Filomat 37:9 (2023), 2807–2821
https://doi.org/10.2298/FIL2309807W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a classification of faithful representations of the Galilean Lie algebra on the polynomial ring in three variables

Liang Wu[a], Youjun Tan[a]

[a]*College of Mathematics, Sichuan University, Chengdu 610064, China*

Abstract. We show a complete classification of faithful representations of the 2 + 1 space-times Galilean Lie algebra on the polynomial ring in three variables, where actions of the Galilean Lie algebra are given by derivations with coefficients of degree at most one. In particular, all such representations of the Galilean Lie algebra are explicitly constructed and classified by one parameter. In a more general setting we show that, with respect to a nonzero abelian ideal of a finite-dimensional Lie algebra, there is at most one such representation up to graded-equivalence.

1. Introduction

Construction and classification of infinite-dimensional representations of Lie algebras are far from being well understood. Lie algebras of vector fields have naturally infinite-dimensional representations on spaces of smooth functions, which have been studied extensively in literature, see, for example, [3][4][5][6][8]. Motivated by the classical representations of the 2 + 1 space-times Galilean Lie algebra $\mathfrak{G}$ [1] (see Example 4.3 below for convenience), we investigate faithful representations of Lie algebras on polynomial rings, with an attempt to deduce some classification results.

Representations of a Lie algebra on a polynomial ring involves Lie algebras consisting of differential operators. Presumably Lie algebras consisting of derivations have relatively simpler structure than those consisting of differential operators of higher orders, and hence it seems more practical to study a Lie algebra's representations given by derivations. Since coefficients of derivations may have higher degrees, it still maybe very difficult to study such representations. So, at present we restrict ourself to representations given by derivations with coefficients of degree at most 1.

For brevity we denote the polynomial ring $\mathbb{F}[x_1, \cdots, x_n]$ over a field $\mathbb{F}$ by $\mathbb{F}[\mathrm{X}_n]$, and denote by $\mathrm{der}_{\leq 1}(\mathbb{F}[\mathrm{X}_n])$ the Lie algebra of derivations on $\mathbb{F}[\mathrm{X}_n]$ with coefficients of degree at most 1. It turns out that the Lie brackets in $\mathrm{der}_{\leq 1}(\mathbb{F}[\mathrm{X}_n])$ can be realized as commutators of matrices (see Lemma 2.1 below), which would simplify some computations. For example, by identifying $\mathrm{der}_{\leq 1}(\mathbb{F}[\mathrm{X}_n])$ with the Lie subalgebra $L_{n+1}(\mathbb{F})$ (see (8)) of the general linear Lie algebra $\mathfrak{gl}_{n+1}(\mathbb{F})$ we can prove easily a result on commutative Lie subalgebras of $\mathrm{der}_{\leq 1}(\mathbb{F}[\mathrm{X}_n])$ (see Proposition 2.3 below), which plays a key role to deduce the main results of this paper. Moreover, such an identification implies directly existence of some faithful representations of finite-dimensional Lie algebras on polynomial rings (see Proposition 3.1 below), which means that any

2020 *Mathematics Subject Classification*. Primary 17B10; Secondary 17B66, 16W25
Keywords. Lie algebras, derivations, polynomial rings, the Galilean Lie algebra.
Received: 25 April 2022; Accepted: 07 September 2022
Communicated by Dijana Mosić
Email addresses: wuliang2468@163.com (Liang Wu), ytan@scu.edu.cn (Youjun Tan)

finite-dimensional Lie algebra can be realized as a subalgebra consisting of derivations on a polynomial ring.

Given a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, we try to deduce some classification results on representations of $\mathfrak{g}$ on $\mathbb{F}[X_n]$ of the form $\mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$. Let $\mathrm{der}_1(\mathbb{F}[X_n])$ be the Lie subalgebra of $\mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ consisting of 0 and derivations with homogenous coefficients of degree 1. We show that, classification of representations of the form $\mathfrak{g} \to \mathrm{der}_1(\mathbb{F}[X_n])$, which maybe called homogeneous representations, is equivalent to classification of finite $\mathfrak{g}$-modules (see Proposition 3.3 below).

Then we consider faithful representations of the form ρ: $\mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ with $\rho(\mathfrak{g}) \cap \mathrm{der}_0(\mathbb{F}[X_n]) \neq 0$, where $\mathrm{der}_0(\mathbb{F}[X_n])$ is the space of derivations of $\mathbb{F}[X_n]$ with constant coefficients. Such representations are non-homogeneous. Note that $\mathrm{der}_0(\mathbb{F}[X_n])$ is an abelian ideal of $\mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$. Since any faithful representation of the form $\mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ has close relation to abelian ideals of $\mathfrak{g}$ (see Proposition 16 below), we shall consider representations of the form ρ: $\mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ satisfying that $\rho(\mathfrak{h}) = \mathrm{der}_0(\mathbb{F}[X_n])$, where $\mathfrak{h}$ is an abelian ideal of $\mathfrak{g}$. As an attempt to classify these representations we introduce the notion of graded-equivalence (Definition 3.5). One of the main results is Theorem 3.7, which states that, given an n-dimensional abelian ideal $\mathfrak{h}$ of $\mathfrak{g}$ ($\mathfrak{g}$ is assumed to be finite-dimensional), there is at most one faithful representation ρ: $\mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ up to graded-equivalence such that $\rho(\mathfrak{h}) = \mathrm{der}_0(\mathbb{F}[X_n])$.

Since Theorem 3.7 concerns abelian ideals we consider solvable Lie algebras to illustrate its applications. There is no general classification result on solvable Lie algebras. We take as an example the $2+1$ space-times Galilean Lie algebra $\mathfrak{G}$, which originated from classical mechanics. See [1][2] for the physical background of $\mathfrak{G}$ and its classical representation. First, we explain that $\mathfrak{G}$ has no faithful representations of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_n])$ for $n = 1, 2$ and of the form $\mathfrak{G} \to \mathrm{der}_1(\mathbb{R}[X_3])$ (see Proposition 4.2 below), which leads us to consider faithful representations of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$. Second, by applying Theorem 3.7 and Lemma 4.6, which states that any faithful representation of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ maps bijectively the 3-dimensional abelian ideal $\mathfrak{H}$ given by (33) to $\mathrm{der}_0(\mathbb{R}[X_3])$, we obtain that any faithful representation of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_3])$ is graded-equivalent to its classical representation (Corollary 4.7). Thirdly, we obtain the other main result which, in terms of equivalence in the usual sense, completely classifies all faithful representations of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_3])$. For details see Theorem 4.10 below. In particular, all such representations of $\mathfrak{G}$ are explicitly given and classified by one parameter.

The paper is organized as follows. In Section 2 we discuss some properties of the Lie algebra $\mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$. In Section 3 we discuss some properties of representations of Lie algebras on polynomial rings and prove Theorem 3.7. In Section 4 we discuss faithful representations of the Galilean algebra $\mathfrak{G}$ on $\mathbb{R}[X_3]$, and give a detailed proof of Theorem 4.10.

2. Derivations with coefficients of degree at most 1

Any derivation d of $\mathbb{F}[X_n]$ has the form $\mathrm{d} = \sum_{i=1}^n d_i \partial_{x_i}$, where $d_i = \mathrm{d}(x_i) \in \mathbb{F}[X_n]$ and $\partial_{x_i} = \partial/\partial x_i$. Let $\mathrm{der}(\mathbb{F}[X_n])$ be the Lie algebra of derivations of $\mathbb{F}[X_n]$, whose Lie bracket is the usual commutator given by

$$[\mathrm{d}_1, \mathrm{d}_2] = \sum_{i=1}^n (\mathrm{d}_1(d_{2,i}) - \mathrm{d}_2(d_{1,i}))\partial_{x_i}, \tag{1}$$

where $\mathrm{d}_1 = \sum_{i=1}^n d_{1,i}\partial_{x_i}$, $\mathrm{d}_2 = \sum_{i=1}^n d_{2,i}\partial_{x_i} \in \mathrm{der}(\mathbb{F}[X_n])$. We shall use the following Lie subalgebras of $\mathrm{der}(\mathbb{F}[X_n])$ given by

$$\mathrm{der}_{\leq 1}(\mathbb{F}[X_n]) := \{\mathrm{d} = \sum_{i=1}^n d_i\partial_{x_i} : \deg(d_i) \leq 1\}, \tag{2}$$

$$\mathrm{der}_1(\mathbb{F}[X_n]) := \{\mathrm{d} = \sum_{i=1}^n d_i\partial_{x_i} : d_i \text{ is homogenous with } \deg(d_i) = 1\} \cup \{0\}, \tag{3}$$

$$\mathrm{der}_0(\mathbb{F}[X_n]) := \{\mathrm{d} = \sum_{i=1}^n d_i\partial_{x_i} : d_i \text{ is a constant}\}. \tag{4}$$

By (1), $\mathrm{der}_{\le 1}(\mathbb{F}[X_n])$ and $\mathrm{der}_1(\mathbb{F}[X_n])$ are Lie subalgebras of $\mathrm{der}(\mathbb{F}[X_n])$, and $\mathrm{der}_0(\mathbb{F}[X_n])$ is an abelian ideal of $\mathrm{der}_{\le 1}(\mathbb{F}[X_n])$. Note that

$$\dim \mathrm{der}_{\le 1}(\mathbb{F}[X_n]) = n^2 + n, \ \ \dim \mathrm{der}_1(\mathbb{F}[X_n]) = n^2, \ \ \dim \mathrm{der}_0(\mathbb{F}[X_n]) = n. \tag{5}$$

The canonical projection $\varpi_n : \ \mathrm{der}_{\le 1}(\mathbb{F}[X_n]) \to \mathrm{der}_1(\mathbb{F}[X_n])$ given by

$$\varpi_n\left(\sum_{i=1}^{n} d_i\partial_{x_i}\right) = \sum_{i=1}^{n}(d_i(x_1,\cdots,x_n) - d_i(0,\cdots,0))\partial_{x_i} \tag{6}$$

is a Lie algebra homomorphism. Let $\mathbb{F}^n$ be the space of n-dimensional *row* vectors on $\mathbb{F}$. Regard $\mathbb{F}^n$ as an abelian Lie algebra. Then, $\mathrm{der}_0(\mathbb{F}[X_n])$ is isomorphic to $\mathbb{F}^n$ via

$$\mathrm{can.}: \ \ \mathrm{der}_0(\mathbb{F}[X_n]) \ni \mathrm{d} = \sum_{i=1}^{n} d_i\partial_{x_i} \mapsto (d_1,\cdots,d_n) \in \mathbb{F}^n. \tag{7}$$

An important property of $\mathrm{der}_{\le 1}(\mathbb{F}[X_n])$ is that its Lie bracket given by (1) can be computed via matrices. More precisely, $\mathrm{der}_{\le 1}(\mathbb{F}[X_n])$ can be identified with a Lie subalgebra of the general linear Lie algebra $\mathfrak{gl}_{n+1}(\mathbb{F})$ in the following way. Consider the following Lie subalgebra

$$L_{n+1}(\mathbb{F}) := \left\{\begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix} : \ A \in \mathfrak{gl}_n(\mathbb{F}), \ \alpha \in \mathbb{F}^n\right\} \tag{8}$$

of $\mathfrak{gl}_{n+1}(\mathbb{F})$. Note that $\mathfrak{gl}_n(\mathbb{F})$ becomes a Lie subalgebra of $L_{n+1}(\mathbb{F})$ via the canonical embedding $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, and the canonical projection

$$\pi_n : \ L_{n+1}(\mathbb{F}) \to \mathfrak{gl}_n(\mathbb{F}) : \ \begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix} \mapsto A \tag{9}$$

is a Lie algebra homomorphism. Moreover, the canonical embedding $\mathbb{F}^n \hookrightarrow L_{n+1}(\mathbb{F})$ given by $\alpha \mapsto \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$ makes $\mathbb{F}^n$ an abelian ideal of $L_{n+1}(\mathbb{F})$. We think the following result should be known, but we did not find a reference in literature at present.

Lemma 2.1. *The mapping given by*

$$\sigma_n : \ x_i\partial_{x_j} \mapsto E_{ij}, \ \partial_{x_k} \mapsto E_{n+1,k}, \ 1 \le i,j,k \le n, \tag{10}$$

is a Lie algebra isomorphism from $\mathrm{der}_{\le 1}(\mathbb{F}[X_n])$ *to* $L_{n+1}(\mathbb{F})$, *where* E_{pq} *is the basic matrix defined by* $E_{pq}(r,s) = \delta_{pr}\delta_{qs}$, *and the restriction* $\overline{\sigma_n}$ *of* σ_n *to* $\mathrm{der}_1(\mathbb{F}[X_n])$ *is a Lie algebra isomorphism from* $\mathrm{der}_1(\mathbb{F}[X_n])$ *to* $\mathfrak{gl}_n(\mathbb{F})$.

Proof. Since $x_i\partial_{x_j}, \partial_{x_k}$ $(1 \le i,j,k \le n)$ form a basis of $\mathrm{der}_{\le 1}(\mathbb{F}[X_n])$, σ_n is a well-defined bijection. Moreover, due to

$$[x_i\partial_{x_j}, x_{i'}\partial_{x_{j'}}] = \delta_{ji'}x_i\partial_{x_{j'}} - \delta_{ij'}x_{i'}\partial_{x_j}, \ [x_i\partial_{x_j}, \partial_{x_k}] = -\delta_{ki}\partial_{x_j}, \ [\partial_{x_k}, \partial_{x_{k'}}] = 0$$

and commutators between basic matrices of $\mathfrak{gl}_{n+1}(\mathbb{F})$, σ_n is a Lie algebra homomorphism. □

So, we have the following commutative diagram of Lie algebras.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{der}_0(\mathbb{F}[X_n]) & \longrightarrow & \mathrm{der}_{\le 1}(\mathbb{F}[X_n]) & \xrightarrow{\varpi_n} & \mathrm{der}_1(\mathbb{F}[X_n]) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{can.}} & & \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \overline{\sigma_n}} & & \\ 0 & \longrightarrow & \mathbb{F}^n & \longrightarrow & L_{n+1}(\mathbb{F}) & \xrightarrow{\pi_n} & \mathfrak{gl}_n(\mathbb{F}) & \longrightarrow & 0 \end{array} \tag{11}$$

To establish equivalence of representations of Lie algebras on $\mathbb{F}[\mathrm{X}_n]$ we shall use some $\mathbb{F}$-automorphisms of $\mathbb{F}[\mathrm{X}_n]$ given by invertible matrices. For any $P \in \mathrm{GL}_n(\mathbb{F})$ define the $\mathbb{F}$-automorphism τ_P of $\mathbb{F}[\mathrm{X}_n]$ by

$$\tau_P(f(\mathrm{X})) = f(\mathrm{X}P), \quad \mathrm{X} = (x_1, \cdots, x_n). \tag{12}$$

In particular, $\tau_P(x_k) = \sum_{t=1}^n P(t,k)x_t$. It's direct that τ_P is also an automorphism of the commutative algebra $\mathbb{F}[\mathrm{X}_n]$. Moreover, we have

$$\tau_P(\partial_{x_i} f(\mathrm{X})) \overset{\mathrm{Y}=\mathrm{X}P}{=} \partial_{y_i}(f(\mathrm{Y})), \tag{13}$$

where $\mathrm{Y} = (y_1, \cdots, y_n) = \mathrm{X}P$ means that $y_k = \tau_P(x_k)$, $1 \le k \le n$. Note that $\partial y_k/\partial x_j = P(j,k)$. We shall use the following result.

Lemma 2.2. *Let $A \in \mathfrak{gl}_n(\mathbb{F})$ and $P \in \mathrm{GL}_n(\mathbb{F})$. Then, as $\mathbb{F}$-linear transformations on $\mathbb{F}[\mathrm{X}_n]$ it holds that*

$$\tau_P\overline{\sigma_n}^{-1}(A) = \overline{\sigma_n}^{-1}(PAP^{-1})\tau_P, \tag{14}$$

where $\overline{\sigma_n}^{-1}$: $\mathfrak{gl}_n(\mathbb{F}) \to \mathrm{der}_1(\mathbb{F}[\mathrm{X}_n])$ is given by Lemma 2.1.

Proof. Fix any $f(\mathrm{X}) \in \mathbb{F}[\mathrm{X}_n]$. Then we have

$$\begin{aligned}
(\overline{\sigma_n}^{-1}(PAP^{-1})\tau_P)(f(\mathrm{X})) = \overline{\sigma_n}^{-1}(PAP^{-1})(f(\mathrm{X}P)) &\overset{\text{Lemma 2.1}}{=} \sum_{i,j=1}^n (PAP^{-1})(i,j)x_i\partial_{x_j}(f(\mathrm{X}P)) \\
= \sum_{i,j=1}^n \left(\sum_{k,\ell=1}^n P(i,k)A(k,\ell)P^{-1}(\ell,j)\right)x_i\partial_{x_j}(f(\mathrm{X}P)) &= \sum_{k,\ell=1}^n A(k,\ell)\left(\sum_{i=1}^n P(i,k)x_i\right)\left(\sum_{j=1}^n P^{-1}(\ell,j)\partial_{x_j} f(\mathrm{X}P)\right) \\
\overset{\mathrm{Y}=\mathrm{X}P}{=} \sum_{k,\ell=1}^n A(k,\ell)\left(\sum_{i=1}^n P(i,k)x_i\right)\left(\sum_{j=1}^n P^{-1}(\ell,j)\sum_{t=1}^n \partial_{y_t} f(\mathrm{Y})\frac{\partial y_t}{\partial x_j}\right) \\
= \sum_{k,\ell=1}^n A(k,\ell)\left(\sum_{i=1}^n P(i,k)x_i\right)\left(\sum_{t=1}^n \left(\sum_{j=1}^n (P^{-1}(\ell,j)P(j,t))\right)\partial_{y_t} f(\mathrm{Y})\right) &= \sum_{k,\ell=1}^n A(k,\ell)y_k\partial_{y_\ell}(f(\mathrm{Y}))
\end{aligned}$$

and

$$\begin{aligned}
\tau_P(\overline{\sigma_n}^{-1}(A)(f(\mathrm{X}))) &= \tau_P\left(\sum_{k,\ell=1}^n A(k,\ell)x_k\partial_{x_\ell}(f(\mathrm{X}))\right) = \sum_{k,\ell=1}^n A(k,\ell)\tau_P(x_k\partial_{x_\ell}(f(\mathrm{X}))) \\
&\overset{(13),\mathrm{Y}=\mathrm{X}P}{=} \sum_{k,\ell=1}^n A(k,\ell)(y_k\partial_{y_\ell}(f(\mathrm{Y}))),
\end{aligned}$$

and hence the proof is completed. □

We shall use the following result on commutative subalgebras of $\mathrm{der}_{\le 1}(\mathbb{F}[\mathrm{X}_n])$.

Proposition 2.3. *Assume that $\mathcal{A}$ is a commutative subalgebra of $\mathrm{der}_{\le 1}(\mathbb{F}[\mathrm{X}_n])$ with $\dim \mathcal{A} > \left[n^2/4\right] + 1$. Then $\mathcal{A} \cap \mathrm{der}_0(\mathbb{F}[\mathrm{X}_n]) \neq 0$.*

Proof. By Lemma 2.1, $\sigma_n(\mathcal{A})$ is a commutative subalgebra of $L_{n+1}(\mathbb{F})$. Let $\{\mathrm{d}_k\}$ be a basis of $\mathcal{A}$. Then $\{\sigma_n(\mathrm{d}_k)\}$ is a basis of $\sigma_n(\mathcal{A})$. Write

$$\sigma_n(\mathrm{d}_k) = \begin{pmatrix} A_k & 0 \\ \alpha_k & 0 \end{pmatrix} \in L_{n+1}(\mathbb{F}),\ A_k \in \mathfrak{gl}_n(\mathbb{F}),\ \alpha_k \in \mathbb{F}^n,\ \ 1 \le k \le \dim \mathcal{A}. \tag{15}$$

Then $\operatorname{span}_{\mathbb{F}}\{A_k\}$ is a commutative subalgebra of $\mathfrak{gl}_n(\mathbb{F})$, and hence $\dim \operatorname{span}_{\mathbb{F}}\{A_k\} \le \left[n^2/4\right]+1$ due to Schur's theorem [7]. It follows that $A_1, \cdots, A_{\dim \mathcal{A}}$ are linearly dependent since $\dim \mathcal{A} > \left[n^2/4\right]+1$. So, we may choose $a_k \in \mathbb{F}$ $(1 \le k \le \dim \mathcal{A})$, at least one of which is nonzero, such that $\sum_{k=1}^{\dim \mathcal{A}} a_k A_k = 0$. Therefore,

$$0 \neq \sigma_n\left(\sum_{k=1}^{\dim \mathcal{A}} a_k \mathrm{d}_k\right) = \sum_{k=1}^{\dim \mathcal{A}} a_k \sigma_n(\mathrm{d}_k) = \begin{pmatrix} 0 & 0 \\ \sum_{k=1}^{\dim \mathcal{A}} a_k \alpha_k & 0 \end{pmatrix} \in \sigma_n(\mathcal{A}),$$

which implies that $\sum_{k=1}^{\dim \mathcal{A}} a_k \mathrm{d}_k \in \operatorname{der}_0(\mathbb{F}[\mathrm{X}_n]) \cap \mathcal{A}$. Moreover, $\sum_{k=1}^{\dim \mathcal{A}} a_k \mathrm{d}_k$ is nonzero since $\mathrm{d}_1, \cdots, \mathrm{d}_{\dim \mathcal{A}}$ are linearly independent. □

3. Properties of representations of Lie algebras given by derivations with coefficients at most one

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. We shall consider representations of $\mathfrak{g}$ on the polynomial ring $\mathbb{F}[\mathrm{X}_n]$ of the form $\rho\colon \mathfrak{g} \to \operatorname{der}_{\le 1}(\mathbb{F}[\mathrm{X}_n])$. We collect some basic facts of such representations as follows.

Proposition 3.1. *Assume that $\mathfrak{g}$ is finite-dimensional. Then, there is an integer m and a faithful representation $\mathfrak{g} \hookrightarrow \operatorname{der}_1(\mathbb{F}[\mathrm{X}_m])$ of $\mathfrak{g}$ on $\mathbb{F}[\mathrm{X}_m]$.*

Proof. By Ado's Theorem there is an embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}_m(\mathbb{F})$ as Lie algebras for some m, and hence there is an injective Lie algebra homomorphism from $\mathfrak{g}$ to $\operatorname{der}_1(\mathbb{F}[\mathrm{X}_m])$ due to Lemma 2.1. □

Recall that $\mathbb{F}[\mathrm{X}_n]$ admits the following filtration given by

$$\mathbb{F} \subset \mathbb{F}[\mathrm{X}_n]_1 \subset \cdots \subset \mathbb{F}[\mathrm{X}_n]_k \subset \cdots, \tag{16}$$

where $\mathbb{F}[\mathrm{X}_n]_k = \{f \in \mathbb{F}[\mathrm{X}_n] : \ \deg(f) \le k\}$, $k = 0, 1, 2, \cdots$.

Proposition 3.2. *Let $\rho\colon \mathfrak{g} \to \operatorname{der}_{\le 1}(\mathbb{F}[\mathrm{X}_n])$ be a representation of $\mathfrak{g}$ on $\mathbb{F}[\mathrm{X}_n]$.*

(i) *For each $k = 0, 1, 2, \cdots$, $\mathbb{F}[\mathrm{X}_n]_k$ becomes a finite-dimensional $\mathfrak{g}$-module via actions of $\rho(\mathrm{g}) \in \operatorname{der}_{\le 1}(\mathbb{F}[\mathrm{X}_n])$ for any $\mathrm{g} \in \mathfrak{g}$.*

(ii) *ρ is uniquely determined by actions of $\rho(\mathrm{g}) \in \operatorname{der}_{\le 1}(\mathbb{F}[\mathrm{X}_n])$ $(\mathrm{g} \in \mathfrak{g})$ on*

$$\mathbb{F}_1[\mathrm{X}_n] = \mathbb{F} \oplus \operatorname{span}_{\mathbb{F}}\{x_1, \cdots, x_n\}.$$

(iii) *$\rho(\ker(\varpi_n \rho)) \subseteq \operatorname{der}_0(\mathbb{F}[X_n])$, where $\varpi_n\colon \operatorname{der}_{\le 1}(\mathbb{F}[\mathrm{X}_n]) \to \operatorname{der}_1(\mathbb{F}[\mathrm{X}_n])$ is the Lie algebra homomorphism given by (6). If ρ is faithful then $\ker(\varpi_n \rho)$ is an abelian ideal of $\mathfrak{g}$.*

Proof. (i) is direct by definitions. (ii) follows since for any $\mathrm{g} \in \mathfrak{g}$ it holds that $\rho(\mathrm{g}) = \sum_{i=1}^{n} \rho(\mathrm{g})(x_i)\partial_{x_i}$, $1 \le i \le n$.

(iii) For any $\mathrm{g} \in \ker(\varpi_n \rho)$, by $\varpi_n(\rho(\mathrm{g})) = 0$ and $\ker \varpi_n = \operatorname{der}_0(\mathbb{F}[\mathrm{X}_n])$ it follows that $\rho(\mathrm{g}) \in \operatorname{der}_0(\mathbb{F}[\mathrm{X}_n])$. Assume that ρ is faithful. For any $\mathrm{g}_1, \mathrm{g}_2 \in \ker(\varpi_n \rho)$, since $\operatorname{der}_0(\mathbb{F}[X_n])$ is abelian, $\rho([\mathrm{g}_1, \mathrm{g}_2]) = [\rho(\mathrm{g}_1), \rho(\mathrm{g}_2)] = 0$, and hence $[\mathrm{g}_1, \mathrm{g}_2] = 0$ since $\ker \rho = 0$. □

The category of finite-dimensional $\mathfrak{g}$-modules can be identified with a full subcategory of the category of representations of $\mathfrak{g}$ of the form $\rho\colon \mathfrak{g} \to \operatorname{der}_1(\mathbb{F}[\mathrm{X}_n])$. In fact, we have the following statement, where $\mathfrak{g}$ is not necessarily finite-dimensional.

Proposition 3.3. *Let M be an n-dimensional $\mathfrak{g}$-module. Then there is a representation of $\mathfrak{g}$ on $\mathbb{F}[\mathrm{X}_n]$ of the form $\rho\colon \mathfrak{g} \to \operatorname{der}_1(\mathbb{F}[\mathrm{X}_n])$ such that $M \cong \operatorname{span}_{\mathbb{F}}\{x_1, \cdots, x_n\}$ as $\mathfrak{g}$-modules. Moreover, such representations of $\mathfrak{g}$ on polynomial ring of n variables are uniquely determined up to equivalence.*

Proof. (*Existence.*) Let $\rho_0 : \mathfrak{g} \to \mathfrak{gl}(M)$ be the $\mathfrak{g}$-module structure on M, and $\mathcal{S}(M)$ the symmetric algebra over M. Then $\mathcal{S}(M)$ becomes a $\mathfrak{g}$-module with its module structure, denoted again by ρ_0, given by $\rho_0(\mathrm{g})(uv) = \rho_0(\mathrm{g})(u)v + u\rho_0(\mathrm{g})(v)$, $\mathrm{g} \in \mathfrak{g}$, $u, v \in \mathcal{S}(M)$.

Fix a basis $\{v_1, \cdots, v_n\}$ of M. Then there is an associative algebra isomorphism σ: $\mathcal{S}(M) \to \mathbb{F}[\mathrm{X}_n]$ given by $v_i \mapsto x_i$, $1 \le i \le n$. For any $\mathrm{g} \in \mathfrak{g}$ set

$$\rho(\mathrm{g}) := \sigma\rho_0(\mathrm{g})\sigma^{-1} : \ \mathbb{F}[\mathrm{X}_n] \to \mathbb{F}[\mathrm{X}_n]. \tag{17}$$

By a direct check we get that $\rho(\mathrm{g}) \in \mathrm{der}(\mathbb{F}[\mathrm{X}_n])$, and the mapping ρ: $\mathfrak{g} \to \mathrm{der}(\mathbb{F}[\mathrm{X}_n])$ given by (17) is a Lie algebra homomorphism. Moreover, for any $\mathrm{g} \in \mathfrak{g}$, we may write $\rho(\mathrm{g}) = \sum\limits_{i=1}^{n} d_i\partial_{x_i}$ with $d_i \in \mathbb{F}[\mathrm{X}_n]$, and hence

$$d_j = \rho(\mathrm{g})(x_j) = \sigma(\rho_0(\mathrm{g})\sigma^{-1}(x_j)) = \sigma(\rho_0(\mathrm{g})(v_j)) \in \mathrm{span}_{\mathbb{F}}\{x_1, \cdots, x_n\}, \ 1 \le j \le n,$$

which means that

$$\rho(\mathrm{g}) = \sum_{i=1}^{n} \sigma(\rho_0(\mathrm{g})(v_i))\partial_{x_i} \in \mathrm{der}_1(\mathbb{F}[\mathrm{X}_n]), \ \ \mathrm{g} \in \mathfrak{g}. \tag{18}$$

Since $\sigma(M) = \mathrm{span}_{\mathbb{F}}\{x_1, \cdots, x_n\}$ and $\rho_0(\mathrm{g})(M) \subseteq M$, the existence part is proved.

(*Uniqueness.*) Let ρ: $\mathfrak{g} \to \mathrm{der}_1(\mathbb{F}[\mathrm{X}_n])$ and ρ': $\mathfrak{g} \to \mathrm{der}_1(\mathbb{F}[\mathrm{Y}_n])$ be representations of $\mathfrak{g}$ such that $M \cong \mathrm{span}_{\mathbb{F}}\{x_1, \cdots, x_n\} \cong \mathrm{span}_{\mathbb{F}}\{y_1, \cdots, y_n\}$ as $\mathfrak{g}$-modules, where $\mathbb{F}[\mathrm{Y}_n]$ is the polynomial ring of $y_1, \cdots, y_n$ and $\mathrm{der}_1(\mathbb{F}[\mathrm{Y}_n])$ is given by (3). Then there is an invertible matrix $P \in \mathrm{GL}_n(\mathbb{F})$ such that the $\mathbb{F}$-isomorphism $(x_1, \cdots, x_n) \mapsto (y_1, \cdots, y_n)P$ is a $\mathfrak{g}$-module isomorphism from $\mathrm{span}_{\mathbb{F}}\{x_1, \cdots, x_n\}$ to $\mathrm{span}_{\mathbb{F}}\{y_1, \cdots, y_n\}$. For any $\mathrm{g} \in \mathfrak{g}$, since $\rho(\mathrm{g}) \in \mathrm{der}_1(\mathbb{F}[\mathrm{X}_n])$ and $\rho'(\mathrm{g}) \in \mathrm{der}_1(\mathbb{F}[\mathrm{Y}_n])$, by Lemma 2.1 we get matrices $A_{\mathrm{g}} = \overline{\sigma_n}(\rho(\mathrm{g}))$ and $B_{\mathrm{g}} = \overline{\sigma_n}(\rho'(\mathrm{g}))$ in $\mathfrak{gl}_n(\mathbb{F})$ satisfying that

$$\begin{cases} PA_{\mathrm{g}} = B_{\mathrm{g}}P, \\ (\rho(\mathrm{g})(x_1), \cdots, \rho(\mathrm{g})(x_n)) = (x_1, \cdots, x_n)A_{\mathrm{g}}, \ \ (\rho'(\mathrm{g})(y_1), \cdots, \rho'(\mathrm{g})(y_n)) = (y_1, \cdots, y_n)B_{\mathrm{g}}. \end{cases} \tag{19}$$

Denote by χ_P the isomorphism of commutative algebras from $\mathbb{F}[\mathrm{X}_n]$ to $\mathbb{F}[\mathrm{Y}_n]$ determined uniquely by

$$\chi_P(x_i) := \sum_{j=1}^{n} P(j, i)y_j, \ \ 1 \le i \le n. \tag{20}$$

Similar to (13), it holds that

$$\chi_P(\partial_{x_i}(f(x_1 \cdots, x_n))) = \frac{\partial f(\chi_P(x_1), \cdots, \chi_P(x_n))}{\partial(\chi_P(x_i))}, \ \ 1 \le i \le n. \tag{21}$$

It remains to check that χ_P is a $\mathfrak{g}$-module homomorphism. Recall that

$$\rho(\mathrm{g}) = \sum_{i=1}^{n} \rho(\mathrm{g})(x_i)\partial_{x_i}, \ \ \rho'(\mathrm{g}) = \sum_{j=1}^{n} \rho(\mathrm{g})(y_j)\partial_{y_j}, \ \ \mathrm{g} \in \mathfrak{g}. \tag{22}$$

Fix any $f(x_1 \cdots, x_n) \in \mathbb{F}[\mathrm{X}_n]$. Then, for any $\mathrm{g} \in \mathfrak{g}$ we have

$$\begin{aligned}
\rho'(\mathrm{g})(\chi_P(f(x_1 \cdots, x_n))) &\overset{(22)}{=} \sum_{j=1}^{n} \rho'(\mathrm{g})(y_j)\partial_{y_j}(f(\chi_P(x_1), \cdots, \chi_P(x_n))) \\
&= \sum_{j=1}^{n} \rho'(\mathrm{g})(y_j) \sum_{i=1}^{n} \frac{\partial f(\chi_P(x_1), \cdots, \chi_P(x_n))}{\partial(\chi_P(x_i))} \cdot \frac{\partial \chi_P(x_i)}{\partial y_j} \\
&\overset{(20),(21)}{=} \sum_{i,j=1}^{n} \rho'(\mathrm{g})(y_j) P(j,i) \chi_P(\partial_{x_i}(f(x_1 \cdots, x_n))) \\
&\overset{(19)}{=} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} B_{\mathrm{g}}(k,j) P(j,i) y_k \chi_P(\partial_{x_i}(f(x_1 \cdots, x_n))) \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} (B_{\mathrm{g}}P)(k,i) y_k \chi_P(\partial_{x_i}(f(x_1 \cdots, x_n))).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\chi_P(\rho(\mathrm{g}))(f(x_1 \cdots, x_n)) &\overset{(22)}{=} \chi_P\left(\sum_{i=1}^{n} \rho(\mathrm{g})(x_i)\partial_{x_i}(f(x_1 \cdots, x_n))\right) \\
&\overset{(19)}{=} \sum_{\ell=1}^{n} \sum_{i=1}^{n} A_{\mathrm{g}}(\ell, i) \chi_P(x_\ell) \chi_P(\partial_{x_i}(f(x_1 \cdots, x_n))) \\
&\overset{(20)}{=} \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{\ell=1}^{n} P(k,\ell) A_{\mathrm{g}}(\ell, i) y_k \chi_P(\partial_{x_i}(f(x_1 \cdots, x_n))) \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} (PA_{\mathrm{g}})(k,i) y_k \chi_P(\partial_{x_i}(f(x_1 \cdots, x_n))).
\end{aligned}$$

So, by $PA_{\mathrm{g}} = B_{\mathrm{g}}P$ we get $\chi_P(\rho(\mathrm{g})) = \rho'(\mathrm{g})\chi_P$ as required. □

Example 3.4. *Consider the 3-dimensional complex simple Lie algebra* $\mathfrak{sl}_2(\mathbb{C})$ *with standard basis* $\{\mathrm{e}, \mathrm{h}, \mathrm{f}\}$, *i.e.,* $[\mathrm{e}, \mathrm{f}] = \mathrm{h}$, $[\mathrm{h}, \mathrm{e}] = 2\mathrm{e}$, $[\mathrm{h}, \mathrm{f}] = -2\mathrm{f}$. *Let* $M = \{v_1, v_2\}$ *be the canonical* 2*-dimensional irreducible module of* $\mathfrak{sl}_2(\mathbb{C})$. *The mapping* ρ: $\mathfrak{sl}_2(\mathbb{C}) \to \mathrm{der}_1(\mathbb{C}[x_1, x_2])$ *given by Proposition 3.3 is (see (18))*

$$\rho(\mathrm{e}) = x_1\partial_{x_2}, \quad \rho(\mathrm{h}) = x_1\partial_{x_1} - x_2\partial_{x_2}, \quad \rho(\mathrm{f}) = x_2\partial_{x_1}, \tag{23}$$

which is a well-known faithful representation of $\mathfrak{sl}_2(\mathbb{C})$ *on* $\mathbb{C}[x_1, x_2]$ *(see [9, Exercise 4, §7]). In particular,* $M \cong \mathrm{span}_{\mathbb{C}}\{x_1, x_2\}$. □

Generally, in terms of equivalence in the usual sense it is difficult to classify representations of the form $\mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[\mathrm{X}_n])$ on $\mathbb{F}[\mathrm{X}_n]$. Given such a representation ρ: $\mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[\mathrm{X}_n])$, by Proposition 3.2 we get a graded $\mathfrak{g}$-module $\oplus_{k=1}^{\infty} \mathbb{F}[\mathrm{X}_n]_k / \mathbb{F}[\mathrm{X}_n]_{k-1}$. For brevity we make the following definition.

Definition 3.5. *Let* ρ, $\tilde{\rho}$: $\mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathrm{F}[\mathrm{X}_n])$ *be representations of* $\mathfrak{g}$ *on* $\mathbb{F}[\mathrm{X}_n]$. *If their induced graded modules are isomorphic as graded modules, then* ρ *and* $\tilde{\rho}$ *are called graded-equivalent.*

In other words, ρ and $\tilde{\rho}$ are called graded-equivalent if for each $k \geq 1$ there is an $\mathbb{F}$-automorphism τ_k of $\mathbb{F}[\mathrm{X}_n]_k / \mathbb{F}[\mathrm{X}_n]_{k-1}$ such that the diagram

$$\begin{array}{ccc}
\mathbb{F}[\mathrm{X}_n]_k / \mathbb{F}[\mathrm{X}_n]_{k-1} & \xrightarrow{\tau_k} & \mathbb{F}[\mathrm{X}_n]_k / \mathbb{F}[\mathrm{X}_n]_{k-1} \\
\rho(\mathrm{g}) \downarrow & & \downarrow \tilde{\rho}(\mathrm{g}) \\
\mathbb{F}[\mathrm{X}_n]_k / \mathbb{F}[\mathrm{X}_n]_{k-1} & \xrightarrow{\tau_k} & \mathbb{F}[\mathrm{X}_n]_k / \mathbb{F}[\mathrm{X}_n]_{k-1}
\end{array}$$

commutes for any $\mathfrak{g} \in \mathfrak{g}$. Moreover, if $\rho, \tilde{\rho}\colon \mathfrak{g} \to \mathrm{der}_1(\mathbb{F}[X_n]) \subset \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ are representations of $\mathfrak{g}$, then graded-equivalence of ρ and $\tilde{\rho}$ is just the quasi-isomorphism of graded modules.

Example 3.6. *Let $\rho\colon \mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ be representations of $\mathfrak{g}$ on $\mathbb{F}[X_n]$. Then ρ and $\overline{\rho} := \varpi_n\rho\colon \mathfrak{g} \to \mathrm{der}_1(\mathbb{F}[X_n])$ are graded-equivalent, where ϖ_n is the Lie algebra homomorphism given by (6). Indeed, for any $\overline{f(X)} \equiv f_0(X) \bmod (\mathbb{F}[X_n]_{k-1})$ with $f_0(X)$ being a homogenous polynomial of degree k, for any $\mathrm{g} \in \mathfrak{g}$ it holds that*

$$\rho(\mathrm{g})(\overline{f(X)}) \equiv \rho(\mathrm{g})(f_0(X)) \bmod (\mathbb{F}[X_n]_{k-1}) \equiv \overline{\rho}(\mathrm{g})(f_0(X)) \bmod (\mathbb{F}[X_n]_{k-1})$$

due to the definition of $\overline{\rho}$.

Motivated by Proposition 3.2 (iii), for finite-dimensional Lie algebras which has abelian ideals, we consider classification in terms of graded-equivalence of their faithful representations on polynomial rings. We have the following main result of this section.

Theorem 3.7. *Assume that $\mathfrak{g}$ is finite-dimensional. Let $\mathfrak{h}$ be an abelian ideal of $\mathfrak{g}$ with $\dim \mathfrak{h} = n$. Then there is at most one faithful representation up to graded-equivalence of $\mathfrak{g}$ of the form $\rho\colon \mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ such that $\rho(\mathfrak{h}) = \mathrm{der}_0(\mathbb{F}[X_n])$.*

Proof. Assume that such a faithful representation $\rho\colon \mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ exists. Recall the Lie algebra isomorphism $\sigma_n\colon \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ given by Lemma 2.1. Fix any basis $\{\mathrm{h}_i\}_{i=1}^n$ of $\mathfrak{h}$. Since ρ is faithful and $\rho(\mathfrak{h}) = \mathrm{der}_0(\mathbb{F}[X_n])$, the vectors $\alpha_i := \sigma_n(\rho(\mathrm{h}_i)) \in \mathbb{F}^n$ $(1 \leq i \leq n)$ are linearly independent, and hence we get an invertible matrix $Q := (\alpha_1, \cdots, \alpha_n)^T$.

Extend $\{\mathrm{h}_i\}_{i=1}^n$ to a basis $\{\mathrm{h}_i\}_{i=1}^n \cup \{\mathrm{g}_j\}_{j=1}^{\ell}$ of $\mathfrak{g}$, and set

$$\sigma_n(\rho(\mathrm{g}_j)) = \begin{pmatrix} A_j & 0 \\ \gamma_j & 0 \end{pmatrix}, \ 1 \leq j \leq \ell. \tag{24}$$

Since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, there are constants $c_{ij}^k \in \mathbb{F}$ such that

$$[\mathrm{h}_i, \mathrm{g}_j] = \sum_{k=1}^{n} c_{ij}^k \mathrm{h}_k, \ \ 1 \leq i \leq n, \ 1 \leq j \leq \ell, \tag{25}$$

from which we deduce that

$$\left[\begin{pmatrix} 0 & 0 \\ \alpha_i & 0 \end{pmatrix}, \begin{pmatrix} A_j & 0 \\ \gamma_j & 0 \end{pmatrix}\right] = [\sigma_n(\rho(\mathrm{h}_i)), \sigma_n(\rho(\mathrm{g}_j))] = \sum_{k=1}^{n} c_{ij}^k \begin{pmatrix} 0 & 0 \\ \alpha_k & 0 \end{pmatrix}.$$

So,

$$\alpha_i A_j = \sum_{k=1}^{n} c_{ij}^k \alpha_k, \ \ 1 \leq i \leq n, \ 1 \leq j \leq \ell. \tag{26}$$

Varying i from 1 to n, we get for each j that

$$QA_j = C_j Q, \ \text{ where } \ C_j = \begin{pmatrix} c_{1j}^1 & c_{1j}^2 & \cdots & c_{1j}^n \\ \cdots & \cdots & \cdots & \cdots \\ c_{nj}^1 & c_{nj}^2 & \cdots & c_{nj}^n \end{pmatrix}. \tag{27}$$

Since Q is invertible, we get that

$$A_j = Q^{-1} C_j Q, \ \ 1 \leq j \leq \ell, \tag{28}$$

which means that $\pi_n\sigma_n(\rho(\mathrm{g}_j))$ (see (9)) is uniquely determined by $\rho(\mathrm{h}_i)$, $1 \leq i \leq n$.

Similarly, if $\tilde{\rho}\colon \mathfrak{g} \to \mathrm{der}_{\leq 1}(\mathbb{F}[X_n])$ is another faithful representation of $\mathfrak{g}$ on $\mathbb{F}[X_n]$ satisfying that $\tilde{\rho}(\mathfrak{h}) = \mathrm{der}_0(\mathbb{F}[X_n])$, then we have the similar identity

$$\tilde{A}_j = \tilde{Q}^{-1} C_j \tilde{Q}, \ \ 1 \leq j \leq \ell, \tag{29}$$

where $\tilde{A}_j = \pi_n\sigma_n(\tilde{\rho}(\mathrm{g}_j))$, $\tilde{Q} = (\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n)^T$ and $\tilde{\alpha}_i = \sigma_n\tilde{\rho}(\mathrm{h}_i) \in \mathbb{F}^n$. So, by (28) and (29) it follows that

$$A_j = P\tilde{A}_jP^{-1}, \ \ 1 \le j \le \ell, \tag{30}$$

where $P := Q^{-1}\tilde{Q} \in \mathrm{GL}_n(\mathbb{F})$. We use P to show that ρ and $\tilde{\rho}$ are graded-equivalent. Indeed, the $\mathbb{F}$-automorphism τ_P on $\mathbb{F}[\mathrm{X}_n]$ given by (12) induces for each $m \ge 1$ an $\mathbb{F}$-automorphism, denoted by τ_m, on $\mathbb{F}[\mathrm{X}_n]_m/\mathbb{F}[\mathrm{X}_n]_{m-1}$. By Definition 3.5, it remains to check that, for $1 \le i \le n$ and $1 \le j \le \ell$,

$$\tau_m\rho(\mathrm{h}_i) = \tilde{\rho}(\mathrm{h}_i)\tau_m \ \text{ and } \ \tau_m\rho(\mathrm{g}_j) = \tilde{\rho}(\mathrm{g}_j)\tau_m \tag{31}$$

as $\mathbb{F}$-linear transformations on $\mathbb{F}[\mathrm{X}_n]_m/\mathbb{F}[\mathrm{X}_n]_m$.

Fix any $\overline{f(\mathrm{X})} \equiv f_0(\mathrm{X}) \bmod(\mathbb{F}[\mathrm{X}_n]_{m-1})$ with $f_0(\mathrm{X})$ being a homogenous polynomial of degree m. Since $\rho(\mathrm{h}_i)(f_0(\mathrm{X}))$ and $\tilde{\rho}(\mathrm{h}_i)\tau_m(f_0(\mathrm{X}))$ has degree $\le m-1$, the first identity of (31) follows. To check the second identity we have

$$\begin{aligned}
(\tau_m\rho(\mathrm{g}_j))(\overline{f(\mathrm{X})}) &\equiv \tau_m(\rho(\mathrm{g}_j)(f_0(\mathrm{X}))) \bmod(\mathbb{F}[\mathrm{X}_n]_{m-1}) \\
&\equiv (\tau_m\overline{\sigma_n}^{-1}(A_j))(f_0(\mathrm{X})) \bmod(\mathbb{F}[\mathrm{X}_n]_{m-1}) \\
&\overset{(30),\text{Lemma 2.2}}{\equiv} (\overline{\sigma_n}^{-1}(P\tilde{A}_jP^{-1})\tau_m)(f_0(\mathrm{X})) \bmod(\mathbb{F}[\mathrm{X}_n]_{m-1}) \\
&\equiv (\overline{\sigma_n}^{-1}(\tilde{A}_j)\tau_m)(f_0(\mathrm{X})) \bmod(\mathbb{F}[\mathrm{X}_n]_{m-1}) \ \equiv \ (\tilde{\rho}(\mathrm{g}_j)\tau_m)(f_0(\mathrm{X})) \bmod(\mathbb{F}[\mathrm{X}_n]_{m-1})
\end{aligned}$$

as required. □

4. Faithful representations of the Galilean Lie algebra on the polynomial ring in three variables

In this section we denote by $\mathfrak{G}$ the Galilean Lie algebra, which is the six-dimensional Lie algebra over $\mathbb{R}$ with a basis $\mathrm{m}, \mathrm{h}, \mathrm{n}_i, \mathrm{p}_i$ $(i = 1, 2)$ under which the Lie bracket is given by [1]

$$\begin{aligned}
&[\mathrm{m}, \mathrm{n}_1] = \mathrm{n}_2,\ [\mathrm{m}, \mathrm{n}_2] = -\mathrm{n}_1,\ [\mathrm{n}_i, \mathrm{h}] = \mathrm{p}_i,\ [\mathrm{m}, \mathrm{p}_1] = \mathrm{p}_2,\ [\mathrm{m}, \mathrm{p}_2] = -\mathrm{p}_1,\ \ [\mathrm{h}, \mathrm{p}_i] = 0, \\
&[\mathrm{p}_1, \mathrm{p}_2] = [\mathrm{n}_1, \mathrm{n}_2] = [\mathrm{m}, \mathrm{h}] = [\mathrm{n}_i, \mathrm{p}_j] = 0.
\end{aligned} \tag{32}$$

In the sequel we shall use the following abelian ideals of $\mathfrak{G}$ given by

$$\mathfrak{H} := \mathrm{span}_{\mathbb{R}}\{\mathrm{h}, \mathrm{p}_1, \mathrm{p}_2\} \ \text{ and } \ \mathfrak{N} := \mathrm{span}_{\mathbb{R}}\{\mathrm{p}_1, \mathrm{p}_2, \mathrm{n}_1, \mathrm{n}_2\} \tag{33}$$

respectively. Since $[\mathfrak{G}, \mathfrak{G}] = \mathfrak{N}$, $\mathfrak{G}$ is solvable, but not nilpotent.

Remark 4.1. *Note that any nonzero ideal $\mathfrak{I}$ of $\mathfrak{G}$ must contain $\mathrm{p}_1, \mathrm{p}_2$. One may use this fact to determine all ideals of $\mathfrak{G}$.*

We have the following result.

Proposition 4.2. *There is no faithful representations of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\le 1}(\mathbb{R}[\mathrm{X}_n])$ for $n = 1, 2$, and there is no faithful representations of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_1(\mathbb{R}[\mathrm{X}_3])$.*

Proof. Since $\dim \mathrm{der}_{\le 1}\mathbb{R}[\mathrm{X}_1] = 2 < \dim \mathfrak{G}$, there is no faithful representation of $\mathfrak{G}$ on $\mathbb{R}[x]$ of the form $\mathfrak{G} \to \mathrm{der}_{\le 1}(\mathbb{R}[x])$. If there were a faithful representation of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\le 1}(\mathbb{R}[\mathrm{X}_2])$, then by $\dim \mathrm{der}_{\le 1}(\mathbb{R}[\mathrm{X}_2]) = 6 = \dim \mathfrak{G}$ we have $\mathfrak{G} \cong \mathrm{der}_{\le 1}(\mathbb{R}[\mathrm{X}_2])$ as Lie algebras, which is impossible since $\mathfrak{G}$ is solvable, while by Lemma 2.1, $\mathrm{der}_{\le 1}(\mathbb{R}[\mathrm{X}_2])$ is isomorphic to $L_3(\mathbb{R})$ which contains a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. If there were a faithful representation ρ: $\mathfrak{G} \to \mathrm{der}_1(\mathbb{R}[\mathrm{X}_3])$ then $\rho(\mathfrak{N})$ is a 4-dimensional commutative Lie subalgebra of $\mathrm{der}_1(\mathbb{R}[\mathrm{X}_3])$, and hence $\rho(\mathfrak{N}) \cap \mathrm{der}_0(\mathbb{R}[\mathrm{X}_3]) \ne 0$ by Proposition 2.3, a contradiction. □

So, we consider faithful representations of $\mathfrak{G}$ on $\mathbb{R}[\mathrm{X}_3]$ of the form $\mathfrak{G} \to \mathrm{der}_{\le 1}(\mathbb{R}[\mathrm{X}_3])$.

Example 4.3. *The classical faithful representation ρ_0: $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3]$, which has concrete physical meaning [1], is given by*

$$\begin{aligned} &\rho_0(\mathrm{m}) = x_2\partial_{x_1} - x_1\partial_{x_2},\ \ \rho_0(\mathrm{h}) = -\partial_{x_3}, \\ &\rho_0(\mathrm{p}_1) = \partial_{x_1},\ \ \rho_0(\mathrm{p}_2) = \partial_{x_2},\ \ \rho_0(\mathrm{n}_1) = x_3\partial_{x_1},\ \ \rho_0(\mathrm{n}_2) = x_3\partial_{x_2}. \end{aligned} \tag{34}$$

The canonical projection ϖ_3 : $\mathrm{der}_{\leq 1}(\mathbb{F}[X_3]) \to \mathrm{der}_1(\mathbb{F}[X_3])$ given by (6) induces a representation $\overline{\rho_0} = \varpi_3\rho_0$: $\mathfrak{G} \to \mathrm{der}_1(\mathbb{R}[X_3])$ of $\mathfrak{G}$ on $\mathbb{R}[X_3]$ given by

$$\begin{aligned} &\overline{\rho_0}(\mathrm{m}) = x_2\partial_{x_1} - x_1\partial_{x_2},\ \ \overline{\rho_0}(\mathrm{h}) = 0, \\ &\overline{\rho_0}(\mathrm{p}_1) = 0,\ \ \overline{\rho_0}(\mathrm{p}_2) = 0,\ \ \overline{\rho_0}(\mathrm{n}_1) = x_3\partial_{x_1},\ \ \overline{\rho_0}(\mathrm{n}_2) = x_3\partial_{x_2}, \end{aligned} \tag{35}$$

Clearly $\overline{\rho_0}$ is not faithful.

We present a series with one parameter of faithful representations ρ_λ ($\lambda \in \mathbb{R}$) of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ as follows.

Lemma 4.4. *For any $\lambda \in \mathbb{R}$ define the map ρ_λ: $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X]_3)$ by*

$$\begin{aligned} &\rho_\lambda(\mathrm{p}_1) = \partial_{x_1},\ \ \rho_\lambda(\mathrm{p}_2) = \partial_{x_2},\ \ \rho_\lambda(\mathrm{h}) = -\partial_{x_3}, \\ &\rho_\lambda(\mathrm{n}_1) = x_3\partial_{x_1} + \lambda\partial_{x_2},\ \ \rho_\lambda(\mathrm{n}_2) = x_3\partial_{x_2} - \lambda\partial_{x_1},\ \rho_\lambda(\mathrm{m}) = x_2\partial_{x_1} - x_1\partial_{x_2}. \end{aligned} \tag{36}$$

Then,

(i) *ρ_λ is a faithful representation of $\mathfrak{G}$ on $\mathbb{R}[X_3]$.*
(ii) *ρ_{λ_1} and ρ_{λ_2} are equivalent if and only if $\lambda_1 = \lambda_2$.*
(iii) *All ρ_λ ($\lambda \in \mathbb{R}$) are graded-equivalent.*

Proof. (i) It follows by a direct check.

(ii) If ρ_{λ_1} and ρ_{λ_2} are equivalent then there is an $\mathbb{R}$-automorphism φ of $\mathbb{R}[X_3]$ such that

$$(\varphi\rho_{\lambda_1}(\mathrm{g}))(f) = (\rho_{\lambda_2}(\mathrm{g})\varphi)(f),\ \ \mathrm{g} \in \mathfrak{G},\ f \in \mathbb{R}[X_3]. \tag{37}$$

Choosing $\mathrm{g} = \mathrm{p}_1, \mathrm{p}_2, \mathrm{h}$ and $f(x_1, x_2, x_3) = x_1$ in (37) we get $\varphi(x_1) \in \mathbb{R}[x_1]$, $\partial_{x_1}(\varphi(x_1)) = \varphi(1)$. Similarly, $\varphi(x_2) \in \mathbb{R}[x_2]$, $\partial_{x_2}(\varphi(x_2)) = \varphi(1)$; $\varphi(x_3) \in \mathbb{R}[x_3]$, $\partial_{x_3}(\varphi(x_3)) = \varphi(1)$. Now, by choosing $\mathrm{g} = \mathrm{n}_1$ and $f(x_1, x_2, x_3) = x_2$ in (37) we get that $\lambda_1\varphi(1) = \lambda_2\varphi(1)$, and hence $\lambda_1 = \lambda_2$ as required since $\varphi(1) \neq 0$.

(iii) It follows by Example 3.6, since $\overline{\rho_\lambda} = \varpi_n\rho_\lambda = \overline{\rho_0}$ holds for any $\lambda \in \mathbb{R}$, where $\overline{\rho_0}$ is given by (35). □

Remark 4.5. *Due to (36), ρ_0 for $\lambda = 0$ coincides with the representation given by Example 4.3. As kindly pointed out by the referee, it would be interesting to elucidate some physical meaning of the representation ρ_λ for any $\lambda \in \mathbb{R}$.*

In the remaining of this section we shall classify all faithful representations of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ in terms of the usual equivalence, not just graded-equivalence. To this end we fix some notations for basis elements $\mathrm{p}_1, \mathrm{p}_2, \mathrm{h}, \mathrm{n}_1, \mathrm{n}_2, \mathrm{m}$ of $\mathfrak{G}$ as follows. Recall the mapping σ_n given by Lemma 2.1. For any representation ρ: $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ set

$$\begin{aligned} &(\sigma_3\rho)(\mathrm{p}_1) = \begin{pmatrix} A_1 & 0 \\ \delta_1 & 0 \end{pmatrix},\ (\sigma_3\rho)(\mathrm{p}_2) = \begin{pmatrix} A_2 & 0 \\ \delta_2 & 0 \end{pmatrix}, (\sigma_3\rho)(\mathrm{h}) = \begin{pmatrix} A_3 & 0 \\ \delta_3 & 0 \end{pmatrix}, \\ &(\sigma_3\rho)(\mathrm{n}_1) = \begin{pmatrix} A_4 & 0 \\ \delta_4 & 0 \end{pmatrix},\ (\sigma_3\rho)(\mathrm{n}_2) = \begin{pmatrix} A_5 & 0 \\ \delta_5 & 0 \end{pmatrix}, (\sigma_3\rho)(\mathrm{m}) = \begin{pmatrix} A_6 & 0 \\ \delta_6 & 0 \end{pmatrix}, \end{aligned} \tag{38}$$

where $A_i \in \mathfrak{gl}_3(\mathbb{R})$, $\delta_i \in \mathbb{R}^3$, $1 \leq i \leq 6$. For example, the identity $(\sigma_3\rho)(\mathrm{m}) = \begin{pmatrix} A_6 & 0 \\ \delta_6 & 0 \end{pmatrix}$ is equivalent to $\rho(\mathrm{m}) = \sum_{i,j=1}^{3} A_6(i, j)x_i\partial_{x_j} + \sum_{i=1}^{3} t_i\partial_{x_i} \in \mathrm{der}_{\leq 1}(\mathbb{R}[X_3]$, where (t_1, t_2, t_3) is the row vector $\delta_6 \in \mathbb{R}^3$, and $A_6(i, j)$ is the (i, j)-entry of A_6. Moreover, for brevity we set

$$A := \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} \in \mathfrak{gl}_3(\mathbb{R}). \tag{39}$$

The following observation plays a key role for our arguments of classification.

Lemma 4.6. *Keep notation as above. Let* $\rho\colon \mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ *be a faithful representation of* $\mathfrak{G}$. *Then* $A_1 = A_2 = A_3 = 0$. *Moreover, the matrix* A *given by (39) is invertible and* $\rho(\mathfrak{H}) = \mathrm{der}_0(\mathbb{R}[X_3])$.

Proof. Since ρ is faithful and $\mathfrak{N}$ is a 4-dimensional abelian ideal of $\mathfrak{G}$, $\rho(\mathfrak{N})$ is a 4-dimensional commutative algebra of $\mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$, and hence $\rho(\mathfrak{N}) \cap \mathrm{der}_0(\mathbb{R}[X_3]) \neq 0$ due to Proposition 2.3. So we can choose $0 \neq \mathrm{g}_1 = r_1\mathrm{n}_1 + r_2\mathrm{n}_2 + s_1\mathrm{p}_1 + s_2\mathrm{p}_2 \in \mathfrak{N}$ such that $\rho(\mathrm{g}_1) \in \rho(\mathfrak{N}) \cap \mathrm{der}_0(\mathbb{R}[X_3])$, $r_i, s_i \in \mathbb{R}$. Then by Lemma 2.1 we get

$$(\sigma_3\rho)(\mathrm{g}_1) = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}, \ \gamma := r_1\delta_4 + r_2\delta_5 + s_1\delta_1 + s_2\delta_2 \in \mathbb{R}^3.$$

Set $\mathrm{g}_2 := [\mathrm{g}_1, \mathrm{m}] = r_2\mathrm{n}_1 - r_1\mathrm{n}_2 + s_2\mathrm{p}_1 - s_1\mathrm{p}_2$. Then $0 \neq \mathrm{g}_2 \in \mathfrak{N}$ since $\mathfrak{N}$ is an ideal of $\mathfrak{G}$ and at least one of r_1, r_2, s_1, s_2 is nonzero. By

$$(\sigma_3\rho)(\mathrm{g}_2) = [(\sigma_3\rho)(\mathrm{g}_1), (\sigma_3\rho)(\mathrm{m})] = \begin{pmatrix} 0 & 0 \\ \gamma A_6 & 0 \end{pmatrix}$$

it follows that $\rho(\mathrm{g}_2) \in \rho(\mathfrak{N}) \cap \mathrm{der}_0(\mathbb{R}[X_3])$ due to Lemma 2.1. Set

$$\mathrm{g}_3 := [\mathrm{g}_1, \mathrm{h}] = r_1\mathrm{p}_1 + r_2\mathrm{p}_2, \ \ \mathrm{g}_4 := [\mathrm{g}_2, \mathrm{h}] = r_2\mathrm{p}_1 - r_1\mathrm{p}_2.$$

By a similar argument we get that $\rho(\mathrm{g}_3), \rho(\mathrm{g}_4) \in \rho(\mathfrak{N}) \cap \mathrm{der}_0(\mathbb{R}[X_3])$. But $\dim \rho(\mathfrak{N}) \cap \mathrm{der}_0(\mathbb{R}[X_3]) \leq 3$. So, g_i $(1 \leq i \leq 4)$ are linearly dependent, which means that

$$\det \begin{pmatrix} r_1 & r_2 & s_1 & s_2 \\ r_2 & -r_1 & s_2 & -s_1 \\ 0 & 0 & r_1 & r_2 \\ 0 & 0 & r_2 & -r_1 \end{pmatrix} = -(r_1^2 + r_2^2) = 0,$$

and hence $r_1 = r_2 = 0$ since $r_i \in \mathbb{R}$. Therefore,

$$\mathrm{g}_1 = s_1\mathrm{p}_1 + s_2\mathrm{p}_2, \ \ \mathrm{g}_2 = s_2\mathrm{p}_1 - s_1\mathrm{p}_2 \in \mathfrak{N},$$

and hence $\rho(\mathrm{p}_1), \rho(\mathrm{p}_2) \in \rho(\mathfrak{N}) \cap \mathrm{der}_0(\mathbb{R}[X_3])$ by $s_1s_2 \neq 0$, from which we deduce that $A_1 = A_2 = 0$ by Lemma 2.1. Moreover, by (32) and (38) we get that

$$\delta_1 A_i = \delta_2 A_i = 0, \ \ [A_i, A_j] = 0, \ \ i, j = 3, 4, 5, \tag{40}$$

Indeed, by $(\sigma_3\rho)([\mathrm{n}_j, \mathrm{h}]) = (\sigma_3\rho)([\mathrm{n}_j, \mathrm{p}_i]) = 0$ and (38) we get the first identity; by $(\sigma_3\rho)([\mathrm{n}_i, \mathrm{h}]) = (\sigma_3\rho)(\mathrm{p}_i)$, $A_1 = A_2 = 0$ and (38) we get the second identity.

Assume contrarily that $A_3 \neq 0$. By (40) it follows that $\mathrm{rank}(A_3) = 1$, since $\delta_1, \delta_2 \in \mathbb{R}^3$ are linearly independent. Therefore, A_3 is similar to a diagonal matrix over $\mathbb{R}$, i.e., there is $0 \neq \lambda_3 \in \mathbb{R}$ and $\delta_0 \in \mathbb{R}^3$ such that

$$TA_3T^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \ \ T := \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_0 \end{pmatrix} \in GL_3(\mathbb{R}). \tag{41}$$

Since $[A_3, A_4] = 0 = [A_3, A_5]$ due to (40), by (41) and $\lambda_3 \neq 0$ we get that

$$TA_4T^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_4 \end{pmatrix}, \ \ TA_5T^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_5 \end{pmatrix}, \tag{42}$$

where $\lambda_4, \lambda_5 \in \mathbb{R}$. So, by $\lambda_3 \neq 0$, (41) and (42) there are $\mu_4, \mu_5 \in \mathbb{R}$ such that $\mu_4 A_3 + A_4 = 0$ and $\mu_5 A_3 + A_5 = 0$, from which we deduce by using (38) that

$$(\sigma_3\rho)(\mu_4\mathrm{h} + \mathrm{n}_1) = \begin{pmatrix} 0 & 0 \\ \mu_4\delta_3 + \delta_4 & 0 \end{pmatrix}, \ \ (\sigma_3\rho)(\mu_5\mathrm{h} + \mathrm{n}_2) = \begin{pmatrix} 0 & 0 \\ \mu_5\delta_3 + \delta_5 & 0 \end{pmatrix},$$

which means that $\rho(\mu_4\mathrm{h} + \mathrm{n}_1)$, $\rho(\mu_5\mathrm{h} + \mathrm{n}_2) \in \mathrm{der}_0(\mathbb{R}[X_3])$ by Lemma 2.1. We have already shown that $\rho(\mathrm{p}_1), \rho(\mathrm{p}_2) \in \mathrm{der}_0(\mathbb{R}[X_3])$. Since $\dim \mathrm{der}_0(\mathbb{R}[X_3]) = 3$, there are $t_i \in \mathbb{R}$, at least one of which is nonzero, such that

$$t_1\rho(\mathrm{p}_1) + t_2\rho(\mathrm{p}_2) + t_3\rho(\mu_4\mathrm{h} + \mathrm{n}_1) + t_4\rho(\mu_5\mathrm{h} + \mathrm{n}_2) = 0.$$

Since $\ker \rho = 0$ we get $t_1\mathrm{p}_1 + t_2\mathrm{p}_2 + (t_3\mu_4 + t_4\mu_5)\mathrm{h} + t_3\mathrm{n}_1 + t_4\mathrm{n}_2 = 0$, from which we deduce that $t_1 = t_2 = t_3 = t_4 = 0$, a contradiction, since $\mathrm{p}_1, \mathrm{p}_2, \mathrm{h}, \mathrm{n}_1, \mathrm{n}_1, \mathrm{n}_2$ are linearly independent. Therefore, we have shown that $A_3 = 0$ as required.

By $A_1 = A_2 = A_3 = 0$ and Lemma 2.1 we get that $\delta_1, \delta_2, \delta_3$ are linearly independent since ρ is faithful. So A is invertible and $\rho(\mathfrak{H}) = \mathrm{der}_0(\mathbb{R}[X_3])$. □

By Theorem 3.7 and Lemma 4.6 we get the following corollary, which extends Lemma 4.4 (iii).

Corollary 4.7. *All faithful representations of the form $\rho\colon \mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ are graded-equivalent to the representation of $\mathfrak{G}$ given by (34).*

We continue to discuss the matrices given in (38) for faithful representations of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$.

Lemma 4.8. *Keep notation as above. If $\rho\colon \mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ is faithful then,*

(i) *For row vectors $\delta_4, \delta_5, \delta_6$ in (38) there are unique $a_1, a_2, c_1, c_2 \in \mathbb{R}$ such that*

$$\delta_4 = a_1\delta_1 + a_2\delta_2,\ \delta_5 = -a_2\delta_1 + a_1\delta_2,\ \delta_6 = c_1\delta_1 + c_2\delta_2. \tag{43}$$

In particular, $\delta_i \in \mathrm{span}_{\mathbb{R}}\{\delta_1, \delta_2\}$, $i = 4, 5, 6$.

(ii) *The matrices A_4, A_5, A_6 in (38) are uniquely determined via*

$$A_4A^{-1} = A^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ A_5A^{-1} = A^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},\ A_6A^{-1} = A^{-1}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

where A is given by (39).

Proof. (i) By Lemma 4.6, $\delta_4, \delta_5, \delta_6$ can be uniquely linearly presented by $\delta_1, \delta_2, \delta_3$. Hence there are unique $a_i, b_i, c_i \in \mathbb{R}$ such that

$$\delta_4 = a_1\delta_1 + a_2\delta_2 + a_3\delta_3,\ \delta_5 = b_1\delta_1 + b_2\delta_2 + b_3\delta_3,\ \delta_6 = c_1\delta_1 + c_2\delta_2 + c_3\delta_3.$$

It remains to show that $a_3 = b_3 = c_3 = 0$ and $b_1 = -a_2$, $b_2 = a_1$. By (38) and

$$[(\sigma_3\rho)(\mathrm{m}), (\sigma_3\rho)(\mathrm{n}_1)] = (\sigma_3\rho)(\mathrm{n}_2),\ [(\sigma_3\rho)(\mathrm{m}), (\sigma_3\rho)(\mathrm{n}_2)] = -(\sigma_3\rho)(\mathrm{n}_1)$$

we get that

$$\delta_5 = \delta_6A_4 - \delta_4A_6,\ \ \delta_4 = -(\delta_6A_5 - \delta_5A_6). \tag{45}$$

Moreover, similar to (40), by $A_1 = A_2 = A_3 = 0$ (see Lemma 4.6) we get that

$$-\delta_1A_6 = \delta_2,\ \ \delta_2A_6 = \delta_1,\ \ \delta_3A_6 = 0,\ \ -\delta_3A_5 = \delta_2,\ \ -\delta_3A_4 = \delta_1. \tag{46}$$

Then we have

$$\begin{aligned} \delta_5 &= b_1\delta_1 + b_2\delta_2 + b_3\delta_3 \overset{(45)}{=} \delta_6A_4 - \delta_4A_6 \\ &= (c_1\delta_1 + c_2\delta_2 + c_3\delta_3)A_4 - (a_1\delta_1 + a_2\delta_2 + a_3\delta_3)A_6 \overset{(40),(46)}{=} (-a_2 - c_3)\delta_1 + a_1\delta_2, \end{aligned}$$

which implies that $b_3 = 0$ and $b_2 = a_1, b_1 = -a_2 - c_3$, since $\delta_1, \delta_2, \delta_3$ are linearly independent. Similarly, we get that $\delta_4 = (a_2 + 2c_3)\delta_2 + a_1\delta_1$, which means that $c_3 = 0$, and hence $b_2 = a_1$, $b_1 = -a_2$ as required.

(ii) By (40) and (46) we have

$$AA_4 = \begin{pmatrix} 0 \\ 0 \\ -\delta_1 \end{pmatrix}, \ AA_5 = \begin{pmatrix} 0 \\ 0 \\ -\delta_2 \end{pmatrix}, \ AA_6 = \begin{pmatrix} -\delta_2 \\ \delta_1 \\ 0 \end{pmatrix}.$$

Write $A^{-1} = (\gamma_1\ \gamma_2\ \gamma_3)$, where γ_i is a 3-dimensional column vector over $\mathbb{R}$, that is, $\delta_i\gamma_j = \delta_{ij}$, the Kronecker notation. It follows that

$$AA_4A^{-1} = \begin{pmatrix} 0 \\ 0 \\ -\delta_1 \end{pmatrix} (\gamma_1\ \gamma_2\ \gamma_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The other two identities in (44) are similar. □

We write down explicitly all faithful representations of $\mathfrak{G}$ of the form $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ as follows.

Lemma 4.9. *Let* $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3 \in \mathrm{der}_0(\mathbb{R}[X_3])$ *be linearly independent,* $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3 \in \mathbb{R}[X_3]$ *be homogeneous of degree* 1 *satisfying* $\mathrm{d}_i(\mathrm{f}_j) = \delta_{ij}$*, and* a_i, c_i *(*$i = 1, 2$*) be real numbers. Set*

$$\rho(\mathrm{p}_1) = \mathrm{d}_1, \ \ \rho(\mathrm{p}_2) = \mathrm{d}_2, \ \ \rho(\mathrm{h}) = \mathrm{d}_3, \tag{47}$$

$$\rho(\mathrm{n}_1) = (-\mathrm{f}_3 + a_1)\mathrm{d}_1 + a_2\mathrm{d}_2, \ \ \rho(\mathrm{n}_2) = -a_2\mathrm{d}_1 + (-\mathrm{f}_3 + a_1)\mathrm{d}_2, \tag{48}$$

$$\rho(\mathrm{m}) = (\mathrm{f}_2 + c_1)\mathrm{d}_1 - (\mathrm{f}_1 - c_2)\mathrm{d}_2. \tag{49}$$

Then (47)-(49) define a faithful of $\mathfrak{G}$ *of the form* $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$*. Conversely, any faithful representation* ρ*:* $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ *has the form given by (47)-(49).*

Proof. By choices of d_i, f_j and (47)-(49) it follows that $\rho(\mathrm{g}) \in \mathrm{der}_{\leq 1}(\mathbb{R}[X_3])$ for any $\mathrm{g} \in \mathfrak{G}$. It's direct to check that (47)-(49) define a representation of $\mathfrak{G}$. For example, we can check $[\rho(\mathrm{m}), \rho(\mathrm{n}_1)] = \rho(\mathrm{n}_2)$ holds as derivations of $\mathbb{R}[X_3]$. Indeed, by $\mathrm{d}_1(\mathrm{f}_3) = 0$ and $\mathrm{d}_1(\mathrm{f}_2) = 0$ we have $[(\mathrm{f}_2 + c_1)\mathrm{d}_1, (-\mathrm{f}_3 + a_1)\mathrm{d}_1] = 0$, and by $\mathrm{d}_1(\mathrm{f}_1) = 1$, $\mathrm{d}_2(\mathrm{f}_3) = 0$ we get

$$\begin{aligned} & [(\mathrm{f}_1 - c_2)\mathrm{d}_2, (-\mathrm{f}_3 + a_1)\mathrm{d}_1] \\ = \ & (\mathrm{f}_1 - c_2)\mathrm{d}_2((-\mathrm{f}_3 + a_1)\mathrm{d}_1) - ((-\mathrm{f}_3 + a_1)\mathrm{d}_1)((\mathrm{f}_1 - c_2)\mathrm{d}_2) \\ = \ & (\mathrm{f}_1 - c_2)(-\mathrm{f}_3 + a_1)\mathrm{d}_2\mathrm{d}_1 - (-\mathrm{f}_3 + a_1)\mathrm{d}_2 - (\mathrm{f}_1 - c_2)(-\mathrm{f}_3 + a_1)\mathrm{d}_1\mathrm{d}_2 \ = \ -(-\mathrm{f}_3 + a_1)\mathrm{d}_2 \end{aligned}$$

since $\mathrm{d}_1\mathrm{d}_2 = \mathrm{d}_2\mathrm{d}_1$ as derivations of $\mathbb{R}[X_3]$. Similarly, we have

$$[(\mathrm{f}_1 - c_2)\mathrm{d}_2, a_2\mathrm{d}_1] = 0, \ \ [(\mathrm{f}_2 + c_1)\mathrm{d}_1, a_2\mathrm{d}_2] = -a_2\mathrm{d}_1.$$

Therefore, by (48) and (49) we get that

$$[\rho(\mathrm{m}), \rho(\mathrm{n}_1)] \ = \ [(\mathrm{f}_2 + c_1)\mathrm{d}_1 - (\mathrm{f}_1 - c_2)\mathrm{d}_2, \ \ (-\mathrm{f}_3 + a_1)\mathrm{d}_1 + a_2\mathrm{d}_2] \ = \ -a_2\mathrm{d}_1 + (-\mathrm{f}_3 + a_1)\mathrm{d}_2 = \rho(\mathrm{n}_2)$$

as required. Moreover, if $\mathrm{g} = u_1\mathrm{p}_1 + u_2\mathrm{p}_2 + u_3\mathrm{h} + u_4\mathrm{n}_1 + u_5\mathrm{p}_2 + u_3\mathrm{m} \in \ker\rho$ ($u_i \in \mathbb{R}$), then by (48) and (49) we have

$$\begin{aligned} \mathrm{der}_{\leq 1}(\mathbb{R}[X_3]) \ni 0 \ = \ & (u_1 + u_4(-\mathrm{f}_3 + a_1) - u_2a_5 + u_6(\mathrm{f}_2 + c_1))\mathrm{d}_1 \\ & + (u_2 + u_4a_2 + u_5(-\mathrm{f}_3 + a_1) - u_6(\mathrm{f}_1 - c_2))\mathrm{d}_2 + u_3\mathrm{d}_3. \end{aligned}$$

Applying both sides to $f(x_1, x_2, x_3) = \mathrm{f}_1$, $f(x_1, x_2, x_3) = \mathrm{f}_2$ and $f(x_1, x_2, x_3) = \mathrm{f}_3$ respectively, and using $\mathrm{d}_i(\mathrm{f}_j) = \delta_{ij}$ we get that

$$\begin{aligned} u_1 + u_4(-\mathrm{f}_3 + a_1) - u_2a_5 + u_6(\mathrm{f}_2 + c_1) &= 0, \\ u_2 + u_4a_2 + u_5(-\mathrm{f}_3 + a_1) - u_6(\mathrm{f}_1 - c_2) &= 0, \\ u_3 &= 0. \end{aligned}$$

By comparing degrees of polynomials we get that $u_i = 0$ ($1 \leq i \leq 6$), which implies that ρ is faithful.

Now we assume that $\rho\colon \mathfrak{G} \to \mathrm{der}_{\le 1}(\mathbb{R}[X_3])$ is a faithful representation of $\mathfrak{G}$. Recall the notation given in (38). Set $\mathrm{d}_i := \sigma_3^{-1}\begin{pmatrix} A_i & 0 \\ \delta_i & 0 \end{pmatrix}$, $i = 1,2,3$. By Lemma 4.6 we get that $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3 \in \mathrm{der}_0(\mathbb{R}[X_3])$ are linearly independent and (47) holds.

By Lemma 2.1 we get that the derivation $\mathrm{D}_j := (\overline{\sigma_3})^{-1}(A_j) \in \mathrm{der}_1(\mathbb{R}[X_3])$ for $j = 4,5,6$, where A_i is given by (38). Then, by Lemma 4.8 (i) and Lemma 2.1 we get that

$$\rho(\mathrm{n}_1) = \mathrm{D}_4 + a_1\mathrm{d}_1 + a_2\mathrm{d}_2,\ \ \rho(\mathrm{n}_2) = \mathrm{D}_5 - a_2\mathrm{d}_1 + a_1\mathrm{d}_2,\ \ \rho(\mathrm{m}) = \mathrm{D}_6 + c_1\mathrm{d}_1 + c_2\mathrm{d}_2. \tag{50}$$

We compute D_i further as follows. By Lemma 4.6 we define homogeneous polynomials $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3 \in \mathbb{R}[X_3]$ of degree 1 by setting $(\mathrm{f}_1\ \mathrm{f}_2\ \mathrm{f}_3) = (x_1\ x_2\ x_3)A^{-1}$, where A is given by (39). By the definition of A (see (39)) we have $\delta_i = (A(i,1)\ A(i,2)\ A(i,3))$, and hence $\mathrm{d}_i = \sum_{k=1}^{3} A(i,k)\partial_{x_k}$. So, for $1 \le i, j \le 3$ we have

$$\mathrm{d}_i(\mathrm{f}_j) = \left(\sum_{k=1}^{3} A(i,k)\partial_{x_k}\right)\left(\sum_{\ell=1}^{3} A^{-1}(\ell,j)x_\ell\right) = \sum_{k=1}^{3} A(i,k)A^{-1}(k,j) = (AA^{-1})(i,j) = \delta_{ij}. \tag{51}$$

Now we claim that

$$\mathrm{D}_4 = -\mathrm{f}_3\mathrm{d}_1,\ \ \mathrm{D}_5 = -\mathrm{f}_3\mathrm{d}_2,\ \ \mathrm{D}_6 = \mathrm{f}_2\mathrm{d}_1 - \mathrm{f}_1\mathrm{d}_2 \tag{52}$$

as derivations of $\mathbb{R}[X_3]$. Since $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3$ and x_1, x_2, x_3 are equivalent in the linear space $\mathrm{span}_{\mathbb{R}}\{x_1, x_2, x_3\}$, it suffices to check both sides of each identity in (52) have the same actions on $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3 \in \mathbb{R}[X_3]$. For example, we have

$$\begin{aligned}\mathrm{D}_4(\mathrm{f}_1\ \mathrm{f}_2\ \mathrm{f}_3) &= \mathrm{D}_4((x_1\ x_2\ x_3)A^{-1}) \overset{\text{Lemma 2.1}}{=} ((x_1\ x_2\ x_3)A_4)A^{-1} \overset{(44)}{=} (x_1\ x_2\ x_3)A^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &= (\mathrm{f}_1\ \mathrm{f}_2\ \mathrm{f}_3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (-\mathrm{f}_3\ 0\ 0) \overset{(51)}{=} -\mathrm{f}_3\mathrm{d}_1(\mathrm{f}_1\ \mathrm{f}_2\ \mathrm{f}_3),\end{aligned}$$

which implies the first identity in (52). The other two identities in (52) can be obtained similarly. By (50) and (52) we get (48) and (49) as required. □

Now we are in the position to prove the main result of this section.

Theorem 4.10. *Let $\rho\colon \mathfrak{G} \to \mathrm{der}_{\le 1}(\mathbb{R}[X_3])$ be a faithful representation of $\mathfrak{G}$ on $\mathbb{R}[X_3]$. Then there is a unique $\lambda \in \mathbb{R}$ such that ρ is equivalent to ρ_λ given by (36).*

Proof. The uniqueness of λ follows by Lemma 4.4 (ii).

It remains to show existence of λ. By Lemma 4.9, $\rho(\mathrm{p}_1)$, $\rho(\mathrm{p}_2)$, $\rho(\mathrm{h})$, $\rho(\mathrm{n}_1)$, $\rho(\mathrm{n}_2)$ and $\rho(\mathrm{m})$ are given by (47)-(49) for some linearly independent derivations $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3 \in \mathrm{der}_0(\mathbb{R}[X_3])$, homogeneous polynomials $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3 \in \mathbb{R}[X_3]$ of degree 1 satisfying $\mathrm{d}_i(\mathrm{f}_j) = \delta_{ij}$, and real numbers a_i, c_i ($i = 1,2$). By the computation in (51) we have $(\mathrm{f}_1\ \mathrm{f}_2\ \mathrm{f}_3) = (x_1\ x_2\ x_3)A^{-1}$ and hence

$$\frac{\partial \mathrm{f}_i}{\partial x_k} = A^{-1}(k,i),\ \ 1 \le i,k \le 3. \tag{53}$$

We show that ρ is equivalent to ρ_λ given by (36) for $\lambda = a_2$. To this end we consider the $\mathbb{R}$-automorphism $\mathcal{T}$ of $\mathbb{R}[X_3]$ given by

$$\mathcal{T}:\ h(x_1, x_2, x_3) \mapsto h(\mathrm{f}_1 - c_2,\ \mathrm{f}_2 + c_1,\ -\mathrm{f}_3 + a_1). \tag{54}$$

Clearly $\mathcal{T}$ is also an automorphism of $\mathbb{R}[X_3]$ as a commutative algebra. Similar to (13), for $i = 1,2,3$ it holds that

$$\frac{\partial(\mathcal{T}(h))}{\partial \mathrm{f}_i} = \mathcal{T}(\partial_{x_i}h),\quad h \in \mathbb{R}[X_3]. \tag{55}$$

It suffices to check that $\mathcal{T}\rho_{a_2}(\mathrm{g}) = \rho(\mathrm{g})\mathcal{T}$ as $\mathbb{R}$-linear transformations on $\mathbb{R}[\mathrm{X}_3]$ holds for g being any one of the basis elements p_1, p_2, h, n_1, n_2 and m of $\mathfrak{G}$.

By the definition of A (see (39)) we have $\delta_i = (A(i,1)\ A(i,2)\ A(i,3))$, and hence $\mathrm{d}_i = \sum_{k=1}^{3} A(i,k)\partial_{x_k}$. Therefore, for any $h \in \mathbb{R}[\mathrm{X}_3]$ it holds that

$$\begin{aligned}
\rho(\mathrm{h})(\mathcal{T}(h)) &\overset{(47)}{=} -\mathrm{d}_3(\mathcal{T}(h)) = \sum_{k=1}^{3} A(3,k)\partial_{x_k}(\mathcal{T}(h)) = -\sum_{k=1}^{3} A(3,k)\left(\sum_{i=1}^{3} \frac{\partial \mathcal{T}(h)}{\partial \mathrm{f}_i} \cdot \frac{\partial \mathrm{f}_i}{\partial x_k}\right) \\
&\overset{(53)}{=} -\sum_{k=1}^{3} A(3,k)\left(\sum_{i=1}^{3} \frac{\partial \mathcal{T}(h)}{\partial \mathrm{f}_i} A^{-1}(k,i)\right) = -\sum_{i=1}^{3}\left(\sum_{k=1}^{3} A(3,k)A^{-1}(k,i)\right)\frac{\partial \mathcal{T}(h)}{\partial \mathrm{f}_i} \\
&= -\frac{\partial \mathcal{T}(h)}{\partial \mathrm{f}_3} \overset{(55)}{=} \mathcal{T}(-\partial_{x_3}(h)) \overset{(36)}{=} \mathcal{T}(\rho_{a_2}(\mathrm{h})(h)).
\end{aligned}$$

So $\rho(\mathrm{h})\mathcal{T} = \mathcal{T}\rho_{a_2}(\mathrm{h})$ follows. Similarly, $\rho(\mathrm{p}_1)\mathcal{T} = \mathcal{T}\rho_{a_2}(\mathrm{p}_1)$, $\rho(\mathrm{p}_2)\mathcal{T} = \mathcal{T}\rho_{a_2}(\mathrm{p}_2)$. Based on these identities we proceed to consider other basis elements of $\mathfrak{G}$.

Since $\mathcal{T}$ is also a commutative algebra automorphism, it holds that

$$\begin{aligned}
\rho(\mathrm{n}_1)\mathcal{T} &\overset{(48)}{=} (-\mathrm{f}_3 + a_1)\mathrm{d}_1\mathcal{T} + a_2\mathrm{d}_2\mathcal{T} \overset{(47)}{=} (-\mathrm{f}_3 + a_1)\rho(\mathrm{p}_1)\mathcal{T} + a_2\rho(\mathrm{p}_2)\mathcal{T} \\
&= (-\mathrm{f}_3 + a_1)\mathcal{T}\rho_{a_2}(\mathrm{p}_1) + a_2\mathcal{T}\rho_{a_2}(\mathrm{p}_2) \\
&\overset{(36)}{=} (-\mathrm{f}_3 + a_1)\mathcal{T}\partial_{x_1} + a_2\mathcal{T}\partial_{x_2} \overset{(54)}{=} \mathcal{T}(x_3)\mathcal{T}\partial_{x_1} + a_2\mathcal{T}\partial_{x_2} = \mathcal{T}(x_3\partial_{x_1} + a_2\partial_{x_2}) \overset{(36)}{=} \mathcal{T}\rho_{a_2}(\mathrm{n}_1)
\end{aligned}$$

as required. Similarly we obtain $\rho(\mathrm{n}_2)\mathcal{T} = \mathcal{T}\rho_{a_2}(\mathrm{n}_2)$ and $\rho(\mathrm{m})\mathcal{T} = \mathcal{T}\rho_{a_2}(\mathrm{m})$. This completes the proof. □

Remark 4.11. *By Lemma 4.4 (iii) and Theorem 4.10, all faithful of* $\mathfrak{G}$ *of the form* $\mathfrak{G} \to \mathrm{der}_{\leq 1}(\mathbb{R}[\mathrm{X}_3])$ *are graded-equivalent. So we can deduce Corollary 4.7 without using Theorem 3.7.*

Acknowledgements. The authors thank the referee for helpful suggestions on the manuscript.

References

[1] S. K. Bose, The Galilean group in 2+1 space-times and its central extension, Commun. Math. Phys. 169 (1995) 385-395.
[2] S. K. Bose, Representations of the (2+1)-dimensional Galilean group, J. Math, Phys. 36 (1995) 875-890.
[3] J. Draisma, Constructing Lie algebras of first order differential operators, J. Symb. Comput. 36 (2003) 685–698.
[4] F. Finkel, A. González-López, N. Kamran, P. J. Olver and M. A. Rodríguez, Lie algebras of differential operators and partial integrability. http://arxiv.org/abs/hep-th/9603139v1.
[5] A. González-López, N. Kamran and P. J. Olver, Lie algebras of vector fields in the real plane, Proc. London Math. Soc. 64 (1992) 339–368.
[6] V. W. Guillemin and S. Sternberg, An algebraic model of transitive differential geometry, Bull. Amer. Math. Soc. 70 (1964) 16–47.
[7] N. Jacobson, Schur's theorems on commutative matrices, Bull. Amer. Math. Soc. 50 (1946) 431–436.
[8] N. Kamran and P. J. Olver, Lie algebras of differential operators and Lie-algebraic potentials, J. Math. Anal. Appl. 145 (1990) 342–356.
[9] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, New York, 1972.