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A new 2-norm generated by bounded linear functionals on a normed space

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Abstract. In this work, we introduce a new 2-norm generated by bounded linear functionals on a normed space X with dimension $\dim(X) \geq 2$, and investigate its relationship with the Gähler's 2-norm [Lineare 2-normierte Räume, Math. Nachr.]. We also derive a norm on X to explore its relation with the usual norm on X.

1. Introduction and preliminaries

We know how to measure the lengths in a normed space $(X, \|.\|)$, since the notion of norm is to be regarded as a generalization of the length. But it's not always easy to measure the area on this space. If we have an inner product on a vector space X with the dimension $\dim(X) \geq 2$, then we can measure the areas of parallelograms spanned by the vectors x and y by the determinant

$$\begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix}^{\frac{1}{2}} \tag{1}$$

which is known as Gramian of linearly independent vectors $\{x, y\}$ in $(X, \langle ., .\rangle)$. Otherwise, at least we need a semi-inner product or orthogonality to measure the area of a parallelogram. Thus we must recognize that the notion of norm has a limitation. To pass this limitation, we need a new notion. One of the treatments is to consider the 2-norm introduced by Gähler [1]. If X is a normed space, then, according to Gähler, the following formula defines a 2-norm on X [1]

$$\|x, y\|^G = \frac{1}{2} \sup_{\substack{f_i \in X',\, \|f_i\| \leq 1 \\ i=1,2}} \begin{vmatrix} f_1(x) & f_2(x) \\ f_1(y) & f_2(y) \end{vmatrix}. \tag{2}$$

Here X' denotes the dual of X, which consists of bounded linear functionals on X. By this way, we can compute the area of the parallelogram spanned by two vectors. The equation (1) is known as standard 2-norm and denoted by $\|x, y\|_S$.

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Now, consider the 2-normed space $(X, \|.,.\|)$. We know how to measure the areas, how can we measure the lengths? At first, this question was asked by Gähler [1]. He defined $\|x\|^* = \|x, a\| + \|x, b\|$ where $\{a, b\}$ is linearly independent set and $\dim(X) \geq 2$. By this way, for $X = \mathbb{R}^2$ the derived norm $\|.\|^*$ is equivalent to the usual norm $\|.\|_{\mathbb{R}^2}$. Later, Gunawan [2] derived a norm for the same purpose in a 2-normed space $(X, \|.,.\|)$ of dimension $\dim(X) \geq 2$ choosing an arbitrary linearly independent set and actually, for 2-normed space l^p, the space of p-summable sequences $(1 \leq p < \infty)$, then obtain that this derived norm $\|.\|_p^*$ is equivalent to the usual norm $\|.\|_p$ on l^p. Indeed, as a result of how to measure the distance; convergence in $\left(l^p, \|.\|_p\right) \Leftrightarrow$ convergence in $\left(l^p, \|.,.\|_p\right)$. For $C[a, b]$, the space of all continuous real valued functions on [a,b], we still don't know whether we may take arbitrary linearly independent set like l^p and L^p (The space of p-integrable functions, $1 \leq p < \infty$). Let us give the definition of 2-normed space:

Let X be a real vector space of dimension $\dim(X) \geq 2$ and $\|.,.\|$ be a real function on $X \times X$ satisfying the following four conditions. The function $\|.,.\|$ is called a 2-norm on X and the pair $(X, \|.,.\|)$ is called a 2-normed space [1].

(1) $\|x, y\| \geq 0$ for every $x, y \in X$; $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(2) $\|x, y\| = \|y, x\|$ for every $x, y \in X$;
(3) $\|\alpha x, y\| = |\alpha| \|x, y\|$ for every $x, y \in X$ and for every $\alpha \in \mathbb{R}$;
(4) $\|x + z, y\| \leq \|x, y\| + \|z, y\|$ for every $x, y, z \in X$.

Euclidean 2-norm on $\mathbb{R}^2$ is given by

$$\|x_1, x_2\|_E = abs\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right), \; x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2 \quad (i = 1, 2),$$

where the subscript E is for Euclidean. The standard 2-norm is exactly same as the Euclidean 2-norm if $X = \mathbb{R}^2$, [3]. For $X = \mathbb{R}^2$, from the equation (1) we obtain a better inequality $\left\|x, y\right\|_S \leq \|x\|_S \left\|y\right\|_S$ which is a special case of Hadamard's inequality (see in [3]) where $\|x\|_S := \sqrt{\langle x, x\rangle}$.

Let X be a normed space and $f : X \to \mathbb{R}$ be a bounded linear functional. Then

$$|f(x)| \leq \|f\| \|x\|. \tag{3}$$

By (3), now observe that for every $x \in X - \{\theta\}$ and $f \in X'$

$$\frac{f(x)}{\|x\|} \leq \frac{|f(x)|}{\|x\|} \leq \|f\|. \tag{4}$$

Consequently, $f(x) \leq \|f\| \|x\|$ or $\frac{f(x)}{\|f\|} \leq \|x\|$ for every $f \in X'$, $f \neq 0$ and $x \in X$. So, we conclude that $\sup_{f \neq 0, f \in X'} \frac{f(x)}{\|f\|} = \sup_{\|f\| \leq 1} f(x) \leq \|x\|$. From the equations (2) and (3) we have $\left\|x, y\right\|^G \leq \|x\| \left\|y\right\| < \infty$.

Let f_1, f_2 be bounded linear functionals. Since $f_1(x), f_1(y), f_2(x), f_2(y) \in \mathbb{R}$, we obtain $f_2(x) f_1(y) = \gamma f_2(x) = f_2(\gamma x) = f_2(f_1(y)x) \in \mathbb{R}$ with $\gamma = f_1(y) \in \mathbb{R}$ and $\gamma x = f_1(y)x \in X$. We also have $f_1(x) f_2(y) = \delta f_2(y) = f_2(\delta y) = f_2(f_1(x)y) \in \mathbb{R}$ with $\delta = f_1(x) \in \mathbb{R}$ and $\delta y = f_1(x)y \in X$. Then from the Gähler's 2-norm, for every $x, y \in X$ we have

$$\left\|x, y\right\|^G = \frac{1}{2} \sup_{\substack{\|f_1\| \leq 1, \, \|f_2\| \leq 1 \\ f_1, f_2 \in X'}} f_2(f_1(x)y - f_1(y)x).$$

Recall that a sequence $(x(n))$ in a 2-normed space $(X, \|\cdot, \cdot\|)$ is called a convergent sequence, if there is an $x \in X$ such that $\|x(n) - x, z\| \longrightarrow 0$, as $n \longrightarrow \infty$ for every $z \in X$. Also, $(x(n))$ is said to be Cauchy sequence with respect to the $\|\cdot, \cdot\|$ if $\|x(m) - x(n), y\| \longrightarrow 0$, as $m, n \longrightarrow \infty$ for every $z \in X$ [4]. A linear 2-normed space in which every Cauchy sequence is convergent is called a 2-Banach space [7]. Throughout the paper, we use standard notation and terminology as in [5].

2. Main results

Let $x, y \in X$. Here we define a mapping $\|.,.\| : X \times X \to \mathbb{R}$

$$\left\|x, y\right\|^{KI} := \sup_{\substack{\|f\|\leq 1 \\ f\in X'}} \left\|xf(y) - yf(x)\right\|_X. \tag{5}$$

Here X' denotes the dual space of X, which consists of bounded linear functionals on X.

Proposition 2.1. *The mapping $\|.,.\|^{KI}$ in (5) defines a 2-norm on X and the pair $(X, \|.,.\|^{KI})$ is a 2-normed space.*

Proof. We need to check that $\|.,.\|^{KI}$ satisfies the four properties of a 2-norm. First note that the (2), (3) and (4) are obvious. To verify the (1), let us choose arbitrary $x, y, z \in X$.

Since $\|.\|_X \geq 0$, then $\|\cdot,\cdot\|^{KI} \geq 0$ holds.

($\Rightarrow$) If $\|x, y\|^{KI} = \sup_{\substack{\|f\|\leq 1 \\ f\in X'}} \left\|xf(y) - yf(x)\right\|_X = 0$, then $xf(y) - yf(x) = 0$. Consequently, $x = \frac{f(x)}{f(y)}\,y$, that is; x and y are linearly dependent.

($\Leftarrow$) If x and y are linearly dependent vectors, then $x = \alpha y$ for $\alpha \in \mathbb{R}$. So

$$\begin{aligned}
\|x, y\|^{KI} &= \sup_{\substack{\|f\|\leq 1 \\ f\in X'}} \left\|xf(y) - yf(x)\right\|_X \\
&= \sup_{\substack{\|f\|\leq 1 \\ f\in X'}} \left\|\alpha yf(y) - yf(\alpha y)\right\|_X \\
&= |\alpha| \sup_{\substack{\|f\|\leq 1 \\ f\in X'}} \left\|yf(y) - yf(y)\right\|_X = 0.
\end{aligned}$$

□

Choosing an arbitrary linearly independent set $\{a, b\}$ in 2-normed space $(X, \|.,.\|^{KI}$ of dimension $\dim(X) \geq 2$, we may define another norm $\|.\|^{KI}$ on X with respect to the set $\{a, b\}$ by

$$\|x\|^{KI} := \|x, a\|^{KI} + \|x, b\|^{KI}. \tag{6}$$

Lemma 2.2. *The mapping $\|.\|^{KI}$ given by (6) is a norm on X with respect to an arbitrary linearly independent set $\{a, b\}$ and the pair $(X, \|.\|^{KI})$ is a normed space.*

Proof. Note that the 'if part' of (1), (2) and (3) are obvious. To verify the 'only if part' of (1), let $x \in X$. If $\|x\|^{KI} = \|x, a\|^{KI} + \|x, b\|^{KI} = 0$, then we have $\|x, a\|^{KI} = 0$ and $\|x, b\|^{KI} = 0$ which mean that both x and a are linearly dependent and x and b are linearly dependent. So there exist scalars α, β such that $x = \alpha a = \beta b$. But from the definition it is known that a and b are linearly independent vectors, hence $x = \theta$. □

Lemma 2.3. *Let $\|.\|^{KI}$ be the derived norm defined by (6) on $(X, \|\cdot,\cdot\|^{KI})$ and $a, b \in X$ be linearly independent vectors, then for every $x \in X$ we have*

$$\|x\|^{KI} \leq 2(\|a\|_X + \|b\|_X)\|x\|_X.$$

Proof. By assumptions in this lemma, take an arbitrary $x \in X$. Using triangle inequality

$$\|x, a\|^{KI} = \sup_{\substack{\|f\|\leq 1 \\ f\in X'}} \left\|xf(a) - af(x)\right\|_X \leq \sup_{\substack{\|f\|\leq 1 \\ f\in X'}} (\|x\|_X\,|f(a)| + \|a\|_X\,|f(x)|).$$

By (4), we obtain $\|x, a\|^{KI} \leq 2\,\|a\|_X\,\|x\|_X$. Replace a with b, so $\|x, b\|^{KI} \leq 2\,\|b\|_X\,\|x\|_X$. Combine two inequalities above, then $\|x\|^{KI} \leq 2(\|a\|_X + \|b\|_X)\|x\|_X$, for every $x \in X$. □

Lemma 2.4. *Suppose that $a, b \in X$ are linearly independent vectors such that $\|a\|_X = \|b\|_X$ in normed space $(X, \|\cdot\|_X)$. If $f_0 \in X'$ such that $f_0(a) \neq 0$, $f_0(b) \neq 0$ and $0 < \|f_0\|_{X'} \leq 1$, then we have $C\|x\|_X \leq \|x\|^{KI}$, for every $x \in X$ with $C = \frac{|f_0(a)|+|f_0(b)|}{1+\frac{3\|a\|^2}{|g(a)h(b)-h(a)g(b)|}}$ and $g, h \in X'$ such that $g(a)h(b) - g(b)h(a) \neq 0$, $0 < \|g\|_{X'} \leq 1$ and $0 < \|h\|_{X'} \leq 1$.*

Proof. Let $a, b \in X$ be linearly independent vectors such that $\|a\|_X = \|b\|_X$ in normed space $(X, \|\cdot\|_X)$ with $\dim(X) \geq 2$. Now, take an arbitrary $x \in X$ and $f_0 \in X'$ such that $f_0(a) \neq 0$, $f_0(b) \neq 0$ and $0 < \|f_0\|_{X'} \leq 1$. Next, observe that

$$\begin{aligned}\|x\|_X|f_0(a)| &= \|xf_0(a) - af_0(x) + af_0(x)\|_X \\ &\leq \|xf_0(a) - af_0(x)\|_X + \|a\|_X|f_0(x)| \\ &\leq \|x, a\|^{KI} + \|a\|_X|f_0(x)|. \end{aligned} \tag{7}$$

Let $g, h \in X'$ such that $g(a)h(b) - g(b)h(a) \neq 0$ and $0 < \|g\|_{X'} \leq 1$ and $0 < \|h\|_{X'} \leq 1$. Then we have

$$\begin{aligned}&(g(a)h(b) - g(b)h(a))f_0(x) \\ &= f_0(x)g(a)h(b) - f_0(x)g(b)h(a) \\ &= f_0(x)g(a)h(b) - f_0(x)g(b)h(a) + (g(x)f_0(a)h(b) - g(x)f_0(a)h(b)) \\ &\quad + (h(x)f_0(a)g(b) - h(x)f_0(a)g(b)) \\ &= (f_0(x)g(a) - g(x)f_0(a))h(b) + (h(x)f_0(a) - f_0(x)h(a))g(b) + (g(x)h(b) - g(b)h(x))f_0(a).\end{aligned}$$

Since $f_0, g, h \in X'$ with $0 < \|f_0\|_{X'} \leq 1$, $0 < \|g\|_{X'} \leq 1$ and $0 < \|h\|_{X'} \leq 1$, then we have the following equation by triangle inequality,

$$\begin{aligned}&|g(a)h(b) - g(b)h(a)|\,|f_0(x)| \\ &\leq |f_0(x)g(a) - g(x)f_0(a)||h(b)| + |h(x)f_0(a) - f_0(x)h(a)||g(b)| + |g(x)h(b) - g(b)h(x)||f_0(a)| \\ &= |f_0(xg(a) - ag(x))||h(b)| + |f_0(ah(x) - xh(a))||g(b)| + |g(xh(b) - bh(x))||f_0(a)| \\ &\leq \left\|f_0\right\|_{X'}\left\|xg(a) - ag(x)\right\|_X\|h\|_{X'}\|b\|_X + \left\|f_0\right\|_{X'}\|ah(x) - xh(a)\|_X\left\|g\right\|_{X'}\|b\|_X \\ &\quad + \left\|g\right\|_{X'}\|xh(b) - bh(x)\|_X\left\|f_0\right\|_{X'}\|a\|_X \\ &\leq \|x, a\|^{KI}\|b\|_X + \|x, a\|^{KI}\|b\|_X + \left\|g\right\|_{X'}\|x, b\|^{KI}\|a\|_X \\ &\leq 2\|x, a\|^{KI}\|b\|_X + \|x, b\|^{KI}\|a\|_X.\end{aligned} \tag{8}$$

Now, check that

$$\begin{aligned}\|a\|_X|f_0(x)| &= \frac{\|a\|_X}{|g(a)h(b) - g(b)h(a)|}|g(a)h(b) - g(b)h(a)||f_0(x)| \\ &\leq \frac{\|a\|_X}{|g(a)h(b) - g(b)h(a)|}(2\|x, a\|^{KI}\|b\|_X + \|x, b\|^{KI}\|a\|_X).\end{aligned}$$

By (7) and (8), we obtain

$$\begin{aligned}\|x\|_X|f_0(a)| &\leq \|x, a\|^{KI} + \|a\|_X|f_0(x)| \\ &\leq \|x, a\|^{KI} + \tfrac{\|a\|_X}{|g(a)h(b)-g(b)h(a)|}(2\|x, a\|^{KI}\|b\|_X + \|x, b\|^{KI}\|a\|_X) \\ &= \left(1 + \tfrac{2\|a\|_X\|b\|_X}{|g(a)h(b)-h(a)g(b)|}\right)\|x, a\|^{KI} + \tfrac{\|a\|_X^2}{|g(a)h(b)-h(a)g(b)|}\|x, b\|^{KI}.\end{aligned} \tag{9}$$

Next replace a with b, we also have

$$\|x\|_X|f_0(b)| \leq \left(1 + \frac{2\|a\|_X\|b\|_X}{|g(a)h(b) - h(a)g(b)|}\right)\|x, b\|^{KI} + \frac{\|b\|_X^2}{|g(a)h(b) - h(a)g(b)|}\|x, a\|^{KI}. \tag{10}$$

Since $a, b \in X$ are linearly independent vectors such that $\|a\|_X = \|b\|_X$, we have the following by combining (9) and (10)

$$\begin{aligned}\|x\|_X \left(\left|f_0(a)\right| + \left|f_0(b)\right|\right) &\leq \left(1 + \tfrac{3\|a\|_X^2}{|g(a)h(b)-h(a)g(b)|}\right)\left(\|x,a\|^{KI} + \|x,b\|^{KI}\right)\\ &= \left(1 + \tfrac{3\|b\|_X^2}{|g(a)h(b)-h(a)g(b)|}\right)\left(\|x,a\|^{KI} + \|x,b\|^{KI}\right)\\ &= \left(1 + \tfrac{3\|a\|_X^2}{|g(a)h(b)-h(a)g(b)|}\right)\|x\|^{KI}\\ &= \left(1 + \tfrac{3\|b\|_X^2}{|g(a)h(b)-h(a)g(b)|}\right)\|x\|^{KI}.\end{aligned}$$

Finally, $C\|x\|_X \leq \|x\|^{KI}$ for every $x \in X$ with $C = \frac{|f_0(a)|+|f_0(b)|}{1+\frac{3\|a\|_X^2}{|g(a)h(b)-h(a)g(b)|}}$. This completes the proof. □

As a result of combining both of Lemma 2.3 and Lemma 2.4, we obtain the equivalence of the norms $\|.\|_X$ and $\|.\|^{KI}$ under the conditions of Lemma 2.4.

Corollary 2.5. *Under the conditions of Lemma 2.4, the derived norm $\|\cdot\|^{KI}$ is equivalent to the norm $\|.\|_X$ on X.*

Lemma 2.6 and Theorem 2.7 arise as a consequence of Corollary 2.5 under the conditions of Lemma 2.4.

Lemma 2.6. *In X, a sequence $(x(n))$ converges to x with respect to $\|.\|_X$ if and only if it converges to x with respect to $\|.,.\|^{KI}$. Similarly, a sequence $(x(n))$ is a Cauchy sequence with respect to $\|.\|_X$ if and only if it is a Cauchy sequence with respect to $\|.,.\|^{KI}$.*

Proof. (⇒) Let $(x(n))$ be a sequence convergent to x with respect to $\|.\|_X$. By Lemma 7, for every $y \in X$, we have $0 \leq \|x(n) - x, y\|^{KI} \leq 2\|x(n) - x\|_X\|y\|_X \longrightarrow 0$, as $n \longrightarrow \infty$. Thus $(x(n))$ converges to x with respect to $\|.,.\|^{KI}$.

(⇐) Suppose that $(x(n))$ is a sequence converges to x with respect to $\|.,.\|^{KI}$. So, for every $y \in X$, we have $\|x(n) - x, y\|^{KI} \longrightarrow 0$, as $n \longrightarrow \infty$. Now, take linearly independent vectors $a, b \in X$ such that $\|a\|_X = \|b\|_X$. By Lemma 8, we have

$$0 \leq C\|x(n) - x\|_X \leq \|x(n) - x\|^{KI} = \|x(n) - x, a\|^{KI} + \|x(n) - x, b\|^{KI} \longrightarrow 0,$$

as $n \longrightarrow \infty$. Thus $x(n)$ converges to x with respect to $\|.\|_X$. The second part of the theorem can be proved in a similar way: one only needs to replace the expressions "convergent to x" with "Cauchy" and "$x(n) - x$" with "$x(n) - x(m)$" and $n \longrightarrow \infty$ with $m, n \longrightarrow \infty$. □

Theorem 2.7. $(X, \|.,.\|^{KI})$ *is a* 2*-Banach space if and only if* $(X, \|.\|_X)$ *is a Banach space.*

Proof. Let $(x(n))$ be a Cauchy sequence in X with respect to $\|.,.\|^{KI}$. Then by Lemma 2.6, $(x(n))$ is Cauchy sequence with respect to the norm $\|.\|_X$. If $(X, \|.\|_X)$ is a Banach space; X is complete with respect to the norm $\|.\|_X$, and then $x(n)$ must converge to some $x \in X$ in $\|.\|_X$. By another application of Lemma 2.6, $x(n)$ also converges to x in $\|.,.\|^{KI}$. This shows that X is complete with respect to the 2-norm $\|.,.\|^{KI}$, that is $(X, \|.,.\|^{KI})$ is a 2-Banach space. □

Relation between the 2-norms $\|.,.\|^{KI}$ and $\|.,.\|^{G}$ is presented in the following theorem.

Theorem 2.8. *For every $x, y \in X$, $\left\|x, y\right\|^{G} \leq \frac{1}{2}\left\|x, y\right\|^{KI}$.*

Proof. From (2), (4) and (5), for every $x, y \in X$ we have the following

$$\begin{aligned}
\left\|x, y\right\|^{G} &= \frac{1}{2} \sup_{\substack{f_i \in X',\ \|f_i\|_{X'} \le 1 \\ i=1,2}} \begin{vmatrix} f_1(x) & f_2(x) \\ f_1(y) & f_2(y) \end{vmatrix} \\
&= \frac{1}{2} \sup_{\substack{f_i \in X',\ \|f_i\|_{X'} \le 1 \\ i=1,2}} \left(f_1(x) f_2(y) - f_1(y) f_2(x)\right) \\
&= \frac{1}{2} \sup_{\substack{f_i \in X',\ \|f_i\|_{X'} \le 1 \\ i=1,2}} f_2\left(y f_1(x)\right) - f_2\left(x f_1(y)\right) \\
&= \frac{1}{2} \sup_{\substack{f_i \in X',\ \|f_i\|_{X'} \le 1 \\ i=1,2}} f_2\left(y f_1(x) - x f_1(y)\right) \qquad (\text{for } f_1 = f) \\
&= \frac{1}{2} \sup_{\substack{f_i \in X',\ \|f_i\|_{X'} \le 1 \\ i=1,2}} f_2\left(y f(x) - x f(y)\right) \\
&\le \frac{1}{2} \sup_{\substack{f_i \in X',\ \|f_i\|_{X'} \le 1 \\ i=1,2}} \left\|f_2\right\|_{X'} \left\|y\, f(x) - x\, f(y)\right\|_{X} \\
&\le \frac{1}{2} \sup_{f \in X',\ \|f\|_{X'} \le 1} \left\|y f(x) - x f(y)\right\|_{X} \\
&= \frac{1}{2} \sup_{f \in X',\ \|f\|_{X'} \le 1} \left\|x f(y) - y f(x)\right\|_{X} \\
&= \frac{1}{2} \left\|x, y\right\|^{KI}.
\end{aligned}$$

Hence, for every $x, y \in X$

$$\left\|x, y\right\|^{G} \le \frac{1}{2} \left\|x, y\right\|^{KI}.$$

□

Theorem 2.9. *In X, if there is a $C > 0$ such that $\sup_{\|g\|_{X'} \le 1} g(x) = C \|x\|_X$ for every $x \in X$, then $C\|x, y\|^{KI} = 2\|x, y\|^{G}$ for every $x, y \in X$.*

Proof. Assume that there is a $C > 0$ such that $\sup_{\|g\|_{X'} \le 1} g(x) = C \|x\|_X$ for every $x \in X$. Recall (2), (4), (5) and take $f_1 = f$, $f_2 = g$, $z = x f(y) - y f(x)$. So

$$f_2(x f_1(y) - y f_1(x)) = g(x f(y) - y f(x)) = g(z).$$

By assumption, we obtain $\sup_{\|g\|_{X'} \le 1} g(z) = C\|z\|_X$. Consequently,

$$\begin{aligned}
2\left\|x, y\right\|^{G} &= \sup_{\substack{\|f_1\|_{X'} \le 1,\ \|f_2\|_{X'} \le 1 \\ f_1, f_2 \in X'}} f_2(x f_1(y) - y f_1(x)) \\
&= \sup_{\substack{\|f\|_{X'} \le 1,\ \|g\|_{X'} \le 1 \\ g, f \in X'}} g(x f(y) - y f(x)) \\
&= \sup_{\substack{\|f\|_{X'} \le 1,\ \|g\|_{X'} \le 1 \\ g, f \in X'}} g(z) = C \sup_{\substack{\|f\|_{X'} \le 1 \\ f \in X'}} \|z\|_X \\
&= C \sup_{\substack{\|f\|_{X'} \le 1 \\ f \in X'}} \left\|x f(y) - y f(x)\right\|_X \\
&= C\left\|x, y\right\|^{KI}
\end{aligned}$$

for every $x, y \in X$. Hence, $C\|x, y\|^{KI} = 2\|x, y\|^{G}$ for every $x, y \in X$. □

Theorem 2.10. *In X, if there exist $C_1, C_2 > 0$ such that $C_1\|x\|_X \le \sup_{\|g\|_{X'}\le 1} g(x) \le C_2\|x\|_X$ for every $x \in X$, then the 2-norms $\|\cdot,\cdot\|^{KI}$ and $\|\cdot,\cdot\|^{G}$ are equivalent.*

Proof. Let $C_1, C_2 > 0$ such that $C_1\|x\|_X \le \sup_{\|g\|_{X'}\le 1} g(x) \le C_2\|x\|_X$ for every $x \in X$. As in the proof of the above theorem, if we take $f_1 = f$, $f_2 = g$ and $z = xf(y) - yf(x)$, then

$$f_2(xf_1(y) - yf_1(x)) = g(xf(y) - yf(x)) = g(z).$$

By assumption, we obtain $C_1\|z\|_X \le \sup_{\|g\|_{X'}\le 1} g(z) \le C_2\|z\|_X$. Consequently,

$$\begin{aligned}
2\left\|x,y\right\|^{G} &= \sup_{\substack{\|f_1\|_{X'}\le 1,\ \|f_2\|_{X'}\le 1\\ f_1,f_2\in X'}} f_2(xf_1(y) - yf_1(x))\\
&= \sup_{\substack{\|f\|_{X'}\le 1,\ \|g\|_{X'}\le 1\\ g,f\in X'}} g(xf(y) - yf(x))\\
&= \sup_{\substack{\|f\|_{X'}\le 1,\ \|g\|_{X'}\le 1\\ g,f\in X'}} g(z) \le C_2 \sup_{\substack{\|f\|_{X'}\le 1\\ f\in X'}} \|z\|_X\\
&= C_2 \sup_{\substack{\|f\|_{X'}\le 1\\ f\in X'}} \left\|xf(y) - yf(x)\right\|_X\\
&= C_2\left\|x,y\right\|^{KI}\\
\Rightarrow 2\left\|x,y\right\|^{G} &\le C_2\left\|x,y\right\|^{KI}
\end{aligned}$$

for every $x, y \in X$. We also have

$$\begin{aligned}
2\left\|x,y\right\|^{G} &= \sup_{\substack{\|f_1\|_{X'}\le 1,\ \|f_2\|_{X'}\le 1\\ f_1,f_2\in X'}} f_2(xf_1(y) - yf_1(x))\\
&= \sup_{\substack{\|f\|_{X'}\le 1,\ \|g\|_{X'}\le 1\\ g,f\in X'}} g(xf(y) - yf(x))\\
&= \sup_{\substack{\|f\|_{X'}\le 1,\ \|g\|_{X'}\le 1\\ g,f\in X'}} g(z) \ge C_1 \sup_{\substack{\|f\|_{X'}\le 1\\ f\in X'}} \|z\|_X\\
&= C_1 \sup_{\substack{\|f\|_{X'}\le 1\\ f\in X'}} \left\|xf(y) - yf(x)\right\|_X\\
&= C_1\left\|x,y\right\|^{KI}\\
\Rightarrow 2\left\|x,y\right\|^{G} &\ge C_1\left\|x,y\right\|^{KI}
\end{aligned}$$

for every $x, y \in X$. We conclude that $\|\cdot,\cdot\|^{KI}$ and $\|\cdot,\cdot\|^{G}$ are equivalent. □

3. Concluding remarks

A vector space can be equipped with several 2-norms. In such a case, we may have an equivalence relation between them. In [6], it is shown that all 2-norms on a finite dimensional vector space are equivalent. If X is a 2-dimensional space, say $X := span\{e_1, e_2\}$, and $\|.,.\|_1$, $\|.,.\|_2$ are two 2-norms on X, then one may verify that the two 2-norms are equivalent. In fact, one can show that $\|x,y\|_2 = A\|x,y\|_1$ with $A = \frac{\|e_1,e_2\|_2}{\|e_1,e_2\|_1}$. Indeed, one may verify for $X = \mathbb{R}^2$ that the 2-norms $\|.,.\|^{G}_{\mathbb{R}^2}$ and $\|.,.\|^{KI}_{\mathbb{R}^2}$ are (strongly) equivalent. Recall the usual norm on $\mathbb{R}^2$, $\|x\|_{\mathbb{R}^2} := (|x_1|^p + |x_2|^2)^{\frac{1}{2}}$ for every $x = (x_1, x_2) \in \mathbb{R}^2$. Let $v, w \in \mathbb{R}^2$, for every $x \in \mathbb{R}^2$,

$f_w(x) := \sum_{k=1}^{2} w_k x_k$ is a bounded linear functional on $\mathbb{R}^2$. Then

$$C_2 \left\|x, y\right\|_2^{KI} \leq \left\|x, y\right\|_2^{G} \leq C_1 \left\|x, y\right\|_2^{KI}$$

can be obtained where $C_1 = \frac{\sup_{\|f_v\|,\|f_w\| \leq 1} |v_1 w_2 - v_2 w_1|}{2 \sup_{\|f_w\| \leq 1} (|w_1|^p + |w_2|^p)^{\frac{1}{p}}}$ and $C_2 = \frac{1}{2 \sup_{\|f_u\| \leq 1} (|u_1|^p + |u_2|^p)^{\frac{1}{p}}}$.

On infinite-dimensional vector spaces there is no quarantee that every two 2-norms are equivalent. In this work, we define a new 2-norm $\|.,.\|^{KI}$ equipped with bounded linear functionals on a normed space $(X, \|.\|_X)$ and investigate its relationship with Gähler's [1] 2-norm $\|.,.\|^G$. We also derive a norm $\|.\|^{KI}$ on X from this 2-norm $\|.,.\|^{KI}$ and explore its relation with the norm $\|.\|_X$ on X. We investigate under which conditions the equivalence of these 2-norms can be satisfied. Corollary 2.5 tells us in particular that $\|.\|_X$ is dominated by derived norm $\|.\|^{KI}$. As we see from Lemma 2.4 equivalence of two norms is obtained with respect to the linearly independent vectors $a, b \in X$ such that $\|a\|_X = \|b\|_X$ in normed space $(X, \|\cdot\|_X)$. We have similar difficulties in proving the strong equivalence between the two 2-norms $\|.,.\|^{KI}$ and $\|.,.\|^G$ on X. As a matter of fact, we do not know whether the two 2-norms are strongly equivalent or not unless we examine it in detail for the special cases of X. This ongoing problem will be continued to research for some special cases of X, for example for l^p, L^p $(1 \leq p < \infty)$ and $C[a, b]$. These all remain as open problems to explore for the readers.

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