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Convergence structures in (L,M)-fuzzy convex spaces

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Abstract. This paper presents the concepts of (L,M)-remotehood spaces and (L,M)-convergence spaces in the framework of (L,M)-fuzzy convex spaces. Firstly, it is shown that the category of (L,M)-remotehood spaces is isomorphic to the category of (L,M)-fuzzy convex spaces. Secondly, it is proved that the category of (L,M)-fuzzy convex spaces can be embedded in the category of (L,M)-convergence spaces as a reflective subcategory. Finally, the concepts of preconvex (L,M)-remotehood spaces and preconvex (L,M)-convergence spaces are introduced and it is shown that the category of preconvex (L,M)-remotehood spaces is isomorphic to the category of preconvex (L,M)-convergence spaces.

1. Introduction

Since the concept of fuzzy sets was proposed by Zadeh in [38], fuzzy set theory has been greatly developed. Many mathematical structures have been endowed with fuzzy sets, such as fuzzy topology [9, 26, 35], fuzzy convergence structures [2–4, 7, 8, 11, 12, 20, 33, 34], fuzzy uniformity [5, 36, 37, 40–42] and so on. Following this approach, convex structures have also been extended to the fuzzy case. Rosa [22] presented the notion of fuzzy convex structures with the unit interval as the truth-value table. Maruyama [15] generalized it to L-lattice valued case, where L denotes a completely distributive lattice. Fuzzy convex structures in the sense of Rosa and Maruyama are called L-convex structures nowadays. Recently, Shi and Xiu [24] proposed the concept of M-fuzzifying convex structures, in which each subset can be regarded as a convex set to some degree. Combining the ideas in [9] and [26], Shi and Xiu [25] introduced the notion of (L,M)-fuzzy convex structures which is a generalization of L-convex structures and M-fuzzifying convex structures. In this sense, each L-fuzzy subset can be regarded as an L-convex set to some degree. Up to know, many scholars have extensively studied L-convex structures [18, 19, 21, 32, 43, 44], M-fuzzifying convex structures [13, 16, 27, 30, 31] and (L,M)-fuzzy convex structures [10, 14].

Characterizations of fuzzy convex structures are important parts of the theory of fuzzy convex structures. Many scholars introduced different types of fuzzy hull operators to characterize the corresponding fuzzy convex structures, including M-fuzzifying restricted hull operators [23] for M-fuzzifying convex structures, L-hull operators [18] for L-convex structures, (L,M)-fuzzy (restricted) hull operators [17] for (L,M)-fuzzy

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convex structures. Recently, M-fuzzy convex structures and L-convex structures have been characterized from the aspect of fuzzy convergence structures. Pang [16] introduced the concept of M-fuzzifying convergence structures and proved that the category of M-fuzzifying convex convergence spaces is isomorphic to the category of M-fuzzifying convex spaces. Xiu et al [29] proposed the concept of L-convex ideals and used it to define L-convergence structures, and proved that the category of convex L-convergence spaces is isomorphic to the category of L-convex spaces. Convexities and concavities are dual concepts and we usually do not distinguish them. From the concave aspect, Xiu [28] proposed the concept of L-convergence structures based on L-concave prefilters, and showed that the category of concave L-convergence spaces is isomorphic to the category of L-concave spaces. Zhang and Pang [39] introduced the notion of (concave) L-convergence structures by using L-ordered c-filters and proved that the category of strong L-concave spaces can be embedded in the category of L-convergence spaces as a reflective subcategory. However, for (L, M)-fuzzy convex structures which is the more general lattice-valued convex structures, there is no corresponding description from the aspect of fuzzy convergence structures. This motivates us to propose a new type of fuzzy convergence structures in the framework of (L, M)-fuzzy convex spaces and study their relationships with (L, M)-fuzzy convex spaces.

Based on the above-mentioned motivations, we will first propose the concept of (L, M)-convex ideals and use it to define (L, M)-convergence structures. Further, we will investigate the categorical relationships between (L, M)-fuzzy convex spaces and (L, M)-convergence spaces.

This paper is organized as follows. In Section 2, we will recall some necessary concepts and notations. In Section 3, we will introduce the notion of (L, M)-convex remotehood systems and study its relationships with (L, M)-fuzzy convex structures. In Section 4, we will propose the concept of (L, M)-convergence structures and study its relationships with (L, M)-fuzzy convex spaces. In Section 5, we will propose concepts of preconvex (L, M)-convergence structures and preconvex (L, M)-remotehood structures, then show the resulting categories are isomorphic.

2. Preliminaries

Throughout this paper, both L and M denote completely distributive lattices and $'$ is an order-reversing involution on L. The largest element and the smallest element in L (M) are denoted by $\top_L$ ($\top_M$) and $\bot_L$ ($\bot_M$), respectively. For $a, b \in M$, we say that a is wedge below b in M, in symbols $a \prec b$, if for every subset $D \subseteq M$, $b \leq \bigvee D$ implies $a \leq d$ for some $d \in D$. Let $\beta(a) = \{b \in M \mid b \prec a\}$. A complete lattice M is completely distributive if and only if $a = \bigvee \beta(a)$ for each $a \in M$. We can define a residual implication operation $\rightarrow: M \times M \longrightarrow M$ corresponding to $\wedge$ by

$$a \rightarrow b = \bigvee \{c \in M \,|\, a \wedge c \leq b\}.$$

Further, $\wedge$ and $\rightarrow$ form an adjoint pair in the sense of

$$a \wedge b \leq c \Longleftrightarrow b \leq a \rightarrow c.$$

For some properties about the adjoint pair $(\wedge, \rightarrow)$, we refer to [6].

Let X be a nonempty set. An L-subset on X is a mapping from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. By $\bot_L^X$ and $\top_L^X$, we denote the constant L-subsets on X taking the value $\bot_L$ and $\top_L$, respectively. L^X is also a completely distributive lattice with an order-reversing involution operation $'$ when it inherits the structure of the lattice L in a natural way, by defining $\bigvee, \bigwedge, \leq$ and $'$ pointwisely. For each $x \in X$ and $\lambda \in L$, the L-subset x_λ, defined by $x_\lambda(y) = \lambda$ if $y = x$, and $x_\lambda(y) = \bot_L$ if $y \neq x$, is called a fuzzy point. Put $J(L^X) = \{x_\lambda \mid x \in X, \lambda \in L \backslash \{\bot\}\}$. For convenience, let $\mathbb{P}(J(L^X))$ denote the powerset of $J(L^X)$. We say $\{A_j\}_{j\in J}$ is a directed subset of L^X, if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j\in J}$, there exists $A_{j_3} \in \{A_j\}_{j\in J}$ such that $A_{j_1} \leq A_{j_3}, A_{j_2} \leq A_{j_3}$. We usually use the symbols $\{A_j\}_{j\in J} \subseteq^{dir} \mathcal{A}$ to denote that $\{A_j\}_{j\in J}$ is a directed subset of $\mathcal{A}$.

Let $\varphi : X \longrightarrow Y$ be a mapping. Define $\varphi^{\rightarrow} : L^X \longrightarrow L^Y$ and $\varphi^{\leftarrow} : L^Y \longrightarrow L^X$ by $\varphi^{\rightarrow}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $\varphi^{\leftarrow}(B) = B \circ \varphi$ for $B \in L^Y$, respectively.

Lemma 2.1. ([1]) *Suppose that* $\mathbb{F} : \mathbf{A} \longrightarrow \mathbf{B}$ *and* $\mathbb{G} : \mathbf{B} \longrightarrow \mathbf{A}$ *are concrete functors. Then the following conclusions are equivalent:*

(1) $\{id_Y : \mathbb{F} \circ \mathbb{G}(Y) \longrightarrow Y \mid Y \in \mathbf{B}\}$ *is a natural transformation from the functor* $\mathbb{F} \circ \mathbb{G}$ *to the identity functor* $id_{\mathbf{B}}$ *on* $\mathbf{B}$, *and* $\{id_X : X \longrightarrow \mathbb{G} \circ \mathbb{F}(X) \mid X \in \mathbf{A}\}$ *is a natural transformation from the identity functor* $id_{\mathbf{A}}$ *on* $\mathbf{A}$ *to the functor* $\mathbb{G} \circ \mathbb{F}$.

(2) *For each* $Y \in \mathbf{B}$, $id_Y : \mathbb{F} \circ \mathbb{G}(Y) \longrightarrow Y$ *is a* $\mathbf{B}$*-morphism, and for each* $X \in \mathbf{A}$, $id_X : X \longrightarrow \mathbb{G} \circ \mathbb{F}(X)$ *is an* $\mathbf{A}$*-morphism.*

In this case, $(\mathbb{F}, \mathbb{G})$ *is called a Galois correspondence between* $\mathbf{A}$ *and* $\mathbf{B}$.

If $(\mathbb{F}, \mathbb{G})$ is a Galois correspondence, then $\mathbb{F}$ is called a left adjoint of $\mathbb{G}$ or equivalently, $\mathbb{G}$ is called a right adjoint of $\mathbb{F}$.

Definition 2.2. ([25]) An (L, M)-fuzzy convex structure on X is a mapping $C : L^X \longrightarrow M$ which satisfies

(LMC1) $C(\bot_L^X) = C(\top_L^X) = \top_M$;

(LMC2) $\bigwedge_{j\in J} C(A_j) \leq C(\bigwedge_{j\in J} A_j)$, for every $\{A_j\}_{j\in J} \subseteq L^X$;

(LMC3) $\bigwedge_{i\in I} C(A_i) \leq C(\bigvee_{i\in I} A_i)$, for every $\{A_i\}_{i\in I} \subseteq^{dir} L^X$.

For an (L, M)-convex structure C on X, the pair (X, C) is called an (L, M)-fuzzy convex space.

A mapping $\varphi : (X, C_X) \longrightarrow (Y, C_Y)$ between (L, M)-fuzzy convex spaces is called *LM*-convexity-preserving (*LM*-CP, in short) provided that $C_Y(B) \leq C_X(\varphi^{\leftarrow}(B))$ for each $B \in L^Y$.

It is easy to verify that all (L, M)-fuzzy convex spaces as objects and all *LM*-convexity-preserving mappings as morphisms form a category, which is denoted by *LM*-**CS**.

3. (L, M)-convex remotehood spaces

In this section, we will propose the concept of (L, M)-convex remotehood spaces and then study its relationships with (L, M)-fuzzy convex spaces.

Definition 3.1. An (L, M)-convex remotehood system on X is a set $Q = \{Q_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, where $Q_{x_\lambda} : L^X \longrightarrow M$ satisfies

(LMCQ1) $Q_{x_\lambda}(\bot_L^X) = \top_M, Q_{x_\lambda}(\top_L^X) = \bot_M$;

(LMCQ2) $Q_{x_\lambda}(A) \neq \bot_M$ implies $\lambda \leq A'(x)$;

(LMCQ3) $B \leq A$ implies $Q_{x_\lambda}(A) \leq Q_{x_\lambda}(B)$;

(LMCQ4) $\forall \{A_j\}_{j\in J} \subseteq^{dir} L^X$, $\bigwedge_{j\in J} Q_{x_\lambda}(A_j) \leq Q_{x_\lambda}(\bigvee_{j\in J} A_j)$;

(LMCQ5) $Q_{x_\lambda}(A) = \bigvee_{x_\lambda \leq B \leq A'} \bigwedge_{y_\mu < B} Q_{y_\mu}(B')$.

For an (L, M)-convex remotehood system Q on X, the pair (X, Q) is called an (L, M)-convex remotehood space.

Definition 3.2. A mapping $\varphi : (X, Q^X) \longrightarrow (Y, Q^Y)$ between (L, M)-convex remotehood spaces is called *LM*-convexity-preserving (*LM*-CP, in short) provided that $Q^Y_{\varphi(x)_\lambda}(B) \leq Q^X_{x_\lambda}(\varphi^{\leftarrow}(B))$ for all $x_\lambda \in J(L^X)$ and $B \in L^Y$.

It is easy to verify that all (L, M)-convex remotehood spaces as objects and all *LM*-convexity-preserving mappings as morphisms form a category, which is denoted by *LM*-**CQ**.

Now let us show that an (L, M)-convex remotehood space can induce an (L, M)-fuzzy convex space.

Proposition 3.3. *Let* $(X, \mathcal{Q})$ *be an* (L, M)*-convex remotehood space on* X*. Then the mapping* $\mathcal{C}^{\mathcal{Q}} : L^X \longrightarrow M$ *defined by*

$$\forall A \in L^X,\ \mathcal{C}^{\mathcal{Q}}(A) = \bigwedge_{x_\lambda \prec A'} \mathcal{Q}_{x_\lambda}(A),$$

is an (L, M)*-fuzzy convex structure on* X*.*

Proof. It suffices to show that $\mathcal{C}^{\mathcal{Q}}$ satisfies (LMC1)–(LMC3).

(LMC1) Take each $x_\lambda \in J(L^X)$. It follows from (LMCQ1) that $\mathcal{Q}_{x_\lambda}(\bot_L^X) = \top_M$. Then we have

$$\mathcal{C}^{\mathcal{Q}}(\bot_L^X) = \bigwedge_{x_\lambda \prec (\bot_L^X)'} \mathcal{Q}_{x_\lambda}(\bot_L^X) \geq \bigwedge_{x_\lambda \in J(L^X)} \mathcal{Q}_{x_\lambda}(\bot_L^X) = \top_M$$

and

$$\mathcal{C}^{\mathcal{Q}}(\top_L^X) = \bigwedge_{x_\lambda \prec (\top_L^X)'} \mathcal{Q}_{x_\lambda}(\top_L^X) = \bigwedge \emptyset = \top_M.$$

(LMC2) Take each $\{A_j\}_{j\in J} \subseteq L^X$. Since

$$\{x_\lambda \mid \exists j \in J, x_\lambda \prec A_j'\} = \{x_\lambda \mid x_\lambda \prec \bigvee_{j\in J} A_j'\},$$

we have

$$\begin{aligned}
\mathcal{C}^{\mathcal{Q}}(\bigwedge_{j\in J} A_j) &= \bigwedge_{x_\lambda \prec (\bigwedge_{j\in J} A_j)'} \mathcal{Q}_{x_\lambda}(\bigwedge_{j\in J} A_j) \\
&= \bigwedge_{x_\lambda \prec \bigvee_{j\in J} A_j'} \mathcal{Q}_{x_\lambda}(\bigwedge_{j\in J} A_j) \\
&= \bigwedge_{j\in J} \bigwedge_{x_\lambda \prec A_j'} \mathcal{Q}_{x_\lambda}(\bigwedge_{j\in J} A_j) \\
&\geq \bigwedge_{j\in J} \bigwedge_{x_\lambda \prec A_j'} \mathcal{Q}_{x_\lambda}(A_j) \quad \text{(by (LMCQ3))} \\
&= \bigwedge_{j\in J} \mathcal{C}^{\mathcal{Q}}(A_j).
\end{aligned}$$

(LMC3) Take each $\{A_j\}_{j\in J} \subseteq^{dir} L^X$. Since

$$\{x_\lambda \mid x_\lambda \prec \bigwedge_{j\in J} A_j'\} \subseteq \{x_\lambda \mid \forall j \in J, x_\lambda \prec A_j'\},$$

we have

$$
\begin{aligned}
C^{Q}(\bigvee_{j\in J} A_j) &= \bigwedge_{x_\lambda \prec (\bigvee_{j\in J} A_j)'} Q_{x_\lambda}(\bigvee_{j\in J} A_j)\\
&= \bigwedge_{x_\lambda \prec \bigwedge_{j\in J} A_j'} Q_{x_\lambda}(\bigvee_{j\in J} A_j)\\
&\geq \bigwedge_{\forall j\in J, x_\lambda \prec A_j'} Q_{x_\lambda}(\bigvee_{j\in J} A_j)\\
&\geq \bigwedge_{\forall j\in J, x_\lambda \prec A_j'} \bigwedge_{j\in J} Q_{x_\lambda}(A_j) \quad \text{(by (LMCQ4))}\\
&\geq \bigwedge_{j\in J} \bigwedge_{x_\lambda \prec A_j'} Q_{x_\lambda}(A_j)\\
&= \bigwedge_{j\in J} C^{Q}(A_j).
\end{aligned}
$$

□

Proposition 3.4. *If $\varphi : (X, Q^X) \longrightarrow (Y, Q^Y)$ is LM-CP, then so is $\varphi : (X, C^{Q^X}) \longrightarrow (Y, C^{Q^Y})$.*

Proof. Since $\varphi : (X, Q^X) \longrightarrow (Y, Q^Y)$ is LM-CP, we have

$$\forall B \in L^Y,\ Q^X_{x_\lambda}(\varphi^{\leftarrow}(B)) \geq Q^Y_{\varphi(x)_\lambda}(B).$$

Then it follows from $\{x_\lambda \mid x_\lambda \prec \varphi^{\leftarrow}(B)'\} \subseteq \{x_\lambda \mid \varphi(x)_\lambda \prec B'\}$ that

$$
\begin{aligned}
C^{Q^X}(\varphi^{\leftarrow}(B)) &= \bigwedge_{x_\lambda \prec \varphi^{\leftarrow}(B)'} Q^X_{x_\lambda}(\varphi^{\leftarrow}(B))\\
&\geq \bigwedge_{\varphi(x)_\lambda \prec B'} Q^X_{x_\lambda}(\varphi^{\leftarrow}(B))\\
&\geq \bigwedge_{\varphi(x)_\lambda \prec B'} Q^Y_{\varphi(x)_\lambda}(B)\\
&\geq \bigwedge_{y_\mu \prec B'} Q^Y_{y_\mu}(B)\\
&= C^{Q^Y}(B).
\end{aligned}
$$

Hence, $\varphi : (X, C^{Q^X}) \longrightarrow (Y, C^{Q^Y})$ is LM-CP. □

Conversely, we can induce an (L, M)-convex remotehood space by an (L, M)-fuzzy convex space.

Proposition 3.5. *Let (X, C) be an (L, M)-fuzzy convex space. Then the set $Q^C = \{Q^C_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ is an (L, M)-convex remotehood system on X, where $Q^C_{x_\lambda}$ is a mapping from L^X to M defined by*

$$\forall A \in L^X,\ Q^C_{x_\lambda}(A) = \bigvee_{x_\lambda \leq B \leq A'} C(B').$$

Proof. It is enough to show that Q^C satisfies (LMCQ1)–(LMCQ5). (LMCQ1)–(LMCQ3) are obvious, so we only need to verify (LMCQ4) and (LMCQ5).

For (LMCQ4), take each $\{A_j\}_{j\in J} \subseteq^{dir} L^X$ and $\alpha \in M$ such that $\alpha \prec \bigwedge_{j\in J} \bigvee_{x_\lambda \le B_j \le A_j'} C(B_j')$. Then $\alpha \prec \bigvee_{x_\lambda \le B_j \le A_j'} C(B_j')$ for each $j \in J$, which means for each $j \in J$, there exists $B_j \in L^X$ such that $x_\lambda \le B_j \le A_j'$ and $\alpha \le C(B_j')$. Let

$$C_j = \bigvee \{D \in L^X \mid x_\lambda \le D \le A_j', \alpha \le C(D')\}$$

for each $j \in J$ and let $C = \bigwedge_{j\in J} C_j$. Then $x_\lambda \le \bigwedge_{j\in J} C_j = C \le \bigwedge_{j\in J} A_j'$. By (LMC2), we have

$$C(C_j') = C(\bigwedge_{x_\lambda \le D \le A_j', \alpha \le C(D')} D') \ge \bigwedge_{x_\lambda \le D \le A_j', \alpha \le C(D')} C(D') \ge \alpha.$$

Since $\{A_j\}_{j\in J}$ is directed, it is easy to check that $\{C_j'\}_{j\in J}$ is directed. Thus, it follows from (LMC3) that

$$\alpha \le \bigwedge_{j\in J} C(C_j') \le C(\bigvee_{j\in J} C_j') = C(C') \le \bigvee_{x_\lambda \le B \le \bigwedge_{j\in J} A_j'} C(B') = \mathcal{Q}^{C}_{x_\lambda}(\bigvee_{j\in J} A_j).$$

By the arbitrariness of α, we obtain $\bigwedge_{j\in J} \mathcal{Q}^{C}_{x_\lambda}(A_j) \le \mathcal{Q}^{C}_{x_\lambda}(\bigvee_{j\in J} A_j)$.

For (LMCQ5), we first prove the following equality

$$\text{(LMCQ0) } \mathcal{Q}^{C}_{x_\lambda} = \bigwedge_{\mu\in\beta(\lambda)} \mathcal{Q}^{C}_{x_\mu}.$$

On one hand, take each $A \in L^X$ and $\mu \in \beta(\lambda)$. Since

$$\{B \in L^X \mid x_\lambda \le B \le A'\} \subseteq \{C \in L^X \mid x_\mu \le C \le A'\},$$

we have

$$\bigvee_{x_\lambda \le B \le A'} C(B') \le \bigvee_{x_\mu \le C \le A'} C(C').$$

This shows

$$\mathcal{Q}^{C}_{x_\lambda}(A) = \bigvee_{x_\lambda \le B \le A'} C(B') \le \bigwedge_{\mu\in\beta(\lambda)} \bigvee_{x_\mu \le C \le A'} C(C') = \bigwedge_{\mu\in\beta(\lambda)} \mathcal{Q}^{C}_{x_\mu}(A).$$

On the other hand, take each $\alpha \in M$ such that

$$\alpha \prec \bigwedge_{\mu\in\beta(\lambda)} \bigvee_{x_\mu \le C \le A'} C(C') = \bigwedge_{\mu\in\beta(\lambda)} \mathcal{Q}^{C}_{x_\mu}(A).$$

Then $\alpha \prec \bigvee_{x_\mu \le C \le A'} C(C')$ for each $\mu \in \beta(\lambda)$. That is to say, for each $\mu \in \beta(\lambda)$, there exists $C_\mu \in L^X$ such that $x_\mu \le C_\mu \le A'$ and $\alpha \le C(C_u')$. Let $C = \bigvee_{\mu\in\beta(\lambda)} C_\mu$. Then

$$x_\lambda = \bigvee_{\mu\in\beta(\lambda)} x_\mu \le \bigvee_{\mu\in\beta(\lambda)} C_\mu = C \le A'.$$

By (LMC2), we have

$$\alpha \le \bigwedge_{\mu\in\beta(\lambda)} C(C_\mu') \le C(\bigwedge_{\mu\in\beta(\lambda)} C_\mu') = C(C') \le \bigvee_{x_\lambda \le B \le A'} C(B') = \mathcal{Q}^{C}_{x_\lambda}(A).$$

This means $\bigwedge_{\mu\in\beta(\lambda)} \mathcal{Q}^{C}_{x_\mu}(A) \le \mathcal{Q}^{C}_{x_\lambda}(A)$. Thus (LMCQ0) holds.

Now let us verify (LMCQ5). Take each $B \in L^X$ such that $x_\lambda \leq B \leq A'$. It follows that $x_\gamma \prec x_\lambda \leq B \leq A'$ for each $\gamma \in \beta(\lambda)$. This implies $\bigwedge_{y_\mu \prec B} \mathcal{Q}^{\mathcal{C}}_{y_\mu}(B') \leq \mathcal{Q}^{\mathcal{C}}_{x_\gamma}(B')$ for each $\gamma \in \beta(\lambda)$. Then by (LMCQ0), we have

$$\bigwedge_{y_\mu \prec B} \mathcal{Q}^{\mathcal{C}}_{y_\mu}(B') \leq \bigwedge_{\gamma \in \beta(\lambda)} \mathcal{Q}^{\mathcal{C}}_{x_\gamma}(B') = \mathcal{Q}^{\mathcal{C}}_{x_\lambda}(B') \leq \mathcal{Q}^{\mathcal{C}}_{x_\lambda}(A).$$

By the arbitrariness of B, we obtain

$$\bigvee_{x_\lambda \leq B \leq A'} \bigwedge_{y_\mu \prec B} \mathcal{Q}^{\mathcal{C}}_{y_\mu}(B') \leq \mathcal{Q}^{\mathcal{C}}_{x_\lambda}(A).$$

In order to show that inverse inequality, take each $B \in L^X$ such that $x_\lambda \leq B \leq A'$. Then it is easy to see $\mathcal{C}(B') \leq \bigvee_{y_\mu \leq C \leq B} \mathcal{C}(C')$ for each $y_\mu \prec B$, which means

$$\mathcal{C}(B') \leq \bigwedge_{y_\mu \prec B} \bigvee_{y_\mu \leq C \leq B} \mathcal{C}(C').$$

This implies

$$\bigvee_{x_\lambda \leq B \leq A'} \mathcal{C}(B') \leq \bigvee_{x_\lambda \leq B \leq A'} \bigwedge_{y_\mu \prec B} \bigvee_{y_\mu \leq C \leq B} \mathcal{C}(C').$$

That is to say,

$$\mathcal{Q}^{\mathcal{C}}_{x_\lambda}(A) \leq \bigvee_{x_\lambda \leq B \leq A'} \bigwedge_{y_\mu \prec B} \mathcal{Q}^{\mathcal{C}}_{y_\mu}(B'),$$

as desired. □

Proposition 3.6. *If $\varphi : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is LM-CP, then so is $\varphi : (X, \mathcal{Q}^{\mathcal{C}_X}) \longrightarrow (Y, \mathcal{Q}^{\mathcal{C}_Y})$.*

Proof. Since $\varphi : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is LM-CP, it follows that $\mathcal{C}_X(\varphi^{\leftarrow}(D)) \geq \mathcal{C}_Y(D)$ for each $D \in L^Y$. Then for each $x_\lambda \in J(L^X)$ and $B \in L^Y$, we have

$$\begin{aligned}
\mathcal{Q}^{\mathcal{C}_Y}_{\varphi(x)_\lambda}(B) &= \bigvee_{\varphi(x)_\lambda \leq D \leq B'} \mathcal{C}_Y(D') \\
&\leq \bigvee_{x_\lambda \leq \varphi^{\leftarrow}(D) \leq \varphi^{\leftarrow}(B)'} \mathcal{C}_Y(D') \\
&\leq \bigvee_{x_\lambda \leq \varphi^{\leftarrow}(D) \leq \varphi^{\leftarrow}(B)'} \mathcal{C}_X(\varphi^{\leftarrow}(D)') \\
&\leq \bigvee_{x_\lambda \leq C \leq \varphi^{\leftarrow}(B)'} \mathcal{C}_X(C') \\
&= \mathcal{Q}^{\mathcal{C}_X}_{x_\lambda}(\varphi^{\leftarrow}(B)).
\end{aligned}$$

This shows $\varphi : (X, \mathcal{Q}^{\mathcal{C}_X}) \longrightarrow (Y, \mathcal{Q}^{\mathcal{C}_Y})$ is LM-CP. □

Next let us establish further relationships between (L, M)-fuzzy convex spaces and (L, M)-convex remotehood spaces.

Theorem 3.7. *LM-***CS** *is isomorphic to LM-***CQ***.*

Proof. It is enough to show (1) $\mathcal{Q}^{\mathcal{C}^{\mathcal{Q}}} = \mathcal{Q}$ and (2) $\mathcal{C}^{\mathcal{Q}^{\mathcal{C}}} = \mathcal{C}$ for each (L,M)-convex remotehood space $(X,\mathcal{Q})$ and each (L,M)-fuzzy convex space $(X,\mathcal{C})$.

For (1), take each $A \in L^X$ and $x_\lambda \in J(L^X)$. By (LMCQ5), we have

$$\mathcal{Q}^{\mathcal{C}^{\mathcal{Q}}}_{x_\lambda}(A) = \bigvee_{x_\lambda \leq B \leq A'} \mathcal{C}^{\mathcal{Q}}(B^{'}) = \bigvee_{x_\lambda \leq B \leq A'} \bigwedge_{y_\mu < B} \mathcal{Q}_{y_\mu}(B^{'}) = \mathcal{Q}_{x_\lambda}(A).$$

For (2), take each $A \in L^X$. On one hand,

$$\mathcal{C}^{\mathcal{Q}^{\mathcal{C}}}(A) = \bigwedge_{x_\lambda \not< A'} \mathcal{Q}^{\mathcal{C}}_{x_\lambda}(A) = \bigwedge_{x_\lambda \not< A'} \bigvee_{x_\lambda \leq C \leq A'} \mathcal{C}(C^{'}) \geq \bigwedge_{x_\lambda \not< A'} \mathcal{C}(A) = \mathcal{C}(A).$$

On the other hand,

$$\begin{aligned}
\mathcal{C}^{\mathcal{Q}^{\mathcal{C}}}(A) &= \bigwedge_{x_\lambda \not< A'} \bigvee_{x_\lambda \leq C \leq A'} \mathcal{C}(C^{'}) \\
&= \bigvee_{f \in \prod_{x_\lambda \not< A'} \mathcal{B}_{x_\lambda}} \bigwedge_{x_\lambda \not< A'} \mathcal{C}(f(x_\lambda)^{'}) \quad \text{(by the completely distributive law)} \\
&\leq \bigvee_{f \in \prod_{x_\lambda \not< A'} \mathcal{B}_{x_\lambda}} \mathcal{C}(\bigwedge_{x_\lambda \not< A'} f(x_\lambda)^{'}) \quad \text{(by (LMC2))} \\
&= \mathcal{C}(A),
\end{aligned}$$

where $\mathcal{B}_{x_\lambda} = \{C \in L^X \mid x_\lambda \leq C \leq A^{'}\}$, as desired. □

4. (L,M)-convergence spaces

In this section, we will introduce the concept of (L,M)-convex ideals and use it to propose the concept of (L,M)-convergence structures in the framework of (L,M)-fuzzy convex spaces. Then we will study the categorical relationships between (L,M)-convergence spaces and (L,M)-fuzzy convex spaces.

Definition 4.1. A mapping $I : L^X \longrightarrow M$ is called an (L,M)-convex ideal on X provided that

(LMCI1) $I(\perp_L^X) = \top_M, I(\top_L^X) = \perp_M$;
(LMCI2) $A \leq B$ implies $I(B) \leq I(A)$;
(LMCI3) $\forall \{A_j\}_{j\in J} \subseteq^{dir} L^X, \ \bigwedge_{j\in J} I(A_j) \leq I(\bigvee_{j\in J} A_j)$.

The family of all (L,M)-convex ideals on X is denoted by $I^C_{LM}(X)$.

On the set $I^C_{LM}(X)$ of all (L,M)-convex ideals on X, we define an order by $I_1 \leq I_2$ if $I_1(A) \leq I_2(A)$ for each $A \in L^X$.

Example 4.2. (1) For each $x_\lambda \in J(L^X)$, the mapping $q(x_\lambda) : L^X \longrightarrow M$ defined by $q(x_\lambda)(A) = \top_M$ if $\lambda \leq A^{'}(x)$, otherwise $q(x_\lambda)(A) = \perp_M$ is an (L,M)-convex ideal on X.
(2) It is easy to check that $\mathcal{Q}^{\mathcal{C}}_{x_\lambda}$ defined in Proposition 3.5 is an (L,M)-convex ideal on X and $\mathcal{Q}^{\mathcal{C}}_{x_\lambda} \leq q(x_\lambda)$.
(3) For each $I \in I^C_{LM}(X)$, $\varphi^{\Rightarrow}(I) : L^Y \longrightarrow M$ defined by $\varphi^{\Rightarrow}(I)(B) = I(\varphi^{\leftarrow}(B))$ is an (L,M)-convex ideal on Y.

In what follows, we will propose the concept of (L,M)-convergence structures by means of (L,M)-convex ideals.

Definition 4.3. An (L,M)-convergence structure on X is a mapping $\lim : I^C_{LM}(X) \longrightarrow M^{J(L^X)}$ which satisfies

(LMCN1) $\lim(q(x_\lambda))(x_\lambda) = \top_M$;
(LMCN2) $I_1 \leq I_2$ implies $\lim(I_1) \leq \lim(I_2)$;

(LMCN3) $Q^{\lim}_{x_\lambda} = \bigwedge_{\mu\in\beta(\lambda)} Q^{\lim}_{x_\mu}$, where $Q^{\lim}_{x_\lambda}(A) = \bigwedge_{I\in I^C_{LM}(X)}(\lim(I)(x_\lambda) \to I(A))$.

For an (L,M)-convergence structure lim on X, the pair $(X, \lim)$ is called an (L,M)-convergence space.

Remark 4.4. It is easy to check that the mapping $Q^{\lim}_{x_\lambda}$ defined above is an (L,M)-convex ideal on X. This provides an example of (L,M)-convex ideals from the aspect of (L,M)-convergence structures.

Definition 4.5. A mapping $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is called LM-convexity-preserving provided that $\lim_X(I)(x_\lambda) \le \lim_Y(\varphi^{\Rightarrow}(I))(\varphi(x)_\lambda)$ for each $I \in I^C_{LM}(X)$ and $x_\lambda \in J(L^X)$.

It is easy to verify that all (L,M)-convergence spaces as objects and all continuous mappings as morphisms form a category, which is denoted by LM-**Conv**.

Let $\lim(X)$ denote the fibre of X, i.e.,

$$\lim(X) := \{\lim_X \mid \lim_X \text{ is an } (L,M)\text{-convergence structure on } X\}.$$

We can define an order on $\lim(X)$ by for each $\lim^1_X, \lim^2_X \in \lim(X)$, $\lim^1_X \le \lim^2_X$ if and only if $id_X : (X, \lim^1_X) \longrightarrow (X, \lim^2_X)$ is LM-convexity-preserving. In this case, we call $\lim^1_X$ is finer than $\lim^2_X$ or $\lim^2_X$ is coarser than $\lim^1_X$.

Example 4.6. (1) The mapping $\lim_{ind} : I^C_{LM}(X) \longrightarrow M^{J(L^X)}$ defined by for any $I \in I^C_{LM}(X), x_\lambda \in J(L^X)$

$$\lim_{ind}(I)(x_\lambda) = \top_M,$$

is the coarsest (L,M)-convergence structure on X, which is called the indiscrete (L,M)-convergence structure on X.

(2) The mapping $\lim_{dis} : I^C_{LM}(X) \longrightarrow M^{J(L^X)}$ defined by for any $I \in I^C_{LM}(X), x_\lambda \in J(L^X)$, $\lim_{dis}(I)(x_\lambda) = \top_M$ whenever $q(x_\lambda) \le I$, and $\lim_{dis}(I)(x_\lambda) = \bot_M$, otherwise, is the finest (L,M)-convergence structure on X, which is called the discrete (L,M)-convergence structure on X.

(3) When $M = \{0,1\}$, an (L,M)-convergence structures is exactly an L-convergence structure in [29].

(4) Here, we call the concept of Definition 4.1 in [16] M-fuzzifying order convergence structure. The category of M-fuzzifying order convergence spaces and continuous mappings is denoted by M-**COS**. The mapping $\lim : F_M(X) \longrightarrow M^X$ is called M-fuzzifying convergence structure when we change (MC2) $\mathcal{S}_F(\mathcal{F},\mathcal{G}) \le \mathcal{S}(\lim\mathcal{F}, \lim\mathcal{G})$ in [16] to

$$\text{(MC2)}' \quad \mathcal{F} \le \mathcal{G} \Longrightarrow \lim\mathcal{F} \le \lim\mathcal{G}.$$

The category of M-fuzzifying convergence spaces and continuous mappings is denoted by M-**CS**. It is easy to check that the category M-**COS** is a reflective subcategory of the category M-**CS**. When $L = \{0,1\}$, the condition (LMCN3) holds naturally. Further, we can easily see that an (L,M)-convergence structure is exactly an M-fuzzifying convergence structure when we replace (L,M)-convex ideals with M-fuzzifying convex filters in [16].

Example 4.7. Let $X = \{x, y\}$ and $L = M = \{0, \frac{1}{2}, 1\}$ be a chain. We define a mapping $\lim : I^C_{LM}(X) \longrightarrow M^{J(L^X)}$, $\forall I \in I^C_{LM}(X), \forall z \in X$,

$$\lim(I)(z_{\frac{1}{2}}) = \begin{cases} 1, & q(z_1) \le I, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\lim(I)(z_1) = \begin{cases} 1, & q(z_1) \le I, \\ \dfrac{1}{2}, & \textit{otherwise}. \end{cases}$$

Then it is easy to see lim is an (L,M)-convergence structure on X.

Next we present another example from the aspect of (L, M)-fuzzy interval operators. Let us first recall the definition of (L, M)-fuzzy interval operators [16], where L and M are completely distributive De Morgan algebra.

Definition 4.8. ([16]) A mapping $\mathcal{I} : J(L^X) \times J(L^X) \longrightarrow M^{J(L^X)}$ is called an (L, M)-fuzzy interval operator on X if it satisfies:

(LMI1) $\mathcal{I}(x_\lambda, y_\mu)(x_\lambda) = \mathcal{I}(x_\lambda, y_\mu)(y_\mu) = \top_M$;
(LMI2) $\mathcal{I}(x_\lambda, y_\mu) = \mathcal{I}(y_\mu, x_\lambda)$;
(LMI3) $\mathcal{I}(x_\lambda, y_\mu)(z_\nu) = \bigwedge_{\gamma<\nu} \bigvee_{\lambda_1<\lambda} \bigvee_{\mu_1<\mu} \mathcal{I}(x_{\lambda_1}, y_{\mu_1})(z_\gamma)$.

For an (L, M)-fuzzy interval operator $\mathcal{I}$ on X, the pair $(X, \mathcal{I})$ is called an (L, M)-fuzzy interval space.

Example 4.9. Let $(X, \mathcal{I})$ be an (L, M)-fuzzy interval space and define $\lim^{\mathcal{I}} : I^C_{LM}(X) \longrightarrow M^{J(L^X)}$ by for any $I \in I^C_{LM}(X), x_\lambda \in J(L^X)$,

$$\lim{}^{\mathcal{I}}(I)(x_\lambda) = \bigwedge_{A\in L^X} \bigwedge_{x_\lambda \leq A'} \Big(\Big(\bigvee_{z_\nu \not\leq A} \bigvee_{s_\alpha \vee t_\beta \leq A} \mathcal{I}(s_\alpha, t_\beta)(z_\nu)\Big)' \to I(A)\Big).$$

Then $\lim^{\mathcal{I}}$ is an (L, M)-convergence structure on X, which is called the underlying convergence structure of $(X, \mathcal{I})$. In particular, when M is a complete Boolean algebra,

$$\lim{}^{\mathcal{I}}(I)(x_\lambda) = \bigwedge_{A\in L^X} \bigwedge_{x_\lambda \leq A'} \Big(\bigvee_{z_\nu \not\leq A} \bigvee_{s_\alpha \vee t_\beta \leq A} \mathcal{I}(s_\alpha, t_\beta)(z_\nu) \vee I(A)\Big).$$

In the sequel, we will establish the relationship between (L, M)-fuzzy convex spaces and (L, M)-convergence spaces. To this end, we first introduce the mapping $\mathcal{S}(-,-) : I^C_{LM}(X) \times I^C_{LM}(X) \longrightarrow M$ as follows:

$$\forall I, H \in I^C_{LM}(X),\ \mathcal{S}(I, H) = \bigwedge_{A\in L^X} I(A) \to H(A).$$

Actually, the order $\leq$ on $I^C_{LM}(X)$ defined before is exactly the classical inclusion order between (L, M)-convex ideals. Here $\mathcal{S}(-,-)$ represents the fuzzy inclusion order between (L, M)-convex ideals. Concretely, $\mathcal{S}(I, H)$ can be interpreted as the degree to which I is a subset of H. Next, we will provide the induced formula from an (L, M)-fuzzy convex space to an (L, M)-convergence structure by using $\mathcal{S}(-,-)$.

Proposition 4.10. *Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space and define a mapping $\lim^{\mathcal{C}} : I^C_{LM}(X) \longrightarrow M^{J(L^X)}$ by*

$$\forall I \in I^C_{LM}(X), \forall x_\lambda \in J(L^X),\ \lim{}^{\mathcal{C}}(I)(x_\lambda) = \mathcal{S}(\mathcal{Q}^{\mathcal{C}}_{x_\lambda}, I).$$

Then $\lim^{\mathcal{C}}$ is an (L, M)-convergence structure on X.

Proof. It is enough to show that $\lim^{\mathcal{C}}$ satisfies (LMCN1)–(LMCN3).

(LMCN1) It follows immediately from $\mathcal{Q}^{\mathcal{C}}_{x_\lambda} \leq q(x_\lambda)$.
(LMCN2) It is obvious.
(LMCN3) We first show $\mathcal{Q}^{\lim^{\mathcal{C}}}_{x_\lambda} = \mathcal{Q}^{\mathcal{C}}_{x_\lambda}$. Take each $A \in L^X$. Then

$$\begin{aligned}
\mathcal{Q}^{\lim^{\mathcal{C}}}_{x_\lambda}(A) &= \bigwedge_{I\in I^C_{LM}(X)} \lim{}^{\mathcal{C}}(I)(x_\lambda) \to I(A) \\
&= \bigwedge_{I\in I^C_{LM}(X)} \mathcal{S}(\mathcal{Q}^{\mathcal{C}}_{x_\lambda}, I) \to I(A) \\
&= \bigwedge_{I\in I^C_{LM}(X)} (\bigwedge_{B\in L^X} \mathcal{Q}^{\mathcal{C}}_{x_\lambda}(B) \to I(B)) \to I(A)
\end{aligned}$$

$$\geq \bigwedge_{I\in I^C_{LM}(X)} (Q^C_{x_\lambda}(A) \to I(A)) \to I(A)$$
$$\geq Q^C_{x_\lambda}(A),$$

and

$$Q^{\lim^C}_{x_\lambda}(A) \leq (\bigwedge_{B\in L^X} Q^C_{x_\lambda}(B) \to Q^C_{x_\lambda}(B)) \to Q^C_{x_\lambda}(A) = Q^C_{x_\lambda}(A).$$

By (LMCQ0) in Proposition 3.5, we can conclude

$$Q^{\lim^C}_{x_\lambda} = Q^C_{x_\lambda} = \bigwedge_{\mu\in\beta(\lambda)} Q^C_{x_\mu} = \bigwedge_{\mu\in\beta(\lambda)} Q^{\lim^C}_{x_\mu},$$

as desired. □

Proposition 4.11. *If $\varphi : (X, C_X) \longrightarrow (Y, C_Y)$ is LM-CP, then so is $\varphi : (X, \lim^{C_X}) \longrightarrow (Y, \lim^{C_Y})$.*

Proof. Since $\varphi : (X, C_X) \longrightarrow (Y, C_Y)$ is LM-CP, it follows from Proposition 3.6 that $\varphi : (X, Q^{C_X}) \longrightarrow (Y, Q^{C_Y})$ is LM-CP, which means $Q^{C_X}_{x_\lambda}(\varphi^\leftarrow(B)) \geq Q^{C_Y}_{\varphi(x)_\lambda}(B)$ for each $B \in L^Y$ and $x_\lambda \in J(L^X)$. Then for each $I \in I^C_{LM}(X)$, we have

$$\begin{aligned}\lim^{C_Y}(\varphi^\Rightarrow(I))(\varphi(x)_\lambda) =&\mathcal{S}(Q^{C_Y}_{\varphi(x)_\lambda}, \varphi^\Rightarrow(I))\\ =&\bigwedge_{B\in L^Y} Q^{C_Y}_{\varphi(x)_\lambda}(B) \to \varphi^\Rightarrow(I)(B)\\ \geq&\bigwedge_{B\in L^Y} Q^{C_X}_{x_\lambda}(\varphi^\leftarrow(B)) \to I(\varphi^\leftarrow(B))\\ \geq&\bigwedge_{A\in L^X} Q^{C_X}_{x_\lambda}(A) \to I(A)\\ =&\mathcal{S}(Q^{C_X}_{x_\lambda}, I)\\ =&\lim^{C_X}(I)(x_\lambda).\end{aligned}$$

This shows $\varphi : (X, \lim^{C_X}) \longrightarrow (Y, \lim^{C_Y})$ is LM-CP. □

By Propositions 4.10 and 4.11, we can get a functor $\mathbb{G} : LM\text{-}\mathbf{CS} \longrightarrow LM\text{-}\mathbf{Conv}$ by

$$\mathbb{G} : \begin{cases} LM\text{-}\mathbf{CS} \longrightarrow LM\text{-}\mathbf{Conv},\\ (X, C) \longmapsto (X, \lim^C),\\ \varphi \longmapsto \varphi.\end{cases}$$

Conversely, we can induce an (L, M)-fuzzy convex structure $C^{\lim}$ by an (L, M)-convergence space $(X, \lim)$.

Proposition 4.12. *Let $(X, \lim)$ be an (L, M)-convergence space and define $C^{\lim} : L^X \longrightarrow M$ as follows*

$$\forall A \in L^X,\ C^{\lim}(A) = \bigwedge_{x_\lambda \prec A'} Q^{\lim}_{x_\lambda}(A).$$

Then $C^{\lim}$ is an (L, M)-fuzzy convex structure on X.

Proof. It suffices to show that $C^{\lim}$ satisfies (LMC1)–(LMC3). (LMC1) is obvious. So we only need to verify (LMC2) and (LMC3).

For (LMC2), since

$$\{x_\lambda \mid x_\lambda \prec \bigvee_{j\in J} A_j'\} = \{x_\lambda \mid \exists j \in J, x_\lambda \prec A_j'\},$$

we have

$$\begin{aligned}
C^{\lim}(\bigwedge_{j\in J} A_j) &= \bigwedge_{x_\lambda \prec (\bigwedge_{j\in J} A_j)'} Q^{\lim}_{x_\lambda}(\bigwedge_{j\in J} A_j) \\
&= \bigwedge_{x_\lambda \prec \bigvee_{j\in J} A_j'} Q^{\lim}_{x_\lambda}(\bigwedge_{j\in J} A_j) \\
&= \bigwedge_{j\in J} \bigwedge_{x_\lambda \prec A_j'} Q^{\lim}_{x_\lambda}(\bigwedge_{j\in J} A_j) \\
&\geq \bigwedge_{j\in J} \bigwedge_{x_\lambda \prec A_j'} Q^{\lim}_{x_\lambda}(A_j) \\
&= \bigwedge_{j\in J} C^{\lim}(A_j).
\end{aligned}$$

For (LMC3), take each $\{A_j\}_{j\in J} \subseteq^{dir} L^X$. Then it follows from $\{x_\lambda \mid x_\lambda \prec \bigwedge_{j\in J} A_j'\} \subseteq \{x_\lambda \mid \forall j \in J, x_\lambda \prec A_j'\}$ that

$$\begin{aligned}
\bigwedge_{j\in J} C^{\lim}(A_j) &= \bigwedge_{j\in J} \bigwedge_{x_\lambda \prec A_j'} Q^{\lim}_{x_\lambda}(A_j) \\
&\leq \bigwedge_{\forall j\in J, x_\lambda \prec A_j'} \bigwedge_{j\in J} Q^{\lim}_{x_\lambda}(A_j) \\
&\leq \bigwedge_{x_\lambda \prec \bigwedge_{j\in J} A_j'} Q^{\lim}_{x_\lambda}(\bigvee_{j\in J} A_j) \\
&= C^{\lim}(\bigvee_{j\in J} A_j).
\end{aligned}$$

□

Proposition 4.13. *If $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is LM-CP, then so is $\varphi : (X, C^{\lim_X}) \longrightarrow (Y, C^{\lim_Y})$.*

Proof. Since $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is LM-CP, it follows that $\lim_Y(\varphi^{\Rightarrow}(I))(\varphi(x)_\lambda) \geq \lim_X(I)(x_\lambda)$ for each $I \in I^C_{LM}(X)$ and $x_\lambda \in J(L^X)$. Then for each $B \in L^X$, we have

$$\begin{aligned}
C^{\lim_X}(\varphi^{\leftarrow}(B)) &= \bigwedge_{x_\lambda \prec \varphi^{\leftarrow}(B)'} Q^{\lim_X}_{x_\lambda}(\varphi^{\leftarrow}(B)) \\
&= \bigwedge_{x_\lambda \prec \varphi^{\leftarrow}(B)'} \Big(\bigwedge_{I\in I^C_{LM}(X)} \lim_X(I)(x_\lambda) \to I(\varphi^{\leftarrow}(B)) \Big) \\
&\geq \bigwedge_{x_\lambda \prec \varphi^{\leftarrow}(B)'} \Big(\bigwedge_{I\in I^C_{LM}(X)} \lim_Y(\varphi^{\Rightarrow}(I))(\varphi(x)_\lambda) \to \varphi^{\Rightarrow}(I)(B) \Big) \\
&\geq \bigwedge_{x_\lambda \prec \varphi^{\leftarrow}(B)'} \Big(\bigwedge_{H\in I^C_{LM}(Y)} \lim_Y(H)(\varphi(x)_\lambda) \to H(B) \Big) \\
&\geq \bigwedge_{\varphi(x)_\lambda \prec B'} \Big(\bigwedge_{H\in I^C_{LM}(Y)} \lim_Y(H)(\varphi(x)_\lambda) \to H(B) \Big)
\end{aligned}$$

$$\geq \bigwedge_{y_\mu \prec B'} \Big(\bigwedge_{H \in I^C_{LM}(Y)} \lim_Y(H)(y_\mu) \to H(B) \Big)$$
$$= \bigwedge_{y_\mu \prec B'} \mathcal{Q}^{\lim_Y}_{y_\mu}(B)$$
$$= \mathcal{C}^{\lim_Y}(B).$$

This shows $\varphi : (X, \mathcal{C}^{\lim_X}) \longrightarrow (Y, \mathcal{C}^{\lim_Y})$ is LM-CP. □

By Propositions 4.12 and 4.13, we can get a functor $\mathbb{F} : LM\text{-}\mathbf{Conv} \longrightarrow LM\text{-}\mathbf{CS}$ by

$$\mathbb{F} : \begin{cases} LM\text{-}\mathbf{Conv} \longrightarrow LM\text{-}\mathbf{CS}, \\ (X, \lim) \longmapsto (X, \mathcal{C}^{\lim}), \\ \varphi \longmapsto \varphi. \end{cases}$$

Theorem 4.14. $(\mathbb{F}, \mathbb{G})$ *is a Galois correspondence and* $\mathbb{F}$ *is a left inverse of* $\mathbb{G}$*, which means that the category* LM-**CS** *can be embedded in* LM-**Conv** *as a reflective subcategory.*

Proof. It is enough to show (1) $\lim^{\mathcal{C}^{\lim}} \geq \lim$ and (2) $\mathcal{C}^{\lim^{\mathcal{C}}} = \mathcal{C}$ for each (L, M)-convergence space $(X, \lim)$ and each (L, M)-fuzzy convex space $(X, \mathcal{C})$.

For (1), take each $I \in I^C_{LM}(X)$ and $x_\lambda \in J(L^X)$. Then

$$\lim^{\mathcal{C}^{\lim}}(I)(x_\lambda) = \mathcal{S}(\mathcal{Q}^{\mathcal{C}^{\lim}}_{x_\lambda}, I)$$
$$= \bigwedge_{A \in L^X} \mathcal{Q}^{\mathcal{C}^{\lim}}_{x_\lambda}(A) \to I(A)$$
$$= \bigwedge_{A \in L^X} \Big(\big(\bigvee_{x_\lambda \leq B \leq A'} \mathcal{C}^{\lim}(B') \big) \to I(A) \Big)$$
$$= \bigwedge_{A \in L^X} \Big(\Big(\bigvee_{x_\lambda \leq B \leq A'} \big(\bigwedge_{y_\mu \prec B} \mathcal{Q}^{\lim}_{y_\mu}(B') \big) \Big) \to I(A) \Big)$$
$$\geq \bigwedge_{A \in L^X} \Big(\Big(\bigvee_{x_\lambda \leq B \leq A'} \big(\bigwedge_{\mu \in \beta(\lambda)} \mathcal{Q}^{\lim}_{x_\mu}(B') \big) \Big) \to I(A) \Big)$$
$$= \bigwedge_{A \in L^X} \Big(\Big(\bigvee_{x_\lambda \leq B \leq A'} \mathcal{Q}^{\lim}_{x_\lambda}(B') \Big) \to I(A) \Big) \quad \text{(by (LMCN3))}$$
$$\geq \bigwedge_{A \in L^X} \mathcal{Q}^{\lim}_{x_\lambda}(A) \to I(A)$$
$$= \bigwedge_{A \in L^X} \Big(\big(\bigwedge_{H \in I^C_{LM}(X)} \lim(H)(x_\lambda) \to H(A) \big) \to I(A) \Big)$$
$$\geq \bigwedge_{A \in L^X} \Big((\lim(I)(x_\lambda) \to I(A)) \to I(A) \Big)$$
$$\geq \lim(I)(x_\lambda).$$

For (2), take each $A \in L^X$. On one hand, it follows from $\mathcal{Q}^{\lim^{\mathcal{C}}}_{x_\lambda} = \mathcal{Q}^{\mathcal{C}}_{x_\lambda}$ that

$$\mathcal{C}^{\lim^{\mathcal{C}}}(A) = \bigwedge_{x_\lambda \prec A'} \mathcal{Q}^{\lim^{\mathcal{C}}}_{x_\lambda}(A) = \bigwedge_{x_\lambda \prec A'} \mathcal{Q}^{\mathcal{C}}_{x_\lambda}(A) = \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \leq B \leq A'} \mathcal{C}(B') \geq \mathcal{C}(A).$$

On the other hand,

$$\mathcal{C}^{\lim^{\mathcal{C}}}(A) = \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \leq B \leq A'} \mathcal{C}(B')$$

$$= \bigvee_{f\in\prod_{x_\lambda<A'}\mathcal{B}_{x_\lambda}} \bigwedge_{x_\lambda<A'} C(f(x_\lambda)') \text{ (by the completely distributive law)}$$
$$\leq \bigvee_{f\in\prod_{x_\lambda<A'}\mathcal{B}_{x_\lambda}} C(\bigwedge_{x_\lambda<A'} f(x_\lambda)') \text{ (by (LMC2))}$$
$$=C(A),$$

as desired. □

In what follows, we will find out what kind of convergence structures in the category of (L,M)-convergence spaces corresponds to the category of (L,M)-fuzzy convex spaces. Firstly, we give the following definition.

Definition 4.15. An (L,M)-convergence structure lim on X is called convex if it satisfies

(LMCNP) $\lim(I)(x_\lambda) = \mathcal{S}(Q_{x_\lambda}^{\lim}, I)$;

(LMCNC) $Q_{x_\lambda}^{\lim}(A) = \bigvee_{x_\lambda\leq B\leq A'} \bigwedge_{y_\mu<B} Q_{y_\mu}^{\lim}(B')$.

For a convex (L,M)-convergence structure lim on X, the pair $(X,\lim)$ is called a convex (L,M)-convergence space.

The full subcategory of LM-**Conv** consisting of convex (L,M)-convergence spaces is denoted by LM-**CConv**.

Proposition 4.16. *Let (X,C) be an (L,M)-fuzzy convex space and define $\lim^C : I_{LM}^C(X) \longrightarrow M^{J(L^X)}$ by*

$$\forall I \in I_{LM}^C(X), x_\lambda \in J(L^X),\ \lim^C(I)(x_\lambda) = \mathcal{S}(Q_{x_\lambda}^C, I).$$

Then $\lim^C$ is a convex (L,M)-convergence structure on X.

Proof. By Proposition 4.10, we only need to show $\lim^C$ satisfies (LMCNP) and (LMCNC).

(LMCNP) It follows from $Q_{x_\lambda}^C = Q_{x_\lambda}^{\lim^C}$ that

$$\lim^C(I)(x_\lambda) = \mathcal{S}(Q_{x_\lambda}^C, I) = \mathcal{S}(Q_{x_\lambda}^{\lim^C}, I).$$

(LMCNC) By Proposition 3.7(2), we have

$$Q_{x_\lambda}^{\lim^C}(A) = Q_{x_\lambda}^C(A) = Q_{x_\lambda}^{C^{Q^C}}(A) = \bigvee_{x_\lambda\leq B\leq A'} C^{Q^C}(B') = \bigvee_{x_\lambda\leq B\leq A'} \bigwedge_{y_\mu<B} Q_{y_\mu}^C(B') = \bigvee_{x_\lambda\leq B\leq A'} \bigwedge_{y_\mu<B} Q_{y_\mu}^{\lim^C}(B'),$$

as desired. □

Proposition 4.17. *If $(X,\lim)$ is a convex (L,M)-convergence space, then $\lim^{C^{\lim}} = \lim$.*

Proof. Take each $A \in L^X$. Then by (LMCNC), we have

$$Q_{x_\lambda}^{C^{\lim}}(A) = \bigvee_{x_\lambda\leq B\leq A'} C^{\lim}(B') = \bigvee_{x_\lambda\leq B\leq A'} \bigwedge_{y_\mu<B} Q_{y_\mu}^{\lim}(B') = Q_{x_\lambda}^{\lim}(A).$$

This means $Q_{x_\lambda}^{\lim} = Q_{x_\lambda}^{C^{\lim}}$. Further, it follows from (LMCNP) that

$$\lim^{C^{\lim}}(I)(x_\lambda) = \mathcal{S}(Q_{x_\lambda}^{C^{\lim}}, I) = \mathcal{S}(Q_{x_\lambda}^{\lim}, I) = \lim(I)(x_\lambda),$$

as desired. □

By Propositions 4.14, 4.16 and 4.17, we have following result.

Theorem 4.18. *The category (L,M)-**CConv** is isomorphic to the category (L,M)-**CS**.*

5. Preconvex (L, M)-convergence spaces

In this section, we will relax the axiomatic conditions of convex (L, M)-convergence spaces and propose the concept of preconvex (L, M)-convergence spaces. Further, we introduce the concept of preconvex (L, M)-remotehood spaces which provide a characterization of preconvex (L, M)-convergence spaces.

Definition 5.1. An (L, M)-convergence structure lim on X is called preconvex if it satisfies

(LMCNP) $\lim(I)(x_\lambda) = \mathcal{S}(Q^{\lim}_{x_\lambda}, I)$.

For a preconvex (L, M)-convergence structure lim on X, the pair $(X, \lim)$ is called a preconvex (L, M)-convergence space.

The full subcategory of LM-**Conv** consisting of preconvex (L, M)-convergence spaces is denoted by LM-**PConv**.

It is easy to check the following lemma holds.

Lemma 5.2. *If* $\lim : I^C_{LM}(X) \longrightarrow M^{J(L^X)}$ *is a preconvex* (L, M)*-convergence structure, then (LMCNP) implies the following axiom:*

$$(LMCNP)' \ \mathcal{S}(I_1, I_2) \leq \bigwedge_{x_\lambda \in J(L^X)} \lim(I_1)(x_\lambda) \to \lim(I_2)(x_\lambda).$$

Not every (L, M)-convergence structure lim on X is preconvex.

Example 5.3. Define lim as in Example 4.7. Then $(X, \lim)$ is not a preconvex (L, M)-convergence space since it does not satisfy (LMCNP)′. To see this, we introduce a mapping $I^* : L^X \longrightarrow M$ by for any $A \in L^X$,

$$I^*(A) = \begin{cases} 1, \text{ if } A = \bot_X, \\ \frac{1}{2}, \text{ if } A(x) = 0, A(y) \neq 0, \\ \frac{1}{2}, \text{ if } A(x) = \frac{1}{2}, \\ 0, \text{ if } A(x) = 1. \end{cases}$$

It is routine to verify $I^* \in I^C_{LM}(X)$.

In this case, by $q(x_1)(A) = 1$ whenever $1 \leq A'(x)$ i.e. $A(x) = 0$, and $q(x_1)(A) = 0$, otherwise, we have

$$\begin{aligned} \mathcal{S}(q(x_1), I^*) &= \bigwedge_{A \in L^X} q(x_1)(A) \to I^*(A) \\ &= (1 \to 1) \wedge (1 \to \frac{1}{2}) \wedge (0 \to \frac{1}{2}) \wedge (0 \to 0) = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} &\bigwedge_{x_\lambda \in J(L^X)} \lim(q(x_1))(x_\lambda) \to \lim(I^*)(x_\lambda) \\ \leq &\Big(\lim(q(x_1))(x_{\frac{1}{2}}) \to \lim(I^*)(x_{\frac{1}{2}})\Big) \wedge \Big(\lim(q(x_1))(x_1) \to \lim(I^*)(x_1)\Big) \\ = &(1 \to 0) \wedge (1 \to \frac{1}{2}) = 0. \end{aligned}$$

It follows that

$$\mathcal{S}(q(x_1), I^*) = \frac{1}{2} \nleq 0 = \bigwedge_{x_\lambda \in J(L^X)} \lim(q(x_1))(x_\lambda) \to \lim(I^*)(x_\lambda).$$

Definition 5.4. A preconvex (L, M)-remotehood system on X is a set $\mathcal{Q} = \{\mathcal{Q}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, where $\mathcal{Q}_{x_\lambda} : L^X \longrightarrow M$ satisfies (LMCQ1)–(LMCQ4) and

(LMCQ0) $\mathcal{Q}_{x_\lambda} = \bigwedge_{\mu\in\beta(\lambda)} \mathcal{Q}_{x_\mu}$.

For a preconvex (L, M)-remotehood system $\mathcal{Q}$ on X, the pair $(X, \mathcal{Q})$ is called a preconvex (L, M)-remotehood space.

A mapping $\varphi : (X, \mathcal{Q}^X) \longrightarrow (Y, \mathcal{Q}^Y)$ between preconvex (L, M)-remotehood spaces is called LM-convexity-preserving (LM-CP, in short) provided that $\mathcal{Q}^Y_{\varphi(x)_\lambda}(B) \leq \mathcal{Q}^X_{x_\lambda}(\varphi^\leftarrow(B))$ for all $x_\lambda \in J(L^X)$ and $B \in L^Y$.

It is easy to verify that all preconvex (L, M)-remotehood spaces as objects and all LM-convexity-preserving mappings as morphisms form a category, which is denoted by LM-**PCQ**.

Next, we will give the mutual induction method between preconvex (L, M)-remotehood spaces and preconvex (L, M)-convergence spaces. Further, we will prove that they are one-to-one corresponding.

Proposition 5.5. *Let $(X, \mathcal{Q})$ be a preconvex (L, M)-remotehood space and define $\lim^{\mathcal{Q}} : I^C_{LM}(X) \longrightarrow M^{J(L^X)}$ by*

$$\forall I \in I^C_{LM}(X), x_\lambda \in J(L^X),\ \lim{}^{\mathcal{Q}}(I)(x_\lambda) = \mathcal{S}(\mathcal{Q}_{x_\lambda}, I).$$

Then $\lim^{\mathcal{Q}}$ is a preconvex (L, M)-convergence structure on X.

Proof. It is enough to show that $\lim^{\mathcal{Q}}$ satisfies (LMCN1)–(LMCN3) and (LMCNP). (LMCN1) and (LMCN2) are obvious.

(LMCN3) Take each $x_\lambda \in J(L^X)$ and $A \in L^X$. Then

$$\mathcal{Q}^{\lim^{\mathcal{Q}}}_{x_\lambda}(A) = \bigwedge_{I\in I^C_{LM}(X)} \lim{}^{\mathcal{Q}}(I)(x_\lambda) \to I(A) = \bigwedge_{I\in I^C_{LM}(X)} \mathcal{S}(\mathcal{Q}_{x_\lambda}, I) \to I(A) = \mathcal{Q}_{x_\lambda}(A).$$

By (LMCQ0), we have

$$\mathcal{Q}^{\lim^{\mathcal{Q}}}_{x_\lambda} = \mathcal{Q}_{x_\lambda} = \bigwedge_{\mu\in\beta(\lambda)} \mathcal{Q}_{x_\mu} = \bigwedge_{\mu\in\beta(\lambda)} \mathcal{Q}^{\lim^{\mathcal{Q}}}_{x_\mu}.$$

(LMCNP) It follows from $\mathcal{Q}^{\lim^{\mathcal{Q}}}_{x_\lambda} = \mathcal{Q}_{x_\lambda}$ that

$$\lim{}^{\mathcal{Q}}(I)(x_\lambda) = \mathcal{S}(\mathcal{Q}_{x_\lambda}, I) = \mathcal{S}(\mathcal{Q}^{\lim^{\mathcal{Q}}}_{x_\lambda}, I),$$

as desired. □

Proposition 5.6. *If $\varphi : (X, \mathcal{Q}^X) \longrightarrow (Y, \mathcal{Q}^Y)$ is LM-CP, then so is $\varphi : (X, \lim^{\mathcal{Q}^X}) \longrightarrow (Y, \lim^{\mathcal{Q}^Y})$.*

Proof. Since $\varphi : (X, \mathcal{Q}^X) \longrightarrow (Y, \mathcal{Q}^Y)$ is LM-CP, it follows that

$$\forall B \in L^Y, \forall x_\lambda \in J(L^X),\ \mathcal{Q}^X_{x_\lambda}(\varphi^\leftarrow(B)) \geq \mathcal{Q}^Y_{\varphi(x)_\lambda}(B).$$

Then for each $I \in I^C_{LM}(X)$, we have

$$\begin{aligned}
\lim{}^{\mathcal{Q}^Y}(\varphi^\Rightarrow(I))(\varphi(x)_\lambda) &= \mathcal{S}(\mathcal{Q}^Y_{\varphi(x)_\lambda}, \varphi^\Rightarrow(I)) \\
&= \bigwedge_{B\in L^Y} \mathcal{Q}^Y_{\varphi(x)_\lambda}(B) \to \varphi^\Rightarrow(I)(B) \\
&\geq \bigwedge_{B\in L^Y} \mathcal{Q}^X_{x_\lambda}(\varphi^\leftarrow(B)) \to I(\varphi^\leftarrow(B)) \\
&\geq \bigwedge_{A\in L^X} \mathcal{Q}^X_{x_\lambda}(A) \to I(A)
\end{aligned}$$

$$=\mathcal{S}(Q^X_{x_\lambda}, I)$$
$$=\lim{}^{Q^X}(I)(x_\lambda).$$

This shows $\varphi : (X, \lim^{Q^X}) \longrightarrow (Y, \lim^{Q^Y})$ is LM-CP. □

Proposition 5.7. *Let* $(X, \lim)$ *be a preconvex* (L, M)*-convergence space. Then* $Q^{\lim} = \{Q^{\lim}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ *is a preconvex* (L, M)*-remotehood system on* X*.*

Proof. By Remark 4.4, we only need to show that $Q^{\lim}_{x_\lambda}$ satisfies (LMCQ2) and (LMCQ0).

(LMCQ2) Suppose that $Q^{\lim}_{x_\lambda}(A) \neq \bot_M$. That is to say,

$$\bigwedge_{I \in I^C_{LM}(X)} \lim(I)(x_\lambda) \to I(A) \neq \bot_M.$$

Then it follows that $\lim(I)(x_\lambda) \to I(A) \neq \bot_M$ for each $I \in I^C_{LM}(X)$. This implies

$$q(x_\lambda)(A) = \top \to q(x_\lambda)(A) = \lim(q(x_\lambda))(x_\lambda) \to q(x_\lambda)(A) \neq \bot_M.$$

By the definition of $q(x_\lambda)$, we conclude that $\lambda \leq A'(x)$.

(LMCQ0) It follows immediately from (LMCN3). □

Proposition 5.8. *If* $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ *is LM-CP, then so is* $\varphi : (X, Q^{\lim_X}) \longrightarrow (Y, Q^{\lim_Y})$*.*

Proof. Since $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is LM-CP, it follows that

$$\forall I \in I^C_{LM}(X), \forall x_\lambda \in J(L^X),\ \lim_Y(\varphi^{\Rightarrow}(I))(\varphi(x)_\lambda) \geq \lim_X(I)(x_\lambda).$$

Then for each $B \in L^Y$, we have

$$\begin{aligned}
Q^{\lim_X}_{x_\lambda}(\varphi^{\leftarrow}(B)) &= \bigwedge_{I \in I^C_{LM}(X)} \lim_X(I)(x_\lambda) \to I(\varphi^{\leftarrow}(B)) \\
&\geq \bigwedge_{I \in I^C_{LM}(X)} \lim_Y(\varphi^{\Rightarrow}(I))(\varphi(x)_\lambda) \to \varphi^{\Rightarrow}(I)(B) \\
&\geq \bigwedge_{H \in I^C_{LM}(Y)} \lim_Y(H)(\varphi(x)_\lambda) \to H(B) \\
&= Q^{\lim_Y}_{\varphi(x)_\lambda}(B).
\end{aligned}$$

This shows $\varphi : (X, Q^{\lim_X}) \longrightarrow (Y, Q^{\lim_Y})$ is LM-CP. □

By Propositions 5.5–5.8, we know LM-**PConv** and LM-**PCQ** can be induced by each other. Finally, we will present the main result of this section.

Theorem 5.9. *The category* LM-**PConv** *is isomorphic to the category* LM-**PCQ***.*

Proof. It is enough to show (1) $\lim^{Q^{\lim}} = \lim$ and (2) $Q^{\lim^Q} = Q$ for each preconvex (L, M)-convergence space $(X, \lim)$ and each preconvex (L, M)-remotehood space (X, Q).

For (1), take each $I \in I^C_{LM}(X)$ and $x_\lambda \in J(L^X)$. By (LMCNP), we have

$$\lim{}^{Q^{\lim}}(I)(x_\lambda) = \mathcal{S}(Q^{\lim}_{x_\lambda}, I) = \lim(I)(x_\lambda).$$

For (2), take each $x_\lambda \in J(L^X)$ and $A \in L^X$. Then

$$Q^{\lim^Q}_{x_\lambda}(A) = \bigwedge_{I \in I^C_{LM}(X)} \lim{}^{Q}(I)(x_\lambda) \to I(A)$$

$$
\begin{aligned}
&= \bigwedge_{I \in I^C_{LM}(X)} \mathcal{S}(\mathcal{Q}_{x_\lambda}, I) \to I(A) \\
&\leq \mathcal{S}(\mathcal{Q}_{x_\lambda}, \mathcal{Q}_{x_\lambda}) \to \mathcal{Q}_{x_\lambda}(A) \\
&= \mathcal{Q}_{x_\lambda}(A)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{Q}^{\lim^{\mathcal{Q}}}_{x_\lambda}(A) &= \bigwedge_{I \in I^C_{LM}(X)} \mathcal{S}(\mathcal{Q}_{x_\lambda}, I) \to I(A) \\
&= \bigwedge_{I \in I^C_{LM}(X)} \Big(\bigwedge_{B \in L^X} \mathcal{Q}_{x_\lambda}(B) \to I(B)\Big) \to I(A) \\
&\geq \bigwedge_{I \in I^C_{LM}(X)} (\mathcal{Q}_{x_\lambda}(A) \to I(A)) \to I(A) \\
&\geq \mathcal{Q}_{x_\lambda}(A),
\end{aligned}
$$

as desired. □

6. Conclusions

In this paper, we presented some new characterizations of (L, M)-fuzzy convex spaces. We first introduced the concept of (L, M)-remotehood systems and showed the resulting category was isomorphic to the category of (L, M)-fuzzy convex spaces. Further, we proposed the notion of (L, M)-convergence structures and studied its relationships with (L, M)-convex structures and (L, M)-remotehood systems from a categorical aspect. Following this paper, we will consider the following problems in the future.

- Separation properties are important part of the theory of convex structures. In the future, we will consider separation properties of (L, M)-fuzzy convex spaces by means of (L, M)-convergence structures.
- We will further consider the subcategories of the category of (L, M)-convergence spaces and study their relationships from a categorical aspect.

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