $\lambda = b_1 = 0$, $b_2 = 1$, $a_1 = m$, and $a_2 = r$, $T_{n,k}$ is the r-Whitney number of the second kind $W_{m,r}(n,k)$. The row generating function*

$$D_{n,m,r}(x) = \sum_{k=0}^{n} W_{m,r}(n,k)x^k$$

is called the r-Dowling polynomial by Choen and Jung [13].

(9) *For $a_2 = b_1 = 0$ and $a_1 = b_2 = \lambda = 1$, $T_{n,k}$ equal the numbers $a_{n,k}$ of set partitions of $[n]$ in which exactly k blocks have been distinguished (see [28, A049020]). The triangle $[a_{n,k}]_{n,k}$ first arose in Riordan's letter [27]. We refer reader to [28, A049020] for more information.*

The aim of this paper is to study asymptotic properties of the Stirling-Whitney-Riordan triangle. We define $T_n = T_n(1) = \sum_{k=0}^{n} T_{n,k}$. In this paper, we first present an asymptotic formula of T_n, and then prove the asymptotic normality of $T_{n,k}$. More precisely, we have the following.

Theorem 1.2. *Let $\{a_1, b_1\} \subseteq \mathbb{R}^{>0}$ and $\{\lambda, a_2, b_2\} \subseteq \mathbb{R}^{\geq 0}$. If R_1 is the unique positive solution of the equation*

$$n = a_2 R + (b_1 + b_2)(1+\lambda)Re^{a_1 R}\left[1 + \frac{b_1}{a_1}(1+\lambda)(1 - e^{a_1 R})\right]^{-1}$$

satisfying the condition $0 < R_1 < \frac{1}{a_1}\ln^{(1+\frac{a_1}{b_1(1+\lambda)})}$, then we have

$$T_n \sim \frac{n!}{R_1^n \psi} e^{a_2 R_1}\left[1 + \frac{b_1(1+\lambda)}{a_1}(1 - e^{a_1 R_1})\right]^{-\left(1+\frac{b_2}{b_1}\right)}, \tag{5}$$

where

$$\psi = \sqrt{2\pi(n + \alpha + \beta)}, \quad \alpha = (b_1 + b_2)(1+\lambda)R_1^2 a_1 e^{a_1 R_1}\left[1 + \frac{b_1}{a_1}(1+\lambda)(1 - e^{a_1 R_1})\right]^{-1},$$

$$\beta = (b_1 + b_2)(1+\lambda)^2 b_1 R_1^2 e^{2a_1 R_1}\left[1 + \frac{b_1}{a_1}(1+\lambda)(1 - e^{a_1 R_1})\right]^{-2}.$$

Theorem 1.3. *Let $\{a_1, b_1\} \subseteq \mathbb{R}^{>0}$, $\{\lambda, a_2, b_2\} \subseteq \mathbb{R}^{\geq 0}$. If $a_1(b_1 + b_2) > b_1 a_2$, then the coefficients $T_{n,k}$ are asymptotically normal.*

2. Proof of Theorem 1.2

In this section, we present a proof of Theorem 1.2.

Proof. [Proof of Theorem 1.2]
When $a_1 \neq 0, b_1 \neq 0$, the exponential generating function of $T_n(q)$ is

$$\sum_{n\geq 0} T_n(q)\frac{t^n}{n!} = e^{a_2 t}\left[1+\frac{b_1(q+\lambda)(1-e^{a_1 t})}{a_1}\right]^{-\left(1+\frac{b_2}{b_1}\right)}, \tag{6}$$

(see [34] for instance). Following Moser and Wyma [26], by Cauchy's formula, we can write T_n as

$$T_n = \frac{n!}{2\pi i}\oint_{|t|=R}\frac{e^{a_2 t}\left[1+\frac{b_1(1+\lambda)(1-e^{a_1 t})}{a_1}\right]^{-\left(1+\frac{b_2}{b_1}\right)}}{t^{n+1}}dt.$$

Set $t = Re^{i\theta}$. Then it yields

$$T_n \quad = \quad \frac{n!}{2\pi R^n}\int_{-\pi}^{\pi} e^{a_2 Re^{i\theta}-in\theta}\Big[1+\frac{b_1(1+\lambda)(1-e^{a_1 Re^{i\theta}})}{a_1}\Big]^{-\left(1+\frac{b_2}{b_1}\right)}d\theta. \tag{7}$$

Let

$$F(\theta) = \ln^{e^{a_2 Re^{i\theta}-in\theta}\left[1+\frac{b_1(1+\lambda)(1-e^{a_1 Re^{i\theta}})}{a_1}\right]^{-\left(1+\frac{b_2}{b_1}\right)}}, \quad \varepsilon = n^{-\frac{1}{4}}.$$

We decompose the integral (7) into three parts

$$(\int_{-\pi}^{-\varepsilon}+\int_{-\varepsilon}^{\varepsilon}+\int_{\varepsilon}^{\pi})\exp\big(F(\theta)\big)d\theta. \tag{8}$$

In what follows we will prove that integrals $\int_{-\pi}^{-\varepsilon}$ and $\int_{\varepsilon}^{\pi}$ are negligible, and then the greatest contribution to (8) comes from the middle part $\int_{-\varepsilon}^{\varepsilon}$. By computing, we derive

$$\begin{aligned}
F'(\theta) \quad &= \quad a_2 Rie^{i\theta} - in + (b_1+b_2)(1+\lambda)Rie^{a_1 Re^{i\theta}+i\theta}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1 Re^{i\theta}})\right]^{-1},\\
F''(\theta) \quad &= \quad -a_2 Re^{i\theta} + (b_1+b_2)(1+\lambda)Ri(a_1 Rie^{i\theta}+i)e^{a_1 Re^{i\theta}+i\theta}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1 Re^{i\theta}})\right]^{-1}\\
&\quad\; -(b_1+b_2)b_1(1+\lambda)^2R^2e^{2(a_1 Re^{i\theta}+i\theta)}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1 Re^{i\theta}})\right]^{-2}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
F(0) \quad &= \quad \ln^{e^{a_2 R}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1 R})\right]^{-\left(1+\frac{b_2}{b_1}\right)}},\\
F'(0) \quad &= \quad a_2 Ri - in + (b_1+b_2)(1+\lambda)Rie^{a_1 R}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1 R})\right]^{-1},\\
F''(0) \quad &= \quad -a_2 R - (b_1+b_2)(1+\lambda)R(a_1 R+1)e^{a_1 R}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1 R})\right]^{-1} -\\
&\quad\; (b_1+b_2)b_1(1+\lambda)^2R^2e^{2a_1 R}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1 R})\right]^{-2}.
\end{aligned}$$

Note that $F'(0) = 0$ is equivalent to the equation

$$\frac{n}{R} = a_2 + (b_1 + b_2)(1 + \lambda)e^{a_1 R}\Big[1 + \frac{b_1}{a_1}(1 + \lambda)(1 - e^{a_1 R})\Big]^{-1}.$$

Let $h(R) = \frac{n}{R}$ and

$$v(R) = a_2 + (b_1 + b_2)(1 + \lambda)e^{a_1 R}\Big[1 + \frac{b_1}{a_1}(1 + \lambda)(1 - e^{a_1 R})\Big]^{-1}.$$

We derive

$$v'(R) = (b_1 + b_2)(1 + \lambda)\frac{[a_1 + b_1(1 + \lambda)]e^{a_1 R}}{[1 + \frac{b_1}{a_1}(1 + \lambda)(1 - e^{a_1 R})]^2} > 0,$$

where a_1, b_1, λ, b_2 is nonnegative. Obviously $v(R)$ is increasing and $h(R)$ is decreasing in the interval $\left(0, \frac{1}{a_1}\ln^{\left(1+\frac{a_1}{b_1(1+\lambda)}\right)}\right)$ respectively. It is not hard to obtain $v(0) = a_2 + \frac{a_1(b_1+b_2)(1+\lambda)}{a_1+b_1(1+\lambda)} \geq 0$ and $h\left(\frac{1}{a_1}\ln^{\left(1+\frac{a_1}{b_1(1+\lambda)}\right)}\right) = \frac{na_1}{\ln^{\left(1+\frac{a_1}{b_1(1+\lambda)}\right)}} \geq 0$. In addition, $v(R) \to +\infty$ as $R \to \frac{1}{a_1}\ln^{\left(1+\frac{a_1}{b_1(1+\lambda)}\right)}$ and $h(R) \to +\infty$ as $R \to 0$. In consequence, there exists a point $R_1 \in \left(0, \frac{1}{a_1}\ln^{\left(1+\frac{a_1}{b_1(1+\lambda)}\right)}\right)$ such that $v(R_1) = h(R_1)$.

Expanding the integral $\int_\varepsilon^\pi$ in the Taylor series about $\theta = 0$, we obtain

$$\begin{aligned}
&\left|\int_\varepsilon^\pi \exp(F(\theta))d\theta\right| \\
= &\left|\int_\varepsilon^\pi \exp\left(F(0) + F'(0)\theta + F''(0)\frac{\theta^2}{2} + o(\theta^2)\right)d\theta\right| \\
= &\exp(F(0))\left|\int_\varepsilon^\pi \exp\left(F'(0)\theta + F''(0)\frac{\theta^2}{2} + o(\theta^2)\right)d\theta\right| \\
\leq &\exp(F(0))\int_\varepsilon^\pi \left|\exp\left(F'(0)\theta + F''(0)\frac{\theta^2}{2} + o(\theta^2)\right)\right|d\theta \\
= &e^{a_2 R_1}\left[1 + \frac{b_1}{a_1}(1 + \lambda)(1 - e^{a_1 R_1})\right]^{-\left(1+\frac{b_2}{b_1}\right)}\int_\varepsilon^\pi \exp\left(F''(0)\frac{\theta^2}{2} + o(\theta^2)\right)d\theta.
\end{aligned}$$

In addition, we also derive

$$\begin{aligned}
F''(0) &= -a_2 R_1 - (b_1 + b_2)(1 + \lambda)R_1(a_1 R_1 + 1)e^{a_1 R_1}\left[1 + \frac{b_1}{a_1}(1 + \lambda)(1 - e^{a_1 R_1})\right]^{-1} - \\
&\quad (b_1 + b_2)b_1(1 + \lambda)^2 R_1^2 e^{2a_1 R_1}\left[1 + \frac{b_1}{a_1}(1 + \lambda)(1 - e^{a_1 R_1})\right]^{-2} \\
&= -\Big\{n + (b_1 + b_2)(1 + \lambda)a_1 R_1^2 e^{a_1 R_1}\left[1 + \frac{b_1}{a_1}(1 + \lambda)(1 - e^{a_1 R_1})\right]^{-1} + \\
&\quad (b_1 + b_2)b_1(1 + \lambda)^2 R_1^2 e^{2a_1 R_1}\left[1 + \frac{b_1}{a_1}(1 + \lambda)(1 - e^{a_1 R_1})\right]^{-2}\Big\}.
\end{aligned}$$

In consequence, for the integral $\int_\varepsilon^\pi$ in (8), we obtain

$$\int_\varepsilon^\pi \exp\left(F''(0)\frac{\theta^2}{2} + o(\theta^2)\right)d\theta \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

The same calculations are valid for $\int_{-\pi}^{-\varepsilon}$. So

$$
\begin{aligned}
T_n &\sim \frac{n!}{2\pi R_1^n}\exp(F(0))\int_{-\varepsilon}^{\varepsilon}\exp\left(F''(0)\frac{\theta^2}{2}+o(\theta^2)\right)d\theta\\
&= \frac{n!}{2\pi R_1^n}e^{a_2R_1}\Big[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1R_1})\Big]^{-\left(1+\frac{b_2}{b_1}\right)}\\
&\quad \int_{-\varepsilon}^{\varepsilon}\exp\Big\{-\frac{\theta^2}{2}\Big\{n+(b_1+b_2)(1+\lambda)a_1R_1^2e^{a_1R_1}\Big[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1R_1})\Big]^{-1}\\
&+ (b_1+b_2)b_1(1+\lambda)^2R_1^2e^{2a_1R_1}\Big[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1R_1})\Big]^{-2}\Big\}+o(\theta^2)\Big\}d\theta.
\end{aligned}
$$

Let

$$\varphi = \sqrt{n+\alpha+\beta}\theta,$$

where

$$\alpha = (b_1+b_2)(1+\lambda)R_1^2a_1e^{a_1R_1}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1R_1})\right]^{-1},$$

$$\beta = (b_1+b_2)b_1(1+\lambda)^2R_1^2e^{2a_1R_1}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1R_1})\right]^{-2}.$$

Observing for large enough n, we integrate on the real axis and get

$$
\begin{aligned}
T_n &\sim \frac{n!}{2\pi R_1^n\sqrt{n+\alpha+\beta}}e^{a_2R_1}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1R_1})\right]^{-\left(1+\frac{b_2}{b_1}\right)}\\
&\quad \int_{-\infty}^{\infty}\exp(-\frac{\varphi^2}{2})d\varphi\\
&= \frac{n!}{R_1^n\psi}e^{a_2R_1}\left[1+\frac{b_1}{a_1}(1+\lambda)(1-e^{a_1R_1})\right]^{-\left(1+\frac{b_2}{b_1}\right)},
\end{aligned}
$$

where $\psi = \sqrt{2\pi(n+\alpha+\beta)}$. □

3. Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. Before it, we need some known results.

A standard approach to demonstrating the asymptotic normality is the following criterion, which was used by Harper [21] to show the asymptotic normality of the Stirling numbers of the second kind. We refer the reader to [3, 9, 14] for the asymptotic normality.

Lemma 3.1. [31] *Suppose that $A_n(x) = \sum_{k=0}^{n} a(n,k)x^k$ have only real zero and $A_n(x) = \prod_{i=1}^{n}(x+r_i)$. Let*

$$u_n = \sum_{i=1}^{n}\frac{1}{1+r_i};$$

$$\sigma_n^2 = \sum_{i=1}^{n}\frac{r_1}{(1+r_i)^2}$$

If $\sigma_n \to +\infty$, then the numbers $a(n,k)$ are asymptotically normal with the mean u_n and the variance σ_n^2.

Remark 3.2. [9] *Suppose that* $A_n(x) = \sum_{k=0}^{n} a(n,k)x^k$. *Then the mean and the variance of* $a(n,k)$ *are given by the following expressions*

$$u_n = \frac{A_n'(1)}{A_n(1)} = \frac{\sum_{k=0}^{n} ka(n,k)}{\sum_{k=0}^{n} a(n,k)},$$

$$\sigma_n^2 = \frac{A_n'(1)}{A_n(1)} + \frac{A_n''(1)}{A_n(1)} - \left(\frac{A_n'(1)}{A_n(1)}\right)^2 = \frac{\sum_{k=0}^{n} k^2 a(n,k)}{\sum_{k=0}^{n} a(n,k)} - u_n^2.$$

Now we are in a position to prove Theorem 1.3.

Proof. [Proof of Theorem 1.3] If $a_1(b_1 + b_2) > b_1 a_1$, then the row generating function $T_n(q) = \sum_{k=0}^{n} T_{n,k} q^k$ has only real zeros [34]. So by Lemma 3.1, it suffices to prove that the variance of $T_{n,k}$ tends to $+\infty$ as $n \to \infty$.
Let $T_n = T_n(1) = \sum_{k\geq 0} T_{n,k}$. By the recurrence (4), we have

$$\sum_{k=0}^{n} kT_{n,k} = \frac{T_{n+1} - [(\lambda+1)(b_1+b_2) + a_2]T_n}{[(\lambda+1)^2 b_1 + (\lambda+1)a_1]}.$$

So the mean and the variance of $T_{n,k}$ are

$$\begin{aligned}
u_n &= \frac{\sum_{k=0}^{n} kT_{n,k}}{T_n} = \frac{1}{[(\lambda+1)^2 b_1 + (\lambda+1)a_1]}\left[\frac{T_{n+1}}{T_n} - [(\lambda+1)(b_1+b_2) + a_2]\right],\\
\sigma_n^2 &= \frac{\sum_{k=0}^{n} k^2 T_{n,k}}{\sum_{k=0}^{n} T_{n,k}} - u_n^2\\
&= \frac{1}{[(\lambda+1)^2 b_1 + (\lambda+1)a_1]^2}\Bigg[\frac{T_{n+2}}{T_n} - [(2\lambda+3-\lambda^2)b_1 + (2\lambda+2)b_2 + 2a_2 - \lambda a_1]\frac{T_{n+1}}{T_n}\\
&\quad -\Big[[(\lambda+1)^2 b_1 + (\lambda+1)a_1](b_1+b_2) - [(\lambda+1)(b_1+b_2)+a_2][(2+\lambda)b_1 + (\lambda+1)b_2\\
&\quad +a_2 - \lambda a_1 - \lambda^2 b_1]\Big]\Bigg] - u_n^2.
\end{aligned}$$

Using the asymptotic formula (5) of T_n, we have

$$\sigma_n^2 = \frac{\sum_{k=0}^{n} k^2 T_{n,k}}{\sum_{k=0}^{n} T_{n,k}} - u_n^2 \quad \sim \quad \frac{n+1}{[(\lambda+1)^2 b_1 + (\lambda+1)a_1]^2}\left[\frac{1 + [(\lambda^2-1)b_1 + \lambda a_1]R_1}{R_1^2}\right]. \tag{9}$$

Now we claim that $1 + [(\lambda^2-1)b_1 + \lambda a_1]R_1 > 0$. It is easy to get that $(\lambda^2-1)b_1 + \lambda a_1$ is increasing in $(0, +\infty)$. So if $\lambda = 0$, then $(\lambda^2-1)b_1 + \lambda a_1 = -b_1$ is minimum in the interval $(0, +\infty)$. Hence we have

$$1 + [(\lambda^2-1)b_1 + \lambda a_1]R_1 > 1 - b_1 R_1 > 1 - \frac{b_1}{a_1}\ln(1 + \frac{a_1}{b_1}) > 0,$$

for $R_1 \in (0, \frac{1}{a_1}\ln(1 + \frac{a_1}{b_1(1+\lambda)})) \subset (0, \frac{1}{a_1}\ln(1 + \frac{a_1}{b_1}))$ and $\ln(1 + \frac{a_1}{b_1}) < \frac{a_1}{b_1}$ when $a_1, b_1 > 0$.
Following (9), we have $\sigma_n^2 \to +\infty$ as $n \to \infty$. □

For $a_1 \neq 0$ or $b_1 = 0$, using continuity of functions, the exponential generating function is

$$\sum_{n\geq 0} T_n(q)\frac{t^n}{n!} = e^{a_2 t + \left[\frac{b_2(q+\lambda)(e^{a_1 t}-1)}{a_1}\right]}$$

(see Zhu [34]). In this case, we also obtain an asymptotic formula of T_n and the asymptotic normality of $T_{n,k}$ in the following theorem. It can be proved by the same technique used in the proof of Theorems 1.2 and 1.3. So we omit its proof for brevity.

Theorem 3.3. *Let* $\{a_1\} \subseteq R^{>0}$ *and* $\{a_2, b_2, \lambda\} \subseteq R^{\geq 0}$. *Then*

(i) *an asymptotic formula of* T_n *is*

$$T_n \sim \frac{n!}{R_2^n \sqrt{2\pi(n + a_1 R_2^2 e^{a_1 R_2})}} \exp\left(a_2 R_2 + \frac{b_2}{a_1}(1+\lambda)(e^{a_1 R_2} - 1)\right),$$

where R_2 *is the unique positive solution* $n = a_2 R + b_2(1+\lambda) R e^{a_1 R}$.

(ii) *the coefficients* $T_{n,k}$ *are asymptotically normal.*

4. Applications

In this section, we give some applications of Theorems 1.2, 1.3 and 3.3, and obtain asymptotic formulas and the asymptotic normality of some combinatorial numbers or polynomials related to the Stirling-Whitney-Riordan triangle.

It is well-known that many classical combinatorial numbers satisfy the following recurrence

$$\mathcal{T}_{n,k} = \lambda(a_0 n + a_1 k + a_2)\mathcal{T}_{n-1,k} + (b_0 n + b_1 k + b_2)\mathcal{T}_{n-1,k-1} + \frac{d(da_1 - b_1)}{\lambda}(n-k+1)\mathcal{T}_{n-1,k-2}; \tag{10}$$

with $\mathcal{T}_{0,0} = 1$ and $\mathcal{T}_{n,k} = 0$ unless $0 \leq k \leq n$. We also denote its row generating function by $\mathcal{T}_n(q) = \sum_{k\geq 0} \mathcal{T}_{n,k} q^k$ for $n \geq 0$.

Now we consider the asymptotic normality of triangles satisfying the following two types of recurrences

(i) $\mathbb{T}_{n,k} = \lambda(a_1 k + a_2)\mathbb{T}_{n-1,k} + [-da_1 n + (b_1 + 2da_1)k + b_2 - b_1 - d(a_1 - a_2)]\mathbb{T}_{n-1,k-1} - \frac{d(da_1+b_1)}{\lambda}(n-k+1)\mathbb{T}_{n-1,k-2}$;

(ii) $\mathbf{T}_{n,k} = \lambda(a_0 n - a_0 k + a_2 - a_0)\mathbf{T}_{n-1,k} + [(b_0 + 2da_0)(n-k) + b_2 + da_2]\mathbf{T}_{n-1,k-1} + \frac{d(b_0+da_0)}{\lambda}(n-k+1)\mathbf{T}_{n-1,k-2}$.

The next relationship was proved by Zhu [35].

Theorem 4.1. [35] *Let* $[\mathcal{T}_{n,k}]_{n,k\geq 0}$ *be defined in (10). Then there exists an array* $[A_{n,k}]_{n,k\geq 0}$ *satisfying the recurrence relation*

$$A_{n,k} = [[b_0 + d(a_1 - a_0)]n + (b_1 - 2da_1)k + b_2 + d(a_1 - a_2)]A_{n-1,k-1} + (a_0 n + a_1 k + a_2)A_{n-1.k}$$

with $A_{0,0} = 1$ *and* $A_{n,k} = 0$ *unless* $0 \leq k \leq n$ *such that their row-generating functions satisfy*

$$\mathcal{T}_n(q) = (\lambda + dq)^n A_n\left(\frac{q}{\lambda + dq}\right)$$

for $n \geq 0$.

For $[\mathbb{T}_{n,k}]_{n,k}$ in (i), by Theorem 4.1, we obtain a corresponding array $[\mathcal{A}_{n,k}]_{n,k}$ as follows: $\mathcal{A}_{n,k} = (a_1 k + a_2)\mathcal{A}_{n-1,k} + [(b_1(k-1) + b_2]\mathcal{A}_{n-1,k-1}$.

Clearly, if $\lambda = 0$ and $b_2 - b_1 \to b_2$ in recurrence relation (4), then the String-Whitney-Riordan triangle $T_{n,k}$ reduces to $\mathcal{A}_{n,k}$. Thus for $\mathcal{A}_n(x) = \sum_{k\geq 0} \mathcal{A}_{n,k} x^k$, we have the exponential generating function

$$\sum_{n\geq 0} \mathcal{A}_n(x)\frac{t^n}{n!} = \exp(a_2 t)\left[\frac{a_1}{a_1 + b_1 x - b_1 x \exp(a_1 t)}\right]^{\frac{b_2}{b_1}}. \tag{11}$$

So we derive

$$\begin{aligned}
\sum_{n\geq 0} \mathbb{T}_n(q)\frac{t^n}{n!} &= \sum_{n\geq 0} (\lambda + dq)^n \mathcal{A}_n\left(\frac{q}{\lambda + dq}\right)\frac{t^n}{n!} \\
&= \exp(a_2 t(\lambda + dq))\left[\frac{a_1(\lambda + dq)}{a_1(\lambda + dq) + b_1 q - b_1 q \exp(a_1(\lambda + dq)t)}\right]^{\frac{b_2}{b_1}}.
\end{aligned} \tag{12}$$

Corollary 4.2. (1)

$$\mathbb{T}_n = \sum_{k=0}^{n} \mathbb{T}_{n,k} \sim \frac{n!\left(\exp(a_2(\lambda+d)R_1)\left[\frac{a_1(\lambda+d)}{a_1(\lambda+d)+b_1-b_1\exp(a_1(\lambda+d)R_1)}\right]^{\frac{b_2}{b_1}}\right)}{R_1^n\sqrt{2\pi(n+\eta)}},$$

where

$$\eta = \frac{b_2a_1^2(\lambda+d)^2R_1^2\exp(a_1(\lambda+d)R_1)}{a_1(\lambda+d)+b_1-b_1\exp(a_1(\lambda+d)R_1)} + \frac{b_1b_2a_1^2(\lambda+d)^2R_1^2\exp(2a_1(\lambda+d)R_1)}{[a_1(\lambda+d)+b_1-b_1\exp(a_1(\lambda+d)R_1)]^2}$$

and R_1 is the positive solution of

$$n = a_2(\lambda+d)R + b_2\frac{a_1(\lambda+d)R\exp a_1(\lambda+d)R}{a_1(\lambda+d)+b_1-b_1\exp a_1(\lambda+d)R}.$$

(2) *If $b_2a_1(\lambda+d) > a_2(\lambda+d)b_1$, then the coefficients $\mathbb{T}_{n,k}$ are asymptotically normal.*

For $[\mathbf{T}_{n,k}]_{n,k}$ in (ii), by Theorem 4.1, we get a corresponding array $[\mathcal{B}_{n,k}]_{n,k}$ satisfying

$$\mathcal{B}_{n,k} = [a_0(n-k-1)+a_2]\mathcal{B}_{n-1,k} + [b_0(n-k)+b_2]\mathcal{B}_{n-1,k-1}.$$

Let $\mathcal{B}^*_{n,k} = \mathcal{B}_{n,n-k}$. It yields

$$\mathcal{B}^*_{n,k} = [a_0(k-1)+a_2]\mathcal{B}^*_{n-1,k-1} + [b_0k+b_2]\mathcal{B}^*_{n-1,k}.$$

So we have $\mathcal{B}^*_n(x) = x^n\mathcal{B}_n(\frac{1}{x})$, where $\mathcal{B}^*_n(x) = \sum_{n\geq 0}\mathcal{B}^*_{n,k}x^k$, $\mathcal{B}_n(x) = \sum_{n\geq 0}\mathcal{B}_{n,k}x^k$.

Combining (11) and $\mathcal{B}^*_n(x) = x^n\mathcal{B}_n(\frac{1}{x})$, we have the exponential generating function:

$$\begin{aligned}
\sum_{n\geq 0}\mathbf{T}_n(q)\frac{t^n}{n!} &= \sum_{n\geq 0}(\lambda+dq)^n\mathcal{B}_n\left(\frac{q}{\lambda+dq}\right)\frac{t^n}{n!}\\
&= \sum_{n\geq 0}\mathcal{B}^*_n\left(\frac{\lambda+dq}{q}\right)\frac{(tq)^n}{n!}\\
&= \exp(b_2tq)\left[\frac{b_0q}{qb_0+a_0(\lambda+dq)-a_0(\lambda+dq)\exp(b_0tq)}\right]^{\frac{a_2}{a_0}}.
\end{aligned}$$

Corollary 4.3. (1)

$$\mathbf{T}_n = \sum_{k=0}^{n}\mathbf{T}_{n,k} \sim \frac{n!\left(\exp(b_2R_1)\left[\frac{b_0}{b_0+a_0(\lambda+d)-a_0(\lambda+d)\exp(b_0R_1)}\right]^{\frac{a_2}{a_0}}\right)}{R_1^n\sqrt{2\pi(n+\mu)}};$$

where

$$\mu = \frac{a_2b_0^2R_1^2(\lambda+d)\exp(b_0R_1)}{b_0+a_0(\lambda+d)-a_0(\lambda+d)\exp(b_0R_1)} + \frac{a_2b_0^2R_1^2(\lambda+d)^2a_0\exp(2b_0R_1)}{[b_0+a_0(\lambda+d)-a_0(\lambda+d)\exp(b_0R_1)]^2}$$

and R_1 is the positive solution of

$$n = b_2R + \frac{a_2b_0R(\lambda+d)\exp(b_0R)}{b_0+a_0(\lambda+d)-a_0(\lambda+d)\exp(b_0R)}.$$

(2) *If $a_2(\lambda+d)b_0 > b_2a_0(\lambda+d)$, then the coefficients $\mathbf{T}_{n,k}$ are asymptotically normal.*

By Theorems 1.2, 1.3 and 3.3, for those combinatorial numbers in Example 1.1, we have the following asymptotic formulas in a unified manner.

Example 4.4. *The following asymptotic formulas hold.*

(1) *The Bell numbers*

$$B_n = \sum_{k=0}^{n} S(n,k) \sim \frac{n!}{R_2^n \sqrt{2\pi(n + R_2^2 e^{R_2})}} \exp(e^{R_2} - 1),$$

where R_2 *is the unique positive solution* $Re^R = n$;

(2) *The ordered Bell numbers*

$$G_n = \sum_{k=0}^{n} k!S(n,k) \sim \frac{n!}{R_1^n \sqrt{2\pi(n(2 - e^{R_1})^2 + 2R_1^2 e^{R_1})}},$$

where R_1 *is the unique solution* $n = Re^R(2 - e^R)^{-1}$ *and satisfying* $0 < R_1 < 1$;

(3) *The Whitney numbers of the second kind*

$$D_n = \sum_{k=0}^{n} W_m(n,k) \sim \frac{n!}{R_2^n \sqrt{2\pi(n + mR_2^2 e^{mR_2})}} \exp\left(R_2 + \frac{1}{m}(e^{mR_2} - 1)\right),$$

where R_2 *is the unique positive solution* $n = R + Re^{mR}$;

(4) *The numbers*

$$F_m = \sum_{k=0}^{n} W_m(n,k)k! \sim \frac{mn!e^{R_1}}{R_1^n \sqrt{2\pi(n(m + 1 - e^{mR_1})^2 + m^2(m + 1)R_1^2 e^{mR_1})}},$$

where R_1 *is the unique solution* $n = R + Re^{mR}(1 + \frac{1}{m}(1 - e^{mR}))^{-1}$ *and satisfies* $0 < R_1 < 1$;

(5) *The numbers of Riordan*

$$A_n = \sum_{k=0}^{n} a_{n,k} \sim \frac{n!}{R_2^n \sqrt{2\pi(n + R_2^2 e^{R_2})}} \exp(2(e^{R_2} - 1)),$$

where R_2 *is the unique positive solution of* $n = 2Re^R$;

(6) *The numbers A154602, let* $b_{n,k} = A154602$

$$E_n = \sum_{k=0}^{n} b_{n,k} \sim \frac{n!}{R_2^n \sqrt{2\pi(n + 2R_2^2 e^{2R_2})}} \exp((e^{2R_2} - 1)),$$

where R_2 *is unique positive solution* $n = 2Re^{2R}$;

(7) *The r-Bell numbers*

$$H_n = \sum_{k=0}^{n} S_r(n,k) \sim \frac{n! \exp(rR_2 + e^{R_2} - 1)}{R_2^n \sqrt{2\pi(n + R_2^2 e^{R_2})}},$$

where R_2 *is the unique positive solution* $n = rR + Re^R$;

(8) *The r-Dowling numbers*

$$L_n = \sum_{k=0}^{n} W_{m,r}(n,k) \sim \frac{n! \exp(rR_2 + \frac{1}{m}(e^{mR_2-1}))}{R_2^n \sqrt{2\pi(n + mR_2^2 e^{mR_2})}},$$

where R_2 is the unique positive solution $n = rR + Re^{mR}$;

(9) *The numbers*

$$V_n = \sum_{k=0}^{n} m^k k! W_m(n,k) \sim \frac{n! e^{R_1}}{R_1^n \sqrt{2\pi(n(2 - e^{mR_1})^2 + 2m^2 R_1^2 e^{mR_1})}},$$

where R_1 is the solution of $R + \frac{mRe^{mR}}{2-e^{mR}} = n$ satisfying $0 < R_1 < 1$;

(10) *The generalized ordered Bell polynomial coefficients $\mathcal{U}_{n,k}$*

$$\mathcal{U}_n = \sum_{k=0}^{n} \mathcal{U}_{n,k} \sim \frac{n!}{R_1^n \psi} e^{a_2 R_1} \left[1 + \frac{b_1}{a_1}(1 - e^{a_1 R_1})\right]^{-\left(1+\frac{b_2}{b_1}\right)},$$

where $\psi = \sqrt{2\pi(n + \alpha + \beta)}$,

$$\alpha = (b_1 + b_2) R_1^2 a_1 e^{a_1 R_1} \left[1 + \frac{b_1}{a_1}(1 - e^{a_1 R_1})\right]^{-1},$$

$$\beta = (b_1 + b_2) b_1 R_1^2 e^{2a_1 R_1} \left[1 + \frac{b_1}{a_1}(1 - e^{a_1 R_1})\right]^{-2},$$

and R_1 is the unique positive solution of $n = a_2 R + (b_1 + b_2) Re^{a_1 R} \left[1 + \frac{b_1}{a_1}(1 - e^{a_1 R})\right]^{-1}$ satisfying $0 < R_1 < \frac{1}{a_1} \ln^{(1+\frac{a_1}{b_1})}$.

Corollary 4.5. *The sequences $(S(n,k))$, $(k!S(n,k))$, $(W_m(n,k))$, $(k!W_m(n,k))$, $(a_{n,k})$, $(b_{n,k})$, $(S_r(n,k))$, $(W_{m,r}(n,k))$, $(m^k k! W_m(n,k))$ are asymptotically normal respectively.*

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