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The spinor representations of framed Bertrand curves

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Abstract. In this study, we intend to examine the framed Bertrand curves in three-dimensional Euclidean space $\mathbb{E}^3$ by using the spinors, which have a fundamental place and importance in different disciplines from mathematics to physics. For this purpose, we investigate the spinor representations of framed Bertrand mates in $\mathbb{E}^3$. Additionally, we present some geometric results and interpretations. Then, we construct numerical examples with illustrative figures in order to support the given materials.

1. Introduction

The concept of the spinor is one of the most popular topics of different disciplines from mathematics especially differential geometry to physics. While according to physicists, spinors are multilinear transformations and with the help of this property, spinors are mathematical structure, but according to the mathematicians this multilinear property does not matter and spinors are a vectorial structure [8, 10, 12]. É. Cartan who introduced the spinor in 1913, but P. Ehrenfest had earned the term spinor to the science in the 1920s [25]. The term spin was asserted by physicists to explain some properties of quantum particles that appeared in the course of various experiments. For considering these properties quantitatively, some new mathematical concepts were investigated which were called spinors [14]. To study and examine the angular momentum of an electron, spinors play a significant and basic role. Moreover, spinors have also an important place in order to describe the spin of the non-relavistic electron, other spin 1/2 particles and state of the relavistic many particle systems in theory of quantum field. Besides electromagnetic theory in physics and spinors are closely related topics [10, 20, 21]. Additionally, spin matrices that can be described the spin in quantum theory were given by W. Pauli in [22], and these matrices is called and known as Pauli matrices. With the help of the Pauli matrices, spin can be represented in the theory of quantum [8]. Pauli said that the wave function of an electron can be represented by using a vector with two complex components whose name is spinor in 1927 [14].

When studying the representation of groups in 1913, Cartan found the mathematical forms of spinors. He indicated that spinors satisfy a linear representation of the groups of rotations of a space of any dimension. Hence, spinors are directly related to geometry in addition to their relationship with physics [14]. Cartan [3] presented the geometric construction of spinor representations. The study [3] is substantially fundamental

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study since gives some basic definitions in terms of geometry of the spinor representations. According to this study, in the vector space $\mathbb{C}^3$, the set of isotropic vectors generates a two-dimensional surface in the space $\mathbb{C}^2$. Considered in reverse, these vectors in $\mathbb{C}^2$, represent the same isotropic vectors. Cartan explained these vectors which are complex as two-dimensional in the space $\mathbb{C}^2$. Moreover, Cartan indicated that spinors satisfy a linear representation of the groups of rotations of a space of any dimension [3, 8]. Additionally, M. D. Vivarelli [26] studied on the spinors in the geometric sense. Vivarelli construct relations between the spinors and quaternions, and established the spinor representation of rotations in $\mathbb{E}^3$ with the help of the relations between the rotations in $\mathbb{E}^3$ and quaternions [8, 26]. In addition to these, spinors also have an important role also in Clifford algebra, which is called geometric algebra by W. K. Clifford [6, 25].

On the other hand, in differential geometry as an important part of mathematics, the theory of curves is a fundamental and significant working area. By using the Frenet-Serret frame, there are several studies for regular curves in existing literature in the classical differential geometry. There exist some associated curves such as Bertrand curves which were defined by J. M. Bertrand [2]. A Bertrand curve is a spatial curve whose principal normal line is the same as the principal normal line of another curve [17]. If a spatial curve has singular points, then Frenet-Serret frame is not established. While the Frenet-Serret frame is constructed for only regular curves with the condition the curvature κ never vanishes along the curve, this frame is not constructed if a curve has singular points. Recently, in order to establish the Frenet-Serret frame for non-regular curves, the concept of the framed curves and framed base curves were discovered by S. Honda and M. Takahashi [16]. Honda and Takahashi defined the framed curve as it is a smooth curve with a moving frame that may has singular points [16, 28]. According to these, framed curves contain both singular and regular curves, and they are a generalization of regular curves with linear independent conditions and Legendre curves in unit tangent bundles, as well [13, 23].

In the literature, lots of studies have been done and ongoing regarding the framed curves such as; existence conditions of framed curves [13], Bertrand and Mannheim curves of framed curves [17], framed rectifying and normal curves [27–29], the surfaces with respect to framed curves and framed helices [15, 18].

del Castillo and Barrales have examined the spinor formulas of the Frenet frame in [5] and this study is a milestone for researchers. Inspired by this study of del Castillo and Barrales [5], some studies have been done and ongoing with respect to the spinor representations of regular curves in both three-dimensional Euclidean space $\mathbb{E}^3$ and Minkowski space $\mathbb{E}_1^3$. We can refer to them as follows: the spinor representations of regular curves with respect to the Sabban, Darboux and Bishop frames [7, 20, 24] have been investigated in $\mathbb{E}^3$. Then, spinor formulas of Bertrand curves [8], spinor representations of involute evolute curves [10] and spinor formulas for successor curves [11] have been studied in $\mathbb{E}^3$. In addition to these, hyperbolic spinor equations of Frenet frame [19], Bishop frame [9] and Darboux frame [1] in $\mathbb{E}_1^3$ have been examined. Besides, in the study [21], spinor representations of the Frenet frame in three-dimensional Lie groups have been introduced, and Fibonacci spinors have been given in [12].

We want to bring together the Bertrand curves of framed curves, which are a part of the framed curves are an attractive topic, and spinors, which are used in various fields from differential geometry to physics. In the existing literature, several studies have been seen with respect to the spinor representations of regular curves in both $\mathbb{E}^3$ and $\mathbb{E}_1^3$. We definitely believe that this study contributes to the literature since this study includes spinor representations of regular Bertrand curves and singular Bertrand curves since we know the Bertrand framed curves is a generalization of regular Bertrand and singular Bertrand curves.

This paper is organized as follows. In the first two sections, Section 1 and Section 2 which are called introduction and basic concepts, we give some fundamental notations and notions with respect to the spinors, framed curves and Bertrand curves of framed curves. In Section 3, we construct the spinor formulas for framed Bertrand curves and some relations. In Section 4, we establish numerical examples with respect to the spinor representations of Bertrand curves of framed curves and they are supported by figures. Then, the conclusions are given in Section 5.

2. Basic Concepts

In this section, some required and basic information and notions are reminded with respect to the spinors, framed curves and Bertrand curves of framed curves in the existing literature.

2.1. Spinors

The spinors are represented as two-dimensional complex vectors as follows:

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

via the vectors $a, b, c \in \mathbb{R}^3$ such that

$$\begin{aligned} a + ib &= \delta^t \sigma \delta, \\ c &= -\widehat{\delta}^t \sigma \delta, \end{aligned} \tag{1}$$

where "t" represents the transposition, $\widehat{\delta}$ is the conjugation (mate) of δ, $\overline{\delta}$ is the complex conjugation of δ, and then the followings are given [5]:

$$\widehat{\delta} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\delta} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\delta}_1 \\ \overline{\delta}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\delta}_2 \\ \overline{\delta}_1 \end{pmatrix}.$$

Additionally, the following 2×2 matrices (Pauli matrices) which are cartesian components for the vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are presented [5]:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

Also, $a + ib$ is an isotropic vector and c is a real vector. Let $y = (y_1, y_2, y_3) \in \mathbb{C}^3$ be an isotropic vector provided that $\langle y, y \rangle = 0$ where $\mathbb{C}^3$ is three-dimensional complex vector space. The isotropic vectors in $\mathbb{C}^3$ produce the two-dimensional surface in $\mathbb{C}^2$. If this surface is parametrized by means of the δ_1 and δ_2, then $y_1 = \delta_1^2 - \delta_2^2$, $y_2 = i(\delta_1^2 + \delta_2^2)$, $y_3 = -2\delta_1\delta_2$ are written. Therefore, we have $\delta_1 = \pm\sqrt{\frac{y_1 - iy_2}{2}}$ and $\delta_2 = \pm\sqrt{\frac{-y_1 - iy_2}{2}}$ [3, 5, 8]. According to the $a + ib = (y_1, y_2, y_3)$ the followings are obtained by using the Eqs. (1) and (2):

$$y_1 = \delta^t \sigma_1 \delta = \delta_1^2 - \delta_2^2, \;\; y_2 = \delta^t \sigma_2 \delta = i(\delta_1^2 + \delta_2^2), \;\; y_3 = \delta^t \sigma_3 \delta = -2\delta_1\delta_2$$

and

$$\begin{aligned} a + ib &= \left(\delta_1^2 - \delta_2^2, i(\delta_1^2 + \delta_2^2), -2\delta_1\delta_2\right), \\ c &= \left(\delta_1\overline{\delta}_2 + \overline{\delta}_1\delta_2, i(\delta_1\overline{\delta}_2 - \overline{\delta}_1\delta_2), |\delta_1|^2 - |\delta_2|^2\right). \end{aligned}$$

Because of the fact that $a + ib \in \mathbb{C}^3$ is an isotropic vector, the vectors a, b, c are mutually orthogonal and then $|a| = |b| = |c| = \overline{\delta}^t \delta$. Hence $\langle a \times b, c \rangle = \det(a, b, c) > 0$. Conversely, if the vectors $a, b, c \in \mathbb{R}^3$ are mutually orthogonal vectors of the same magnitude ($\det(a, b, c) > 0$), then there is a spinor which is defined as in the Eq. (1) [3, 5, 8, 20].

Let δ and ϕ be any two spinors and then the following properties are satisfied:

$$\delta^t \sigma \phi = \phi^t \sigma \delta, \tag{3}$$

$$\overline{\delta^t \sigma \phi} = -\widehat{\delta}^t \sigma \widehat{\phi}, \tag{4}$$

$$\widehat{\left(z_1\delta + z_2\phi\right)} = \overline{z}_1\widehat{\delta} + \overline{z}_2\widehat{\phi}, \tag{5}$$

where $z_1, z_2 \in \mathbb{C}$ [5]. Since the matrices $\sigma_1, \sigma_2, \sigma_3$ which are presented in the Eq. (2) are symmetric, the Eq. (3) holds. Since the spinors δ and $-\delta$ correspond to the same ordered orthogonal basis $\{a, b, c\}$ with $|a| = |b| = |c|$ and $\det(a, b, c) > 0$, the correspondence between the spinors and orthogonal bases which are presented in the Eq. (1) is two-to-one. It is very essential to attention that the ordered triads $\{a, b, c\}, \{b, c, a\}, \{c, a, b\}$ correspond to different spinors. Additionally, the set $\{\delta, \overline{\delta}\}$ is linearly independent if $\delta \neq 0$ [3, 5, 8, 20]. For more detailed information about especially spinors, the studies [3–5, 14, 25, 26] can be examined.

2.2. Framed curves

Let $\gamma : I \to \mathbb{R}^3$ be a curve which may has singular points and I be an interval of $\mathbb{R}$. The following three-dimensional smooth manifold structure is given to construct framed curves as:

$$\Delta_2 = \{\mu = (\mu_1, \mu_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle \mu_i, \mu_j \rangle = \delta_{ij},\ i, j = 1, 2\}$$

and $\nu = \mu_1 \times \mu_2$ is a unit vector [16]. It should be noted that the symbol dot " $\cdot$ " will be used to show the derivative throughout this study.

Definition 2.1. ([16]) $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ is called as a framed curve if $\langle \dot{\gamma}(s), \mu_i(s) \rangle = 0$ for every parameter $s \in I$ and $i = 1, 2$. Additionally, $\gamma : I \to \mathbb{R}^3$ is called as a framed base curve if there exists $\mu : I \to \Delta_2$ such that (γ, μ_1, μ_2) is a framed curve.

Also, $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve and $\{\nu, \mu_1, \mu_2\}$ be a moving frame, then the Frenet-type derivative formulas are given as follows:

$$\begin{pmatrix} \dot{\mu}_1(s) \\ \dot{\mu}_2(s) \\ \dot{\nu}(s) \end{pmatrix} = \begin{pmatrix} 0 & l(s) & m(s) \\ -l(s) & 0 & n(s) \\ -m(s) & -n(s) & 0 \end{pmatrix} \begin{pmatrix} \mu_1(s) \\ \mu_2(s) \\ \nu(s) \end{pmatrix} \tag{6}$$

where $l(s) = \langle \dot{\mu}_1(s), \mu_2(s) \rangle,\ m(s) = \langle \dot{\mu}_1(s), \nu(s) \rangle,\ n(s) = \langle \dot{\mu}_2(s), \nu(s) \rangle$. Besides, a smooth mapping $\alpha : I \to \mathbb{R}$ characterizing the singular points of this curve such that $\dot{\gamma}(s) = \alpha(s)\nu(s)$. Moreover, s_0 is a singular point of the framed curve if and only if $\alpha(s) = 0$. Then, $(l(s), m(s), n(s), \alpha(s))$ are called as the curvatures of the framed curve (γ, μ_1, μ_2). Additionally, $(\breve{\mu}_1, \breve{\mu}_2) \in \Delta_2$ is determined as:

$$\begin{pmatrix} \breve{\mu}_1(s) \\ \breve{\mu}_2(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \mu_1(s) \\ \mu_2(s) \end{pmatrix} \tag{7}$$

where $\theta(s)$ is a smooth function. Then, Bishop frame of the framed curves with the condition $l(s) = \dot{\theta}(s)$ and the Frenet frame with the condition $m(s)\sin\theta(s) = -n(s)\cos\theta(s)$ are attained by using the Eq. (7). For more detailed and fundamental information with respect to the framed curves, we refer to the studies [13, 15–18, 27–29].

2.3. Bertrand curves of framed curves

Let (γ, μ_1, μ_2) and $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be framed curves with the curvatures (l, m, n, α) and $(\widetilde{l}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$, respectively. Assume that the curves γ and $\widetilde{\gamma}$ be different curves, namely; $\gamma \neq \widetilde{\gamma}$ [17].

Definition 2.2. ([17]) If there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\widetilde{\gamma}(s) = \gamma(s) + \lambda(s)\mu_1(s)$ and $\mu_1(s) = \widetilde{\mu}_1(s)$ for every $s \in I$, then the framed curves (γ, μ_1, μ_2) and $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2)$ are named as Bertrand mates, where the framed curve (γ, μ_1, μ_2) is called as Bertrand curve if there exists a framed curve $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2)$.

It should be noted that if (γ, μ_1, μ_2) and $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2)$ are Bertrand mates, then $\lambda \neq 0$. Also, the following equality holds since $\mu_1(s) = \widetilde{\mu}_1(s)$ and there exists a function $\theta : I \to \mathbb{R}$ [17]:

$$\begin{pmatrix} \widetilde{\mu}_2(s) \\ \widetilde{\nu}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \mu_2(s) \\ \nu(s) \end{pmatrix}. \tag{8}$$

Additionally, the curvature $(\widetilde{l}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$ of $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2)$ is written as [17]:

$$\begin{aligned} \widetilde{l}(s) &= l(s)\cos\theta(s) - m(s)\sin\theta(s), && (9) \\ \widetilde{m}(s) &= l(s)\sin\theta(s) + m(s)\cos\theta(s), && (10) \\ \widetilde{n}(s) &= n(s) - \dot{\theta}(s), && (11) \\ \widetilde{\alpha}(s) &= \lambda l(s)\sin\theta(s) + (\alpha(s) + \lambda m(s))\cos\theta(s). \end{aligned}$$

Theorem 2.3. ([17]) *Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with the curvature (l, m, n, α). In that case, (γ, μ_1, μ_2) is a Bertrand curve if and only if there exist a nonzero constant λ and a smooth function $\theta : I \to \mathbb{R}$ such that*

$$\lambda l(s) \cos \theta(s) - (\alpha(s) + \lambda m(s)) \sin \theta(s) = 0 \tag{12}$$

for every $s \in I$.

3. The Spinor Representations of Framed Bertrand Curves

In this section, we introduce the spinor representations of Bertrand curves of framed curves, and we achieved some properties and results about them.

Definition 3.1. Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed Bertrand curve and the spinor δ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$. Then, the spinor representations of the framed Bertrand curve are given as follows:

$$\mu_2 + i\nu = \delta^t \sigma \delta, \tag{13}$$

$$\mu_1 = -\widehat{\delta}^t \sigma \delta, \tag{14}$$

where $\overline{\delta}^t \delta = 1$.

Theorem 3.2. *Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed Bertrand curve and the spinor δ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$. The single spinor equation that contains the curvatures of framed curve is given as:*

$$\frac{d\delta}{ds} = \frac{1}{2}\left[-in\delta + (l + im)\,\widehat{\delta}\right]. \tag{15}$$

Proof. By taking the derivative of the Eq. (13) according to the parameter s, then the following is obtained:

$$\frac{d\mu_2}{ds} + i\frac{d\nu}{ds} = \left(\frac{d\delta}{ds}\right)^t \sigma\delta + \delta^t \sigma \frac{d\delta}{ds}. \tag{16}$$

Since $\{\delta, \widehat{\delta}\}$ is a basis for spinors, then the spinor $\dfrac{d\delta}{ds}$ can be expressed as follows:

$$\frac{d\delta}{ds} = f\delta + g\widehat{\delta} \tag{17}$$

where f and g are any complex valued functions. By means of the equations (6), (16) and (17), we have:

$$\begin{aligned} n\,\nu - l\,\mu_1 + i\,(-m\,\mu_1 - n\,\mu_2) &= \left(f\delta + g\widehat{\delta}\right)^t \sigma\delta + \delta^t\sigma\left(f\delta + g\widehat{\delta}\right) \\ &= 2f\delta^t\sigma\delta + g\widehat{\delta}^t\sigma\delta + g\delta^t\sigma\widehat{\delta}. \end{aligned}$$

Then, by using the equations (3), (13) and (14), we get:

$$-i\,n\,(\mu_2 + i\,\nu) - (l + i\,m)\,\mu_1 = 2f\,(\mu_2 + i\,\nu) - 2g\mu_1.$$

Subsequently, we achieved:

$$f = \frac{-\,i\,n}{2} \quad \text{and} \quad g = \frac{l + i\,m}{2}. \tag{18}$$

By substituting the Eq. (18) in the Eq. (17), then the Eq. (15) is obtained. □

Theorem 3.3. *Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed Bertrand curve and the spinor δ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$. Then, the spinor equations for the framed vectors are presented as follows:*

$$\nu = -\frac{i}{2}\left(\delta^t\sigma\delta + \widehat{\delta}^t\sigma\widehat{\delta}\right), \tag{19}$$

$$\mu_1 = -\widehat{\delta}^t\sigma\delta, \tag{20}$$

$$\mu_2 = \frac{1}{2}\left(\delta^t\sigma\delta - \widehat{\delta}^t\sigma\widehat{\delta}\right). \tag{21}$$

Proof. Let the spinor δ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$ of the framed Bertrand curve (γ, μ_1, μ_2). According to the Eq. (13), we can write $\mu_2 = Re(\delta^t\sigma\delta)$ and $\nu = Im(\delta^t\sigma\delta)$. By using the properties of complex numbers: $Re(z) = \frac{1}{2}(z + \overline{z})$ and $iIm(z) = \frac{1}{2}(z - \overline{z})$ for any $z \in \mathbb{C}$, then we get:

$$\mu_2 = \frac{1}{2}\left(\delta^t\sigma\delta + \overline{\delta^t\sigma\delta}\right),$$

$$\nu = -\frac{i}{2}\left(\delta^t\sigma\delta - \overline{\delta^t\sigma\delta}\right).$$

By using the Eq. (4), we have

$$\mu_2 = \frac{1}{2}\left(\delta^t\sigma\delta - \widehat{\delta}^t\sigma\widehat{\delta}\right),$$

$$\nu = -\frac{i}{2}\left(\delta^t\sigma\delta + \widehat{\delta}^t\sigma\widehat{\delta}\right),$$

and also we know $\mu_1 = -\widehat{\delta}^t\sigma\delta$ from the Eq. (14). □

Corollary 3.4. *Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed Bertrand curve and the spinor δ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$. In that case, the spinor components for framed vectors are expressed as:*

$$\nu = -\frac{i}{2}\left(\delta_1^2 - \delta_2^2 - \overline{\delta}_1^2 + \overline{\delta}_2^2, i\left(\delta_1^2 + \delta_2^2 + \overline{\delta}_1^2 + \overline{\delta}_2^2\right), 2\left(\overline{\delta}_1\overline{\delta}_2 - \delta_1\delta_2\right)\right),$$

$$\mu_1 = \left(\delta_1\overline{\delta}_2 + \overline{\delta}_1\delta_2, i\left(\delta_1\overline{\delta}_2 - \overline{\delta}_1\delta_2\right), |\delta_1|^2 - |\delta_2|^2\right),$$

$$\mu_2 = \frac{1}{2}\left(\delta_1^2 - \delta_2^2 + \overline{\delta}_1^2 - \overline{\delta}_2^2, i\left(\delta_1^2 + \delta_2^2 - \overline{\delta}_1^2 - \overline{\delta}_2^2\right), -2\left(\delta_1\delta_2 + \overline{\delta}_1\overline{\delta}_2\right)\right).$$

Proof. For the spinor equation $\delta^t\sigma\delta$, we have:

$$\begin{cases} \delta^t\sigma_1\delta = \begin{pmatrix}\delta_1 & \delta_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = \delta_1^2 - \delta_2^2, \\ \delta^t\sigma_2\delta = \begin{pmatrix}\delta_1 & \delta_2\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = i(\delta_1^2 + \delta_2^2), \\ \delta^t\sigma_3\delta = \begin{pmatrix}\delta_1 & \delta_2\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = -2\delta_1\delta_2. \end{cases} \tag{22}$$

Also, for the spinor equation $\widehat{\delta}^t\sigma\widehat{\delta}$, we get:

$$\begin{cases} \widehat{\delta}^t\sigma_1\widehat{\delta} = \begin{pmatrix}-\overline{\delta}_2 & \overline{\delta}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-\overline{\delta}_2\\ \overline{\delta}_1\end{pmatrix} = \overline{\delta}_2^2 - \overline{\delta}_1^2, \\ \widehat{\delta}^t\sigma_2\widehat{\delta} = \begin{pmatrix}-\overline{\delta}_2 & \overline{\delta}_1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}-\overline{\delta}_2\\ \overline{\delta}_1\end{pmatrix} = i\left(\overline{\delta}_2^2 + \overline{\delta}_1^2\right), \\ \widehat{\delta}^t\sigma_3\widehat{\delta} = \begin{pmatrix}-\overline{\delta}_2 & \overline{\delta}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-\overline{\delta}_2\\ \overline{\delta}_1\end{pmatrix} = 2\overline{\delta}_1\overline{\delta}_2. \end{cases} \tag{23}$$

Then, for the spinor equation $\widehat{\delta}^t\sigma\delta$, we obtain:

$$\begin{cases}\widehat{\delta}^t\sigma_1\delta=\begin{pmatrix}-\overline{\delta}_2 & \overline{\delta}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}=-\overline{\delta}_2\delta_1-\overline{\delta}_1\delta_2,\\ \widehat{\delta}^t\sigma_2\delta=\begin{pmatrix}-\overline{\delta}_2 & \overline{\delta}_1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}=i\left(-\overline{\delta}_2\delta_1+\overline{\delta}_1\delta_2\right),\\ \widehat{\delta}^t\sigma_3\delta=\begin{pmatrix}-\overline{\delta}_2 & \overline{\delta}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}=-|\overline{\delta}_1|^2+|\overline{\delta}_2|^2.\end{cases}\tag{24}$$

By substituting the equations (22), (23) and (24) in the equations (19), (20) and (21), we complete the proof. □

Since the framed curves (γ,μ_1,μ_2) and $(\widetilde{\gamma},\widetilde{\mu}_1,\widetilde{\mu}_2)$ are Bertrand mates, we want to examine also the framed curve $(\widetilde{\gamma},\widetilde{\mu}_1,\widetilde{\mu}_2)$ which is called as the expression "framed Bertrand partner curve". Now, let us investigate the spinor representations of framed Bertrand partner curve $(\widetilde{\gamma},\widetilde{\mu}_1,\widetilde{\mu}_2)$ of framed Bertrand curve (γ,μ_1,μ_2).

Definition 3.5. Let $(\widetilde{\gamma},\widetilde{\mu}_1,\widetilde{\mu}_2):I\to\mathbb{R}^3\times\Delta_2$ be a framed Bertrand partner curve and the spinor ϕ corresponds to the triad $\{\widetilde{\mu}_2,\widetilde{\nu},\widetilde{\mu}_1\}$. Then, the spinor representations of the framed Bertrand partner curve of the framed Bertrand curve are obtained as:

$$\widetilde{\mu}_2+i\widetilde{\nu} = \phi^t\sigma\phi,\tag{25}$$

$$\widetilde{\mu}_1 = -\widehat{\phi}^t\sigma\phi,\tag{26}$$

where $\overline{\phi}^t\phi=1$.

Theorem 3.6. *Let $(\widetilde{\gamma},\widetilde{\mu}_1,\widetilde{\mu}_2):I\to\mathbb{R}^3\times\Delta_2$ be a framed Bertrand partner curve and the spinor ϕ corresponds to the triad $\{\widetilde{\mu}_2,\widetilde{\nu},\widetilde{\mu}_1\}$. Then, the following single spinor equation which includes the curvatures of the framed curve is given as:*

$$\frac{d\phi}{ds}=\frac{1}{2}\left[-i\widetilde{n}\phi+\left(\widetilde{l}+i\widetilde{m}\right)\widehat{\phi}\right].\tag{27}$$

Proof. We can complete the proof using the same way as in the proof of Theorem 3.2. For the sake of brevity, we omit the proof. □

Theorem 3.7. *Let $(\widetilde{\gamma},\widetilde{\mu}_1,\widetilde{\mu}_2):I\to\mathbb{R}^3\times\Delta_2$ be a framed Bertrand partner curve and the spinor ϕ corresponds to the triad $\{\widetilde{\mu}_2,\widetilde{\nu},\widetilde{\mu}_1\}$. Then, the spinor representations of the framed vectors are given as:*

$$\begin{aligned}\widetilde{\nu} &= -\frac{i}{2}\left(\phi^t\sigma\phi+\widehat{\phi}^t\sigma\widehat{\phi}\right),\\ \widetilde{\mu}_1 &= -\widehat{\phi}^t\sigma\widehat{\phi},\\ \widetilde{\mu}_2 &= \frac{1}{2}\left(\phi^t\sigma\phi-\widehat{\phi}^t\sigma\widehat{\phi}\right).\end{aligned}$$

Proof. This proof is completed similarly to the proof of Theorem 3.3. □

Corollary 3.8. *Let $(\widetilde{\gamma},\widetilde{\mu}_1,\widetilde{\mu}_2):I\to\mathbb{R}^3\times\Delta_2$ be a framed Bertrand partner curve and the spinor ϕ corresponds to the triad $\{\widetilde{\mu}_2,\widetilde{\nu},\widetilde{\mu}_1\}$. Then the spinor components of the framed vectors are written as:*

$$\begin{aligned}\widetilde{\nu} &= -\frac{i}{2}\left(\phi_1^2-\phi_2^2-\overline{\phi}_1^2+\overline{\phi}_2^2,\ i\left(\phi_1^2+\phi_2^2+\overline{\phi}_1^2+\overline{\phi}_2^2\right),\ 2\left(\overline{\phi}_1\overline{\phi}_2-\phi_1\phi_2\right)\right),\\ \widetilde{\mu}_1 &= \left(\phi_1\overline{\phi}_2+\overline{\phi}_1\phi_2,\ i\left(\phi_1\overline{\phi}_2-\overline{\phi}_1\phi_2\right),\ |\phi_1|^2-|\phi_2|^2\right),\\ \widetilde{\mu}_2 &= \frac{1}{2}\left(\phi_1^2-\phi_2^2+\overline{\phi}_1^2-\overline{\phi}_2^2,\ i\left(\phi_1^2+\phi_2^2-\overline{\phi}_1^2-\overline{\phi}_2^2\right),\ -2\left(\phi_1\phi_2+\overline{\phi}_1\overline{\phi}_2\right)\right).\end{aligned}$$

Proof. The proof of this corollary can be completed by using the same method of the proof of the Corollary 3.4. □

Theorem 3.9. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. Then, the relation between these spinors is given as:*

$$\phi = \pm e^{i\frac{\theta}{2}} \delta \tag{28}$$

where θ is angle between the vectors $\widetilde{\mu}_2$ and μ_2.

Proof. By using the Eq. (8), the following can be written for the vector $\widetilde{\mu}_2 + i\widetilde{\nu} \in \mathbb{C}^3$:

$$\begin{aligned} \widetilde{\mu}_2 + i\widetilde{\nu} &= \cos\theta\mu_2 - \sin\theta\nu + i(\sin\theta\mu_2 + \cos\theta\nu) \\ &= (\cos\theta + i\sin\theta)\mu_2 + i(\cos\theta + i\sin\theta)\nu \\ &= (\cos\theta + i\sin\theta)(\mu_2 + i\nu) \\ &= e^{i\theta}(\mu_2 + i\nu). \end{aligned}$$

Then, we have:

$$\widetilde{\mu}_2 + i\widetilde{\nu} = e^{i\theta}(\mu_2 + i\nu)$$

where θ is the angle between the vectors $\widetilde{\mu}_2$ and μ_2. By using the Eqs. (13) and (25), we obtain as follows:

$$\phi^t\sigma\phi = e^{i\theta}\left(\delta^t\sigma\delta\right).$$

According to these, the followings can be written:

$$\begin{aligned} \phi^t\sigma\phi &= \left(\phi_1^2 - \phi_2^2, i\left(\phi_1^2 + \phi_2^2\right), -2\phi_1\phi_2\right), \\ \delta^t\sigma\delta &= \left(\delta_1^2 - \delta_2^2, i\left(\delta_1^2 + \delta_2^2\right), -2\delta_1\delta_2\right), \end{aligned}$$

and then we get $\phi_1^2 = e^{i\theta}\delta_1^2$ and $\phi_2^2 = e^{i\theta}\delta_2^2$. Also, $\phi_1 = \pm e^{i\frac{\theta}{2}}\delta_1$ and $\phi_2 = \pm e^{i\frac{\theta}{2}}\delta_2$ are obtained. Since the spinors ϕ and $-\phi$ correspond to the vector $\widetilde{\mu}_2 + i\widetilde{\nu}$, and also the spinors δ and $-\delta$ correspond to the vector $\mu_2 + i\nu$, then $\phi = \pm e^{i\frac{\theta}{2}}\delta$ is obtained. □

Corollary 3.10. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. Then, the angle between ϕ and δ is $\dfrac{\theta}{2}$.*

Theorem 3.11. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. The relation between the spinors is satisfied:*

$$\widehat{\phi} = \pm e^{-i\frac{\theta}{2}}\widehat{\delta}. \tag{29}$$

Proof. Let us take the conjugate of both sides of the Eq. (28), then we get $\widehat{\phi} = \pm \widehat{e^{i\frac{\theta}{2}}\delta}$. By using the Eq. (5), we obtain $\widehat{\phi} = \pm e^{-i\frac{\theta}{2}}\widehat{\delta}$. □

Corollary 3.12. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. While the $\widehat{\delta}$ makes an opposite rotation to the $\widehat{\phi}$, the spinor δ rotates to the spinor ϕ. Additionally, the angle of rotation is $\dfrac{\theta}{2}$.*

Theorem 3.13. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. Then, the derivative of the spinor ϕ that corresponds to the framed Bertrand partner curve $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2)$ regarding to the curvatures of (γ, μ_1, μ_2) can be expressed as:*

$$\frac{d\phi}{ds} = \frac{1}{2}\left[i\left(\dot{\theta} - n\right)\phi + e^{i\theta}(l + im)\widehat{\phi}\right]. \tag{30}$$

Proof. By using the Eqs. (9), (10), (11) and (27), we have:

$$\begin{aligned}
\frac{d\phi}{ds} &= \frac{1}{2}\left[-i\widetilde{n}\phi + \left(\widetilde{l} + i\widetilde{m}\right)\widehat{\phi}\right] \\
&= \frac{1}{2}\left[-i(n - \dot{\theta})\phi + (l\cos\theta - m\sin\theta + i(l\sin\theta + m\cos\theta))\widehat{\phi}\right] \\
&= \frac{1}{2}\left[i(\dot{\theta} - n)\phi + (l(\cos\theta + i\sin\theta) + im(\cos\theta + i\sin\theta))\widehat{\phi}\right] \\
&= \frac{1}{2}\left[i\left(\dot{\theta} - n\right)\phi + e^{i\theta}(l + im)\widehat{\phi}\right].
\end{aligned}$$

□

Corollary 3.14. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. If the angle between the $\widetilde{\mu}_2$ and μ_2 is θ, then the angle between $\frac{d\phi}{ds}$ and $\widehat{\phi}$ is also θ provided that $\dot{\theta} = n$.*

Corollary 3.15. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. Then, the following equation is satisfied:*

$$\frac{d\phi}{ds} = \pm\frac{1}{2}e^{i\frac{\theta}{2}}\left[i\left(\dot{\theta} - n\right)\delta + (l + im)\widehat{\delta}\right].$$

Proof. By using the Eqs. (28), (29) and (30), we can write:

$$\begin{aligned}
\frac{d\phi}{ds} &= \frac{1}{2}\left[i\left(\dot{\theta} - n\right)\phi + e^{i\theta}(l + im)\widehat{\phi}\right] \\
&= \frac{1}{2}\left[i\left(\dot{\theta} - n\right)\left(\pm e^{i\frac{\theta}{2}}\delta\right) + e^{i\theta}(l + im)\left(\pm e^{-i\frac{\theta}{2}}\widehat{\delta}\right)\right] \\
&= \pm\frac{1}{2}e^{i\frac{\theta}{2}}\left[i\left(\dot{\theta} - n\right)\delta + (l + im)\widehat{\delta}\right].
\end{aligned}$$

□

Corollary 3.16. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. Then, the angle between the spinors $\frac{d\phi}{ds}$ and $\widehat{\delta}$ is $\frac{\theta}{2}$ provided that $\dot{\theta} = n$.*

Theorem 3.17. *Let the framed curves (γ, μ_1, μ_2), $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be Bertrand mates, and the spinors δ and ϕ correspond to the triads $\{\mu_2, \nu, \mu_1\}$ and $\{\widetilde{\mu}_2, \widetilde{\nu}, \widetilde{\mu}_1\}$, respectively. The spinor $\frac{d\phi}{ds}$ can be expressed in terms of the framed curvatures (l, m, n, α) as follows:*

$$\frac{d\phi}{ds} = \frac{1}{2}\left[-i\left(n - \dot{\theta}\right)\phi + \left(\frac{\alpha\sin\theta}{\lambda} + i(l\sin\theta + m\cos\theta)\right)\widehat{\phi}\right]. \tag{31}$$

Proof. By using the Eqs. (9), (10), (11) and (27), we have:

$$\begin{aligned}\frac{d\phi}{ds} &= \frac{1}{2}\left[-i\widetilde{n}\phi+\left(\widetilde{l}+i\widetilde{m}\right)\widehat{\phi}\right]\\ &= \frac{1}{2}\left[-i\left(n-\dot{\theta}\right)\phi+\left[(l\cos\theta-m\sin\theta)+i\left(l\sin\theta+m\cos\theta\right)\right]\widehat{\phi}\right].\end{aligned}$$

Then, by substituting the Eq. (12) in the last equation, the proof is completed. □

4. Application

In this section, we construct numerical examples with respect to the spinor representations of framed Bertrand curves in order to support found materials.

Example 4.1. Consider the framed curve $(\gamma,\mu_1,\mu_2):I\to\mathbb{R}^3\times\Delta_2$ as

$$\begin{aligned}\gamma(s) &= \left(\frac{s^3}{3},\frac{s^4}{4},\frac{s^6}{6}\right), \quad (32)\\ \mu_1(s) &= \frac{1}{\sqrt{1+s^2}}(-s,1,0),\\ \mu_2(s) &= \frac{1}{\sqrt{(1+s^2)(1+s^2+s^6)}}\left(-s^3,-s^4,1+s^2\right).\end{aligned}$$

Since $\dot{\gamma}(s)=(s^2,s^3,s^5)$, we have $\dot{\gamma}(s)=(0,0,0)$. Namely, γ is not a regular curve since this curve has singular point at $s=0$. By using the straightforward calculations, we have the followings:

$$\begin{aligned}\nu(s) &= \frac{1}{\sqrt{1+s^2+s^6}}\left(1,s,s^3\right),\\ l(s) &= \frac{s^3}{(1+s^2)\sqrt{1+s^2+s^6}},\\ m(s) &= \frac{-1}{\sqrt{(1+s^2)(1+s^2+s^6)}},\\ n(s) &= \frac{-s^2(3+2s^2)}{(1+s^2+s^6)\sqrt{1+s^2}},\\ \alpha(s) &= s^2\sqrt{1+s^2+s^6}.\end{aligned}$$

Let the spinor δ corresponds to the triad $\{\mu_2,\nu,\mu_1\}$ of the framed curve (γ,μ_1,μ_2). Then the followings can be calculated:

$$\begin{aligned}\delta_1(s) &= \pm\frac{\sqrt{-s^3+s\sqrt{1+s^2}+i\left(\sqrt{1+s^2}+s^4\right)}}{\sqrt[4]{4\,(1+s^2)\,(1+s^2+s^6)}},\\ \delta_2(s) &= \pm\frac{\sqrt{s^3+s\sqrt{1+s^2}+i\left(-\sqrt{1+s^2}+s^4\right)}}{\sqrt[4]{4\,(1+s^2)\,(1+s^2+s^6)}}.\end{aligned}$$

With the help of the Eq. (15), we can write:

$$\frac{d\delta}{ds}=\frac{1}{2}\left[\frac{is^2(3+2s^2)\delta}{(1+s^2+s^6)\sqrt{1+s^2}}+\frac{\left(s^3-i\sqrt{1+s^2}\right)\widehat{\delta}}{(1+s^2)\sqrt{1+s^2+s^6}}\right].$$

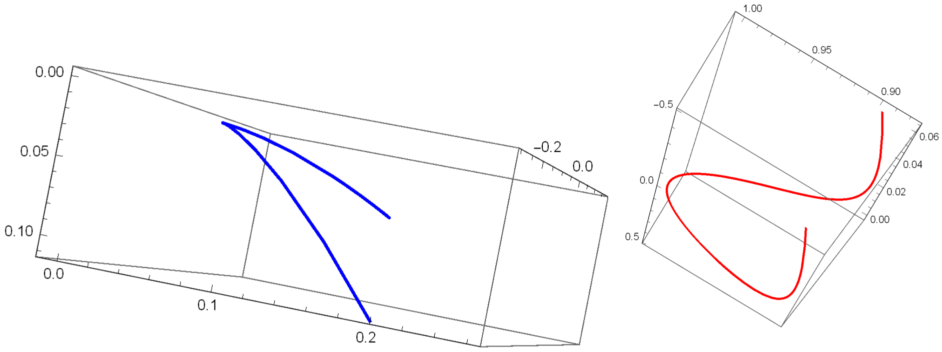
In addition to these, the framed Bertrand partner curve $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2)$ of the framed Bertrand curve (γ, μ_1, μ_2) is as follows:

$$\widetilde{\gamma}(s) = \left(\frac{s^3}{3} - \lambda \frac{s}{\sqrt{1+s^2}}, \frac{s^4}{4} + \lambda \frac{1}{\sqrt{1+s^2}}, \frac{s^6}{6}\right). \tag{33}$$

It can be seen that γ has singular point at $s = 0$, $\widetilde{\gamma}$ has not a singular point, and $\widetilde{\gamma}$ is a regular curve. Since we have

$$\dot{\widetilde{\gamma}}(s) = \left(s^2 + \lambda \frac{s^2}{(1+s^2)^{3/2}} - \lambda \frac{1}{\sqrt{1+s^2}}, s^3 - \lambda \frac{s}{(1+s^2)^{3/2}}, s^5\right).$$

In the following Figure 1, we can see these curves which are given in the Eqs. (32) and (33), respectively. Here we take $\lambda = 1$.



(a) γ (b) $\widetilde{\gamma}$

Figure 1: The curves γ and $\widetilde{\gamma}$ given in the Eqs. (32) and (33)

If the required calculations are made, the spinor representation of the Bertrand pair curve $\widetilde{\gamma}$ of γ can be constructed easily.

Example 4.2. Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve where

$$\gamma(s) = \left(\frac{s^2}{2}, \frac{s^4}{4}, \frac{s^5}{5}\right), \tag{34}$$

$$\mu_1(s) = \frac{1}{\sqrt{1+s^4}}\left(-s^2, 1, 0\right),$$

$$\mu_2(s) = \frac{1}{\sqrt{(1+s^4)(1+s^4+s^6)}}(-s^3, -s^5, 1+s^4).$$

Then, $s = 0$ a singular point. Therefore, we get:

$$\nu(s) = \frac{1}{\sqrt{1+s^4+s^6}}\left(1, s^2, s^3\right),$$

$$l(s) = \frac{2s^4}{(1+s^4)\sqrt{1+s^4+s^6}},$$

$$
\begin{aligned}
m(s) &= -\frac{2s}{\sqrt{(1+s^4)(1+s^4+s^6)}},\\
n(s) &= -\frac{s^2\left(3+s^4\right)}{(1+s^4+s^6)\sqrt{1+s^4}},\\
\alpha(s) &= s\sqrt{1+s^4+s^6}.
\end{aligned}
$$

Let the spinor δ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$ of the framed curve (γ, μ_1, μ_2). In that case, the followings can be given:

$$
\begin{aligned}
\delta_1(s) &= \pm\frac{\sqrt{-s^3+s^2\sqrt{1+s^4}+i\left(\sqrt{1+s^4}+s^5\right)}}{\sqrt[4]{4(1+s^4)(1+s^4+s^6)}},\\
\delta_2(s) &= \pm\frac{\sqrt{s^3+s^2\sqrt{1+s^4}+i\left(-\sqrt{1+s^4}+s^5\right)}}{\sqrt[4]{4(1+s^4)(1+s^4+s^6)}}.
\end{aligned}
$$

Via the Eq. (15), we have:

$$
\frac{d\delta}{ds}=\frac{1}{2}\left[\frac{is^2(3+s^4)\delta}{(1+s^4+s^6)\sqrt{1+s^4}}+\frac{2s\left(s^3-i\sqrt{1+s^4}\right)\widehat{\delta}}{(1+s^4)\sqrt{1+s^4+s^6}}\right].
$$

On the other hand, the framed Bertrand partner curve $(\widetilde{\gamma}, \widetilde{\mu}_1, \widetilde{\mu}_2)$ of the framed Bertrand curve (γ, μ_1, μ_2) is as follows:

$$
\widetilde{\gamma}(s) = \left(\frac{s^2}{2}-\lambda\frac{s^2}{\sqrt{1+s^4}}, \frac{s^4}{4}+\lambda\frac{1}{\sqrt{1+s^4}}, \frac{s^5}{5}\right). \tag{35}
$$

It is clear that framed Bertrand partner curve $\widetilde{\gamma}$ has singular point at $s=0$. Here we take $\lambda = 1$.

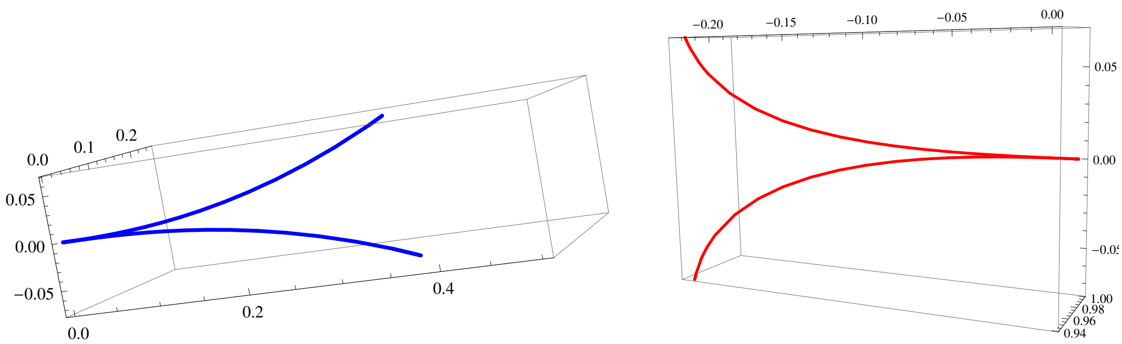


Figure 2: The curves γ and $\widetilde{\gamma}$ given in the Eqs. (34) and (35)

In the above Figure 2, we can see these curves which are presented in the Eqs. (34) and (35), respectively. If the required calculations are made, the spinor representation of the Bertrand pair curve $\widetilde{\gamma}$ of γ can be obtained easily.

5. Conclusions

In this study, we investigated a new type representation for framed Bertrand curves in $\mathbb{E}^3$. While the representation had done, we used the spinors which have many workframes in both mathematics and physics. In other words, we brought together the spinors and framed Bertrand curves. We introduced the spinor formulas of framed Bertrand mates and also gave some geometric interpretations. Moreover, this study is a wide study which includes spinor representations of both regular Bertrand curves and also singular Bertrand curves. Also, we construct numerical examples which are supported by figures.

We do not doubt that this study will led to a new perspective and innovation for future studies with respect to the spinors, framed curves and Bertrand curves of framed curves. In addition to these, some physical results can be examined in the future works by using the physical construction of spinors.

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