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# On the set of $(b, c)$-invertible elements

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**Abstract.** Let $b$ and $c$ be two elements in a semigroup $S$. This paper is devoted to studying the structures of $S^{\|(b,c)}$ and $H_{(b,c)}$ in a semigroup $S$, where $S^{\|(b,c)}$ stands for the set of all $(b,c)$-invertible elements and $H_{(b,c)} = \{y \in S \mid bS^1 = yS^1,\ S^1y = S^1c\}$. Denote the $(b,c)$-inverse of $a \in S^{\|(b,c)}$ by $a^{\|(b,c)}$. If $S^{\|(b,c)} \neq \emptyset$, then $H_{(b,c)} = \{a^{\|(b,c)} \mid a \in S^{\|(b,c)}\}$. We first find some new equivalent conditions for $H_{(b,c)}$ to be a group and analyze its structure from the viewpoint of generalized inverses. Then a necessary and sufficient condition under which $S^{\|(b,c)}$ is a subsemigroup of $S$ with the reverse order law holding for $(b,c)$-inverses is presented. At last, given $a, b, c, d, x, y, z \in S$ and $y \in S^{\|(b,c)}$, we prove that any two of the conditions $x \in S^{\|(a,c)}$, $z \in S^{\|(b,d)}$ and $zy^{\|(b,c)}x \in S^{\|(a,d)}$ imply the rest one.

## 1. Introduction

In a monoid, if two elements $a$ and $b$ are invertible, then their product $ab$ is also invertible with

$$(ab)^{-1} = b^{-1}a^{-1}.$$

The above equality is called the reverse order law for classical inverses. However, the reverse order law is not true for generalized inverses in general. This leads to a question: under what condition the reverse order law holds for generalized inverses. It has become a hot topic in the research of generalized inverses and has been studied from two different aspects: elements and subsets. For instance, Greville [13] proved that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $A^\dagger A$ commutes with $BB^*$ and $A^*A$ commutes with $BB^\dagger$, for complex matrices $A$ and $B$. Cao et al. [4] provided some necessary and sufficient conditions such that $(AB)^\# = B^\# A^\#$ holds for group invertible complex matrices $A$ and $B$. We refer to [5–10, 14, 21, 25] for more results on this topic from elements aspect.

In contrast, for a semigroup $S$, Mary [16] gave an equivalent condition for the subset of all group invertible elements in $S$ to be a semigroup. Furthermore, Mary [17] proved that a completely regular semigroup $S$ (i.e., every element in it is group invertible) is a Clifford semigroup if and only if $(ab)^\# = b^\# a^\#$ for all $a, b \in S$. This is the main motivation of our paper. The aim of this paper is to study the structure of the set of generalized invertible elements and the set of corresponding generalized inverses in a semigroup

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$S$. The inverse along an element and $(b,c)$-inverse provide the possibility for reaching our aim because they unify various different generalized inverses.

In 2011, Mary [15] defined that $a \in S$ is invertible along $d \in S$ if there exists $y \in S$ satisfying the following relations:

$$y \in dS \cap Sd,\ yad = d,\ day = d.$$

If $y$ exists, then it is unique and called the inverse of $a$ along $d$ (denoted by $a^{\|d}$). He proved that the Moore-Penrose inverse of an element $a$ is equal to $a^{\|a^*}$ and the group inverse of $a$ is equal to $a^{\|a}$. The set of all elements which are invertible along $d$ is denoted by $S^{\|d}$.

In 2012, Drazin [11] introduced the $(b,c)$-inverse of an element in a semigroup. Let $b, c \in S$. An element $a \in S$ is $(b,c)$-invertible if there exists $y \in S$ such that

$$y \in bS \cap Sc,\ yab = b,\ cay = c.$$

If $y$ exists, then it is unique and called the $(b,c)$-inverse of $a$ (denoted by $a^{\|(b,c)}$). When $b = c$, we know that $a^{\|(b,b)} = a^{\|b}$. As is well-known, the core inverse of an element $a$ is equal to $a^{\|(a,a^*)}$. The set of all $(b,c)$-invertible elements is denoted by $S^{\|(b,c)}$.

Drazin proved the following fact which connects the $(b,c)$-inverse with Green's relations.

**Lemma 1.1.** ([11, Proposition 6.1]) *Let $a, b, c \in S$. Then $a \in S^{\|(b,c)}$ if and only if there exists $y \in S$ satisfying*

$$yay = y,\ bS^1 = yS^1,\ S^1y = S^1c.$$

In 1951, Green [12] defined the following equivalent relations on $S$:

- $b\mathcal{L}c \Leftrightarrow S^1b = S^1c$;
- $b\mathcal{R}c \Leftrightarrow bS^1 = cS^1$;
- $b\mathcal{H}c \Leftrightarrow b\mathcal{L}c$ and $b\mathcal{R}c$;
- $b\mathcal{D}c \Leftrightarrow$ there exists $a \in S$ such that $bS^1 = aS^1$ and $S^1a = S^1c$,

where $S^1$ stands for the monoid generated by $S$. All above relations are called Green's relations. For Green's relation $\mathcal{K}$, the $\mathcal{K}$-class of $b$ is denoted by $K_b$. For convenience, we denote the $\mathcal{H}$-class $R_b \cap L_c$ by $H_{(b,c)}$. Clearly, $H_{(a,a)} = H_a$.

Lemma 1.1 shows that $S^{\|(b,c)}$ is nonempty if and only if $H_{(b,c)}$ contains a regular element. In this case, $H_{(b,c)} = \{a^{\|(b,c)} \mid a \in S^{\|(b,c)}\}$. The structure of an $\mathcal{H}$-class has been studied by Green [12]. He proved that $H_a$ is a group if and only if $a\mathcal{H}a^2$. We want to find some new equivalent conditions for $H_{(b,c)}$ to be a group from the perspective of generalized inverses. We prove that $H_{(b,c)}$ is a group if and only if $H_{(b,c)} \cap S^{\|(b,c)} \neq \emptyset$ if and only if $H_{(b,c)}S^{\|(b,c)} \cap S^{\|(b,c)} \neq \emptyset$ if and only if $S^{\|(b,c)}H_{(b,c)} \cap S^{\|(b,c)} \neq \emptyset$. In this case,

$$H_{(b,c)}S^{\|(b,c)} = eS^{\|(b,c)}\ ,\ S^{\|(b,c)}H_{(b,c)} = S^{\|(b,c)}e$$

and

$$H_{(b,c)} = eS^{\|(b,c)}e = eS^{\|(b,c)} \cap S^{\|(b,c)}e = \{a \in S^{\#} \cap S^{\|(b,c)} \mid a^{\#} = a^{\|(b,c)}\},$$

where $e$ is the identity element of $H_{(b,c)}$. Furthermore, we show that $S^{\|(b,c)}$ is a subsemigroup of $S$ with the reverse order law holding for $(b,c)$-inverses if and only if $H_{(b,c)}$ contains an idempotent $e$ such that $eade = eaede$ for all $a, d \in S^{\|(b,c)}$. Meanwhile, some semigroups between $H_{(b,c)}$ and $S^{\|(b,c)}$ are presented.

If $a \in S^{\|d}$, Zhu et al. [26, Theorem 3.19] proved that $b \in S^{\|d}$ if and only if $bda \in S^{\|d}$ if and only if $adb \in S^{\|d}$. If $y \in S^{\|(b,c)}$, we prove that any two of the conditions $x \in S^{\|(a,c)}$, $z \in S^{\|(b,d)}$ and $zy^{\|(b,c)}x \in S^{\|(a,d)}$ imply the rest one. Moreover, we get that $y \in S^{\|(b,c)}$ if and only if $H_{(a,c)}yH_{(b,d)} \subseteq H_{(a,d)}$ if and only if $H_{(a,c)}yH_{(b,d)} = H_{(a,d)}$, which generalizes [19, Corollary 2.5].

## 2. Groups and semigroups in $S^{\|(b,c)}$

Throughout this paper, $S$ is a semigroup unless otherwise specified.

Recall that $a \in S$ is said to be regular if there exists $b \in S$ such that $aba = a$, in which case $b$ is called an inner inverse (or a $\{1\}$-inverse) of $a$. The sets of all inner inverses of $a$ and all regular elements of $S$ are denoted by $a\{1\}$ and $S^{\{1\}}$, respectively.

A basic lemma that we will frequently use but without further comment should be noted.

**Lemma 2.1.** *Let $a \in S$. If $a$ is regular and $bS^1 = aS^1$ and $S^1a = S^1c$ for some $b, c \in S$, then $b, c$ are regular and*

$$aa^-b = b,\ bb^-a = a,\ ca^-a = c,\ ac^-c = a$$

*for any $a^- \in a\{1\}$, $b^- \in b\{1\}$ and $c^- \in c\{1\}$.*

Lemma 1.1 shows some close relation between $S^{\|(b,c)}$ and $H_{(b,c)}$ as follows.

**Lemma 2.2.** *Let $b, c \in S$.*

(1) *$S^{\|(b,c)} \neq \emptyset$ if and only if $S^{\{1\}} \cap H_{(b,c)} \neq \emptyset$. In this case, $H_{(b,c)} = \{a^{\|(b,c)} \mid a \in S^{\|(b,c)}\}$.*

(2) *If $S^{\|(b,c)} \neq \emptyset$, then $h$ is the $(b, c)$-inverse of all elements in $h\{1\}$, for any $h \in H_{(b,c)}$.*

*Proof.* The proof is straightforward. □

**Lemma 2.3.** *Let $b, c \in S$. If $a \in S^{\|(b,c)}$, then $d \in S^{\|(b,c)}$ with $d^{\|(b,c)} = a^{\|(b,c)}$ if and only if $d \in a^{\|(b,c)}\{1\}$.*

*Proof.* If $d \in S^{\|(b,c)}$ with $d^{\|(b,c)} = a^{\|(b,c)}$, then we have $a^{\|(b,c)}da^{\|(b,c)} = d^{\|(b,c)}dd^{\|(b,c)} = d^{\|(b,c)}$. Conversely, if $d \in a^{\|(b,c)}\{1\}$, then $a^{\|(b,c)}da^{\|(b,c)} = a^{\|(b,c)}$. Since $a^{\|(b,c)} \in H_{(b,c)}$, it follows that $d \in S^{\|(b,c)}$ with $d^{\|(b,c)} = a^{\|(b,c)}$ by Lemma 1.1. □

The previous lemmas inspire us to define an equivalent relation on $S^{\|(b,c)}$.

**Proposition 2.4.** *Let $b, c \in S$ such that $S^{\|(b,c)} \neq \emptyset$. For any $a, d \in S^{\|(b,c)}$, define a binary relation $\tau$ as:*

$$\tau = \{(a, d) \in S^{\|(b,c)} \times S^{\|(b,c)} \mid a^{\|(b,c)} = d^{\|(b,c)}\}.$$

*Then $\tau$ is an equivalent relation on $S^{\|(b,c)}$ and $\{h\{1\} \mid h \in H_{(b,c)}\}$ is a partition of $S^{\|(b,c)}$.*

*In this case,*

$$S^{\|(b,c)} = \bigcup_{a \in S^{\|(b,c)}} a^{\|(b,c)}\{1\} = \bigcup_{h \in H_{(b,c)}} h\{1\}.$$

*Proof.* The reflexivity, symmetry and transitivity of $\tau$ are easy to verify. So $\tau$ is an equivalent relation on $S^{\|(b,c)}$. For any $a \in S^{\|(b,c)}$, $a^{\|(b,c)}\{1\}$ is its equivalent class by Lemma 2.3. Then we get that

$$S^{\|(b,c)} = \bigcup_{a \in S^{\|(b,c)}} a^{\|(b,c)}\{1\}.$$

Meanwhile, by Lemma 2.2, we know that $H_{(b,c)} = \{a^{\|(b,c)} \mid a \in S^{\|(b,c)}\}$. It follows that

$$\bigcup_{a \in S^{\|(b,c)}} a^{\|(b,c)}\{1\} = \bigcup_{h \in H_{(b,c)}} h\{1\}.$$

Clearly, $h\{1\} \neq \emptyset$ for any $h \in H_{(b,c)}$. If $h\{1\} \cap g\{1\} \neq \emptyset$ for some $h, g \in H_{(b,c)}$, suppose that $a \in h\{1\} \cap g\{1\}$. Then we have $h = a^{\|(b,c)} = g$ by Lemma 2.2. Thus $\{h\{1\} \mid h \in H_{(b,c)}\}$ is a partition of $S^{\|(b,c)}$. □

From Lemma 2.2 and Proposition 2.4, we can see that the structure of $H_{(b,c)}$ is easier to handle than $S^{\|(b,c)}$. The following well-known result, which is called Green's theorem, gives an equivalent condition for an $\mathcal{H}$-class to be a group.

**Lemma 2.5.** ([12, Theorem 7]) *Let $a \in S$. Then $H_a$ is a group if and only if $a\mathcal{H}a^2$.*

A direct corollary of Lemma 2.5 is: if $H_a$ is a subsemigroup of $S$, then it must be a group. So we only discuss the group structure of an $\mathcal{H}$-class.

Recall that $a \in S$ is group invertible if there exists $x \in S$ such that

$$xa^2 = a, \ ax^2 = x, \ ax = xa.$$

If such $x$ exists, then it is unique and called the group inverse of $a$ (denoted by $a^\#$). An element $e$ satisfying $e^2 = e$ is called an idempotent, obviously $e$ is group invertible with $e^\# = e$. The sets of all idempotents and group invertible elements in $S$ are denoted by $E(S)$ and $S^\#$, respectively.

We note that $a \in S$ is group invertible if and only if $a\mathcal{H}a^2$.

**Lemma 2.6.** ([1, Lemma 1]) *Let $a \in S$. Then $a \in S^\#$ if and only if $a \in Sa^2 \cap a^2S$. In this case, $a^\# = uav$ for any $u, v \in S$ such that $ua^2 = a^2v = a$.*

Combining the previous results, it is easy to see that

$$E(S) \cap H_{(b,c)} \neq \emptyset \Leftrightarrow H_{(b,c)} \text{ is a group} \Leftrightarrow S^\# \cap H_{(b,c)} \neq \emptyset.$$

We want to give some equivalent conditions for $H_{(b,c)}$ to be a group from the perspective of generalized inverses.

We know that $H_{(b,c)}$ is an $\mathcal{H}$-class in $S$, which can be restated as follows: if $w \in H_{(b,c)}$, then $H_{(b,c)} = H_w$. The following lemma comes directly from this fact and can also be found in [3, Remark 2.2 (1)] and [18, Proposition 1.4].

**Lemma 2.7.** *Let $b, c, d, w \in S$. If $w \in H_{(b,c)}$, then $d \in S^{\|(b,c)}$ if and only if $d \in S^{\|w}$. In this case,*

$$d^{\|(b,c)} = d^{\|w}.$$

Lemma 2.7 provides a way to express the $(b, c)$-inverse as the inverse along some element, so we can use it to get equivalent conditions for the $(b, c)$-inverse of some element to be group invertible.

**Proposition 2.8.** *Let $b, c \in S$. If $a \in S^{\|(b,c)}$, then the following conditions are equivalent:*

(1) $a^{\|(b,c)} \in S^{\|(b,c)}$;

(2) $a^{\|(b,c)} \in S^\#$;

(3) $E(S) \cap H_{(b,c)} \neq \emptyset$.

*In this case, $H_{(b,c)}$ contains only one idempotent $e$ and*

$$(a^{\|(b,c)})^{\|(b,c)} = (a^{\|(b,c)})^\# = eae, \text{ and } e = a^{\|(b,c)}(a^{\|(b,c)})^\#.$$

*Proof.* (1) $\Leftrightarrow$ (2). By Lemma 2.7, we know that $a^{\|(b,c)} \in S^{\|(b,c)}$ if and only if $a^{\|(b,c)}$ is invertible along $a^{\|(b,c)}$, which is also equivalent to $a^{\|(b,c)} \in S^\#$ by [15, Theorem 11]. In this case,

$$(a^{\|(b,c)})^\# = (a^{\|(b,c)})^{\|a^{\|(b,c)}} = (a^{\|(b,c)})^{\|(b,c)}.$$

(2) $\Rightarrow$ (3). If $a^{\|(b,c)} \in S^\#$, then we have

$$a^{\|(b,c)}(a^{\|(b,c)})^\# S^1 = a^{\|(b,c)}S^1 = bS^1 \text{ and } S^1c = S^1a^{\|(b,c)} = S^1(a^{\|(b,c)})^\# a^{\|(b,c)}.$$

It means that $a^{\|(b,c)}(a^{\|(b,c)})^\# \in E(S) \cap H_{(b,c)}$. If $e$ and $f$ are two idempotents in $H_{(b,c)}$, then $e = f$ because $e\mathcal{H}f$.

(3) $\Rightarrow$ (2). Since $a^{\|(b,c)} \in H_{(b,c)} = H_e$, suppose that $e = xa^{\|(b,c)} = a^{\|(b,c)}y$. It follows that

$$(a^{\|(b,c)})^2y = a^{\|(b,c)}e = a^{\|(b,c)} = ea^{\|(b,c)} = x(a^{\|(b,c)})^2.$$

Thus, by Lemma 2.6, $a^{\|(b,c)} \in S^\#$ with

$$(a^{\|(b,c)})^\# = xa^{\|(b,c)}y = xa^{\|(b,c)}aa^{\|(b,c)}y = eae.$$

□

**Remark 2.9.** *According to [20, Theorem 3], we know that $E(S) \cap H_{(b,c)} \neq \emptyset$ if and only if $cb \in H_{(c,b)}$. If $S$ is a monoid, then $cb \in H_{(c,b)}$ is equivalent to $1 \in S^{\|(b,c)}$ by [11, Theorem 2.2]. As $1^{\|(b,c)}1^{\|(b,c)} = 1^{\|(b,c)}$, which means that $1^{\|(b,c)}$ is the identity element of $H_{(b,c)}$ in this case.*

If $H_{(b,c)}$ contains an idempotent $e$, we know that $e$ is the identity element in group $H_{(b,c)}$. An interesting fact is that $e$ is also $(b,c)$-invertible and it will play an important role in sequel discussion.

**Lemma 2.10.** *Let $b, c \in S$. If $H_{(b,c)}$ contains an idempotent $e$, then*

(1) *$e \in S^{\|(b,c)}$ with $e^{\|(b,c)} = e$ and $a^{\|(b,c)}ae = e = eaa^{\|(b,c)}$ for any $a \in S^{\|(b,c)}$;*

(2) *$ea, ae \in S^{\|(b,c)}$ with $(ea)^{\|(b,c)} = (ae)^{\|(b,c)} = a^{\|(b,c)}$ for any $a \in S^{\|(b,c)}$;*

(3) *$aed \in S^{\|(b,c)}$ with $(aed)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$ for any $a, d \in S^{\|(b,c)}$.*

*Proof.* (1) Since $eee = e$ and $e \in H_{(b,c)}$, it is clear that $e \in S^{\|(b,c)}$ with $e^{\|(b,c)} = e$. For any $a \in S^{\|(b,c)}$, we have $a^{\|(b,c)}ae = e = eaa^{\|(b,c)}$ by Lemma 2.1.

(2) Since $e$ is the identity element of $H_{(b,c)}$, it follows that

$$a^{\|(b,c)}aea^{\|(b,c)} = a^{\|(b,c)} = a^{\|(b,c)}eaa^{\|(b,c)},$$

which shows that $ae, ea \in a^{\|(b,c)}\{1\}$. By Lemma 2.3, $ae, ea \in S^{\|(b,c)}$ with $(ae)^{\|(b,c)} = (ea)^{\|(b,c)} = a^{\|(b,c)}$.

(3) It suffices to show that $d^{\|(b,c)}a^{\|(b,c)}$ is the $(b,c)$-inverse of $aed$. In fact, noting that $d^{\|(b,c)}a^{\|(b,c)} \in H_{(b,c)}$, we get that

$$d^{\|(b,c)}a^{\|(b,c)}aedd^{\|(b,c)}a^{\|(b,c)} = d^{\|(b,c)}edd^{\|(b,c)}a^{\|(b,c)} = d^{\|(b,c)}ea^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}.$$

So $aed \in S^{\|(b,c)}$ with $(aed)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$ by Lemma 2.3. □

Based on the previous results, we obtain some new equivalent conditions for $H_{(b,c)}$ to be a group.

**Theorem 2.11.** *Let $b, c \in S$. Then the following conditions are equivalent:*

(1) *$H_{(b,c)}$ is a group;*

(2) *$H_{(b,c)} \cap S^{\#} \neq \emptyset$;*

(3) *$H_{(b,c)} \cap S^{\|(b,c)} \neq \emptyset$;*

(4) *$\langle H_{(b,c)} \rangle \cap S^{\|(b,c)} \neq \emptyset$;*

(5) *$\{a \in S^{\#} \cap S^{\|(b,c)} \mid a^{\#} = a^{\|(b,c)}\} \neq \emptyset$;*

(6) *$\{a \in S^{\|(b,c)} \mid aa^{\|(b,c)} = a^{\|(b,c)}a\} \neq \emptyset$;*

(7) *$H_{(b,c)}S^{\|(b,c)} \cap S^{\|(b,c)} \neq \emptyset$;*

(8) *$S^{\|(b,c)}H_{(b,c)} \cap S^{\|(b,c)} \neq \emptyset$.*

*In this case, $H_{(b,c)} \subseteq S^{\#} \cap S^{\|(b,c)}$, $XS^{\|(b,c)} = eS^{\|(b,c)}$ and $S^{\|(b,c)}X = S^{\|(b,c)}e$ for any nonempty $X \subseteq H_{(b,c)}$, where $e$ is the identity element of $H_{(b,c)}$.*

*Proof.* (1) ⇔ (2). It is clear by Lemma 2.5.

(2) ⇔ (3). Suppose that $h \in H_{(b,c)}$. By a similar discussion as the proof of (1) ⇔ (2) in Proposition 2.8, we can prove that $h \in S^{\|(b,c)}$ if and only if $h \in S^{\#}$.

Thus, if $H_{(b,c)}$ is a group, then it is obvious that $H_{(b,c)} \subseteq S^{\#}$, which also implies that $H_{(b,c)} \subseteq S^{\|(b,c)}$ by the proof above.

(3) ⇒ (4). Obviously.

(4) ⇒ (2). Suppose that $y \in H_{(b,c)}$. Then $H_{(b,c)} \subseteq yS^1$ by definition. Noting that $yS^1$ is a subsemigroup of $S$, we have $\langle H_{(b,c)}\rangle \subseteq yS^1$. If $x \in \langle H_{(b,c)}\rangle \cap S^{\|(b,c)}$, then $x = ys$ for some $s \in S^1$. Meanwhile, since $x \in S^{\|(b,c)}$, it follows that $x \in S^{\|y}$ by Lemma 2.7. So

$$y = yxx^{\|y} = y^2sx^{\|y} \in y^2S.$$

Dually, we can prove that $y \in Sy^2$. This proves that $y \in S^{\#}$ by Lemma 2.6.

(1) ⇒ (5). If $H_{(b,c)}$ is a group, then its identity element $e \in S^{\|(b,c)}$ with $e^{\|(b,c)} = e$ by Lemma 2.10. And $e^2 = e$, so $e \in S^{\#}$ with $e^{\#} = e$. It follows that $e \in \{a \in S^{\#} \cap S^{\|(b,c)} \mid a^{\#} = a^{\|(b,c)}\}$.

(5) ⇒ (6). If $a \in S^{\#} \cap S^{\|(b,c)}$ such that $a^{\#} = a^{\|(b,c)}$, then $aa^{\|(b,c)} = aa^{\#} = a^{\#}a = a^{\|(b,c)}a$.

(6) ⇒ (2). If $a \in S^{\|(b,c)}$ such that $aa^{\|(b,c)} = a^{\|(b,c)}a$, combining with $a^{\|(b,c)}aa^{\|(b,c)} = a^{\|(b,c)}$, then we have $a^{\|(b,c)} \in S^{\#}$ by Lemma 2.6.

(1) ⇒ (7). If $H_{(b,c)}$ is a group with $e$ as its identity element, then $xa = xea$ for any $x \in H_{(b,c)}$ and $a \in S^{\|(b,c)}$. Since $x \in S^{\|(b,c)}$ by Proposition 2.8, it follows that $xa = xea \in S^{\|(b,c)}$ by Lemma 2.10.

Without loss of generality, we may prove $xS^{\|(b,c)} = eS^{\|(b,c)}$ for $x \in X$. It is clear that $xS^{\|(b,c)} \subseteq eS^{\|(b,c)}$ because $xa = exa$ and $xa \in S^{\|(b,c)}$, for any $a \in S^{\|(b,c)}$. Noting that $x \in S^{\|(b,c)}$ by Proposition 2.8, we have $ea = exx^{\|(b,c)}a = xx^{\|(b,c)}a \in xS^{\|(b,c)}$, for any $a \in S^{\|(b,c)}$. Since the choice of $x \in X$ is arbitrary, it follows that $XS^{\|(b,c)} = eS^{\|(b,c)}$.

(7) ⇒ (2). If $xa \in S^{\|(b,c)}$ for some $x \in H_{(b,c)}$ and $a \in S^{\|(b,c)}$, then $xa \in S^{\|x} = S^{\|a^{\|(b,c)}}$ by Lemma 2.7. On one hand, $x = xxa(xa)^{\|(b,c)} \subseteq x^2S$. On the other hand, we have $a^{\|(b,c)} = (xa)^{\|(b,c)}xaa^{\|(b,c)} = (xa)^{\|(b,c)}x$. It follows that

$$x = xaa^{\|(b,c)} = xa(xa)^{\|(b,c)}x = xa(xa)^{\|(b,c)}x^-x^2 \in Sx^2,$$

where $x^- \in x\{1\}$. So $x \in S^{\#}$ by Lemma 2.6.

(1) ⇒ (8) ⇒ (2). The proof is dual to that of (1) ⇒ (7) ⇒ (2). □

From the previous discussion, we know that $H_{(b,c)} \subseteq S^{\#} \cap S^{\|(b,c)}$ when $H_{(b,c)}$ is a group. However, the converse inclusion is not right in general.

**Example 2.12.** *In the semigroup* $S = \{\left[\begin{smallmatrix} x_1 & x_2 \\ 0 & 0 \end{smallmatrix}\right] | x_1, x_2 \in \mathbb{C}\}$. *Let* $a = \left[\begin{smallmatrix} 1 & -i \\ 0 & 0 \end{smallmatrix}\right]$ *and* $b = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. *Then* $a \in S^{\#}$ *with* $a^{\#} = \left[\begin{smallmatrix} 1 & -i \\ 0 & 0 \end{smallmatrix}\right]$ *and* $a \in S^{\|(a,b)}$ *with* $a^{\|(a,b)} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. *Obviously,* $a^{\|(a,b)} \in S^{\#}$, *it follows that* $H_{(a,b)}$ *is a group. But* $S^1a \neq S^1b$, *which means that* $a \notin H_{(a,b)}$.

It is natural to ask: when does a group invertible (or $(b,c)$-invertible) element belong to $H_{(b,c)}$? The following proposition answers this question and shows the structure of $H_{(b,c)}$.

**Proposition 2.13.** *Let* $b, c \in S$. *If* $H_{(b,c)}$ *contains an idempotent* $e$, *then*

(1) $eS^{\|(b,c)}$ *is a subsemigroup of* $S$ *and*

$$eS^{\|(b,c)} = \{a \in S^{\|(b,c)} \mid a = ea\};$$

(2) $S^{\|(b,c)}e$ *is a subsemigroup of* $S$ *and*

$$S^{\|(b,c)}e = \{a \in S^{\|(b,c)} \mid a = ae\};$$

(3)

$$\begin{aligned} H_{(b,c)} &= eS^{\|(b,c)}e = eS^{\|(b,c)} \cap S^{\|(b,c)}e \\ &= \{a \in S^{\|(b,c)} \mid a = ae = ea\} \\ &= \{a \in S^{\#} \cap S^{\|(b,c)} \mid a^{\#} = a^{\|(b,c)}\}. \end{aligned}$$

*Proof.* (1) It is not hard to check that

$$eS^{\|(b,c)} = \{a \in S^{\|(b,c)} \mid a = ea\}.$$

For any $a, d \in S^{\|(b,c)}$, we know that $eaed \in eS^{\|(b,c)}$ by Lemma 2.10.

(2) It can be proved similarly.

(3) Since $H_{(b,c)}$ is a group, we have that $H_{(b,c)} \subseteq S^{\|(b,c)}$, which leads to

$$H_{(b,c)} \subseteq \{a \in S^{\|(b,c)} \mid a = ae = ea\}.$$

It is obvious that $\{a \in S^{\|(b,c)} \mid a = ae = ea\} \subseteq eS^{\|(b,c)}e$.

And for any $a \in S^{\|(b,c)}$, since $H_{(b,c)}$ contains an idempotent $e$, we have $a^{\|(b,c)} \in S^{\#}$ with $(a^{\|(b,c)})^{\#} = eae$ by Proposition 2.8. Then $eae \in S^{\#}$ with $(eae)^{\#} = a^{\|(b,c)}$. Meanwhile, $eae \in S^{\|(b,c)}$ with $(eae)^{\|(b,c)} = a^{\|(b,c)}$ by Lemma 2.10. This proves that

$$eS^{\|(b,c)}e \subseteq \{a \in S^{\#} \cap S^{\|(b,c)} \mid a^{\#} = a^{\|(b,c)}\}.$$

If $a \in S^{\#} \cap S^{\|(b,c)}$ and $a^{\#} = a^{\|(b,c)}$, we know that

$$aS^1 = aa^{\#}S^1 = a^{\#}aS^1 = a^{\|(b,c)}aS^1 = bS^1$$

and

$$S^1a = S^1a^{\#}a = S^1aa^{\#} = S^1aa^{\|(b,c)} = S^1c.$$

So $\{a \in S^{\#} \cap S^{\|(b,c)} \mid a^{\#} = a^{\|(b,c)}\} \subseteq H_{(b,c)}$.

Combining the above inclusions, we have that

$$H_{(b,c)} = \{a \in S^{\|(b,c)} \mid a = ae = ea\} = eS^{\|(b,c)}e = \{a \in S^{\#} \cap S^{\|(b,c)} \mid a^{\#} = a^{\|(b,c)}\}.$$

It is clear that $eS^{\|(b,c)}e = eS^{\|(b,c)} \cap S^{\|(b,c)}e$. $\square$

If $H_{(b,c)}$ contains an idempotent, we know that $H_{(b,c)} \subseteq S^{\|(b,c)}$. It is easy to see that $H_{(b,c)}$ is a maximal subgroup of $S$, so we have the following result.

**Proposition 2.14.** *Let $b, c \in S$ such that $E(S) \cap H_{(b,c)} \neq \emptyset$. Then $S^{\|(b,c)}$ is a proper subgroup of $S$ if and only if $S^{\|(b,c)} = H_{(b,c)}$.*

If $S$ has the identity element, then we get an interesting result as follows.

**Corollary 2.15.** *Let $S$ be a monoid and $b, c \in S$. Then $S^{\|(b,c)}$ is a proper subgroup of $S$ such that $1 \in S^{\|(b,c)}$ if and only if $b$ is right invertible and $c$ is left invertible.*

Next we consider under what condition $S^{\|(b,c)}$ becomes a subsemigroup of $S$.

Wang et al. [23, Theorem 4.4] gave a criterion for two given $(b, c)$-invertible elements satisfying the reverse order law of $(b, c)$-inverses. We want to find a criterion for all elements in $S^{\|(b,c)}$ to satisfy the reverse order law of $(b, c)$-inverses.

Let $R$ be a unitary ring and $d, x, y \in R$. If $d \in R^{\#}$ and $x, y \in R^{\|d}$, Benítez and Boasso [2, Theorem 6.3] proved that $xy \in R^{\|d}$ with $(xy)^{\|d} = y^{\|d}x^{\|d}$ if and only if $dd^{\#}x(1 - dd^{\#})ydd^{\#} = 0$ by using Pierce decomposition. We have the following result for a semigroup $S$.

**Theorem 2.16.** *Let $b, c \in S$. Then $S^{\|(b,c)}$ is a subsemigroup of $S$ with reverse order law holding for $(b, c)$-inverses if and only if $H_{(b,c)}$ contains an idempotent $e$ such that*

$$eade = eaede$$

*for all $a, d \in S^{\|(b,c)}$.*

*Proof.* If $S^{\|(b,c)}$ is a subsemigroup of $S$ with the reverse order law holding for $(b, c)$-inverses. Suppose that $a \in S^{\|(b,c)}$. Since the reverse order law holds for $(b, c)$-inverse, it follows that $(a^2)^{\|(b,c)} = (a^{\|(b,c)})^2$. Meanwhile, according to Lemma 2.7, $a^2 \in S^{\|a^{\|(b,c)}}$. Then we have

$$a^{\|(b,c)} = a^{\|(b,c)}a^2(a^{\|(b,c)})^2 \in S(a^{\|(b,c)})^2.$$

Similarly, $a^{\parallel(b,c)} \in (a^{\parallel(b,c)})^2 S$. This proves that $a^{\parallel(b,c)} \in S^{\#}$. So there exists an idempotent $e$ in $H_{(b,c)}$ by Proposition 2.8.

If $d \in S^{\parallel(b,c)}$, then by reverse order law we know that $(ad)^{\parallel(b,c)} = d^{\parallel(b,c)} a^{\parallel(b,c)}$. Because $ad \in S^{\parallel e}$, we have

$$e = eadd^{\parallel(b,c)} a^{\parallel(b,c)}.$$

Multiplying by $aede$ on the right of above equality yields that

$$eaede = eadd^{\parallel(b,c)} a^{\parallel(b,c)} aede = eadd^{\parallel(b,c)} de = eade.$$

Conversely, if there exists an idempotent $e$ in $H_{(b,c)}$ such that $eade = eaede$ for all $a, d \in S^{\parallel(b,c)}$, then $d^{\parallel(b,c)} a^{\parallel(b,c)} \in H_{(b,c)}$ because $H_{(b,c)}$ is a group. Meanwhile,

$$d^{\parallel(b,c)} a^{\parallel(b,c)} add^{\parallel(b,c)} a^{\parallel(b,c)} = d^{\parallel(b,c)} a^{\parallel(b,c)} aedd^{\parallel(b,c)} a^{\parallel(b,c)} = d^{\parallel(b,c)} a^{\parallel(b,c)}.$$

Thus, $ad \in S^{\parallel(b,c)}$ with $(ad)^{\parallel(b,c)} = d^{\parallel(b,c)} a^{\parallel(b,c)}$ by Lemma 2.3. □

We have given equivalent conditions for $H_{(b,c)}$ to be a group and $S^{\parallel(b,c)}$ to be a semigroup with the reverse order law holding. Next we give some semigroups between $H_{(b,c)}$ and $S^{\parallel(b,c)}$.

**Proposition 2.17.** *Let $b, c \in S$. If $H_{(b,c)}$ contains an idempotent $e$, then*

$$M_{(b,c)} := \{a \in S^{\parallel(b,c)} \mid ae = ea\} = \{a \in S^{\parallel(b,c)} \mid aa^{\parallel(b,c)} = a^{\parallel(b,c)} a\},$$

$$M^r_{(b,c)} := \{a \in S^{\parallel(b,c)} \mid ae = eae\} = \{a \in S^{\parallel(b,c)} \mid aa^{\parallel(b,c)} = e\}$$

*and*

$$M^l_{(b,c)} := \{a \in S^{\parallel(b,c)} \mid ea = eae\} = \{a \in S^{\parallel(b,c)} \mid a^{\parallel(b,c)} a = e\}$$

*are semigroups containing $H_{(b,c)}$. Moreover, $M_{(b,c)} = M^r_{(b,c)} \cap M^l_{(b,c)}$.*

*Proof.* It is obvious that $H_{(b,c)} \subseteq M_{(b,c)}$. We first prove that $\{a \in S^{\parallel(b,c)} \mid ae = ea\} = \{a \in S^{\parallel(b,c)} \mid aa^{\parallel(b,c)} = a^{\parallel(b,c)} a\}$. Suppose that $a \in S^{\parallel(b,c)}$ such that $aa^{\parallel(b,c)} = a^{\parallel(b,c)} a$. Since $a^{\parallel(b,c)} aa^{\parallel(b,c)} = a^{\parallel(b,c)}$, it follows that $a^{\parallel(b,c)} \in S^{\#}$ with $(a^{\parallel(b,c)})^{\#} = aa^{\parallel(b,c)} a$ by Lemma 2.6. Then $e = a^{\parallel(b,c)} (a^{\parallel(b,c)})^{\#} = a^{\parallel(b,c)} a = aa^{\parallel(b,c)}$ by Proposition 2.8. So we have $ae = aa^{\parallel(b,c)} a = ea$. Conversely, if $ae = ea$ for some $a \in S^{\parallel(b,c)}$, then

$$aa^{\parallel(b,c)} = aea^{\parallel(b,c)} = eaa^{\parallel(b,c)} = e = a^{\parallel(b,c)} ae = a^{\parallel(b,c)} ea = a^{\parallel(b,c)} a.$$

Then we check that $\{a \in S^{\parallel(b,c)} \mid ae = eae\} = \{a \in S^{\parallel(b,c)} \mid aa^{\parallel(b,c)} = e\}$. In fact, multiplying by $a^{\parallel(b,c)}$ on the right of $ae = eae$ yields that $aa^{\parallel(b,c)} = e$. Conversely, if $aa^{\parallel(b,c)} = e$, then we have $ae = aa^{\parallel(b,c)} ae = eae$.

Similarly, $\{a \in S^{\parallel(b,c)} \mid ea = eae\} = \{a \in S^{\parallel(b,c)} \mid a^{\parallel(b,c)} a = e\}$.

Next, we prove that $M_{(b,c)}$ is a subsemigroup of $S$. If $a, d \in M_{(b,c)}$, then $eade = eeade = eaede$, which follows that $ad \in S^{\parallel(b,c)}$ according to the proof of Theorem 2.16. Meanwhile, we have $ade = aed = ead$. So $ad \in M_{(b,c)}$. It can be proved similarly that $M^r_{(b,c)}$ and $M^l_{(b,c)}$ are subsemigroups of $S$.

Finally, we show that $M_{(b,c)} = M^r_{(b,c)} \cap M^l_{(b,c)}$. It is clear that $M_{(b,c)} \subseteq M^r_{(b,c)} \cap M^l_{(b,c)}$ because $ae = ea = eea = eae$ and $ea = ae = aee = eae$ for any $a \in M_{(b,c)}$. If $a \in M^r_{(b,c)} \cap M^l_{(b,c)}$, then $ae = eae = ea$. This proves that $M^r_{(b,c)} \cap M^l_{(b,c)} \subseteq M_{(b,c)}$. □

**Remark 2.18.** *(1) $H_{(b,c)}$ may be a proper subgroup of $M_{(b,c)}$. For example, in the semigroup $S = \{\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ 0 & 0 & 0 \end{bmatrix} \mid x_i \in \mathbb{C}$ for $1 \leq i \leq 6\}$, let $a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $a \in S^{\parallel(b,b)}$ with $a^{\parallel(b,b)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Evidently, $a^{\parallel(b,b)} \in S^{\#}$, it follows that $H_{(b,b)}$ is a group with the identity element $e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We can verify that $ae = ea$, but $S^1 a \neq S^1 b$. This illustrates that $a$ is in $M_{(b,b)}$ but not in $H_{(b,b)}$.*

*(2) $M_{(b,c)}$ may be a proper subsemigroup of $M^r_{(b,c)}$. For example, considering in the semigroup $S = \mathbb{Z}^{2\times 2}$, let $a = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Then $a \in S^{\parallel(a,b)}$ with $a^{\parallel(a,b)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Clearly, $a^{\parallel(a,b)} \in S^{\#}$, so $H_{(a,b)}$ is a group with the identity element $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to check that $aa^{\parallel(a,b)} = e \neq a^{\parallel(a,b)} a$, which means that $a \in M^r_{(a,b)}$ but $a \notin M_{(a,b)}$.*

## 3. The $(b, c)$-invertibility of a special triple product

Let $a, b, c, d, e \in S$. If $a \in S^{\|d}$, Zhu et al. [26, Theorem 3.19] proved that $b \in S^{\|d}$ if and only if $bda \in S^{\|d}$ if and only if $adb \in S^{\|d}$. Mosić et al. [22, Theorem 2.9] proved that $a \in S^{\|(b,c)}$ if and only if $abd \in S^{\|(b,c)}$ when $d \in S^{\|(b,b)}$, as well as $a \in S^{\|(b,c)}$ if and only if $eca \in S^{\|(b,c)}$ when $e \in S^{\|(c,c)}$. The following theorem generalizes above results.

**Theorem 3.1.** *Let $a, b, c, d, x, y, z \in S$.*

(1) *If $x \in S^{\|(a,c)}$ and $y \in S^{\|(b,c)}$, then $z \in S^{\|(b,d)}$ if and only if $zy^{\|(b,c)}x \in S^{\|(a,d)}$.*

(2) *If $z \in S^{\|(b,d)}$ and $y \in S^{\|(b,c)}$, then $x \in S^{\|(a,c)}$ if and only if $zy^{\|(b,c)}x \in S^{\|(a,d)}$.*

*In these cases,*

$$(zy^{\|(b,c)}x)^{\|(a,d)} = x^{\|(a,c)}yz^{\|(b,d)},$$
$$z^{\|(b,d)} = y^{\|(b,c)}x(zy^{\|(b,c)}x)^{\|(a,d)}$$

*and*

$$x^{\|(a,c)} = (zy^{\|(b,c)}x)^{\|(a,d)}zy^{\|(b,c)}.$$

*Proof.* (1) If $z \in S^{\|(b,d)}$, we only need to show that $zy^{\|(b,c)}x \in S^{\|(a,d)}$ with $(zy^{\|(b,c)}x)^{\|(a,d)} = x^{\|(a,c)}yz^{\|(b,d)}$. Actually, we have

$$d(zy^{\|(b,c)}x)(x^{\|(a,c)}yz^{\|(b,d)}) = dzy^{\|(b,c)}yz^{\|(b,d)} = dzz^{\|(b,d)} = d$$

and

$$(x^{\|(a,c)}yz^{\|(b,d)})(zy^{\|(b,c)}x)a = x^{\|(a,c)}yy^{\|(b,c)}xa = x^{\|(a,c)}xa = a.$$

Obviously, $x^{\|(a,c)}yz^{\|(b,d)} \in aS \cap Sd$.

Conversely, if $zy^{\|(b,c)}x \in S^{\|(a,d)}$, it can be verified that

$$dz[y^{\|(b,c)}x(zy^{\|(b,c)}x)^{\|(a,d)}] = d$$

and

$$\begin{aligned}
& [y^{\|(b,c)}x(zy^{\|(b,c)}x)^{\|(a,d)}]zb \\
= \; & y^{\|(b,c)}x(zy^{\|(b,c)}x)^{\|(a,d)}zy^{\|(b,c)}yb \\
= \; & y^{\|(b,c)}x(zy^{\|(b,c)}x)^{\|(a,d)}zy^{\|(b,c)}xx^{\|(a,c)}yb \\
= \; & y^{\|(b,c)}xx^{\|(a,c)}yb \\
= \; & y^{\|(b,c)}yb \\
= \; & b.
\end{aligned}$$

Meanwhile, it is easy to see that $y^{\|(b,c)}x(zy^{\|(b,c)}x)^{\|(a,d)} \in bS \cap Sd$. So $z \in S^{\|(b,d)}$ with $z^{\|(b,d)} = y^{\|(b,c)}x(zy^{\|(b,c)}x)^{\|(a,d)}$.

(2) The proof is dual to that of (1). □

By Lemma 2.2, we can restate Theorem 3.1 as follows.

**Corollary 3.2.** *Let $a, b, c, d, x, z \in S$.*

(1) *If $b\mathcal{D}c$ and $S^{\|(a,c)} \neq \emptyset$, then $z \in S^{\|(b,d)}$ if and only if $zH_{(b,c)}S^{\|(a,c)} \subseteq S^{\|(a,d)}$.*

(2) *If $b\mathcal{D}c$ and $S^{\|(b,d)} \neq \emptyset$, then $x \in S^{\|(a,c)}$ if and only if $S^{\|(b,d)}H_{(b,c)}x \subseteq S^{\|(a,d)}$.*

As a special case of Theorem 3.1, we have the following result.

**Corollary 3.3.** *Let $b, c, d, x, z \in S$.*

(1) *If $x \in S^{\|(b,c)}$, then $z \in S^{\|(c,d)}$ if and only if $zcx \in S^{\|(b,d)}$.*

(2) *If* $z \in S^{\|(c,d)}$*, then* $x \in S^{\|(b,c)}$ *if and only if* $zcx \in S^{\|(b,d)}$.

*In these cases,*

$$(zcx)^{\|(b,d)} = x^{\|(b,c)}c^{-}z^{\|(c,d)},\ z^{\|(c,d)} = cx(zcx)^{\|(b,d)},\ x^{\|(b,c)} = (zcx)^{\|(b,d)}zc,$$

*for any* $c^{-} \in c\{1\}$.

Let $a, d \in S$. Mary and Patrício [19, Corollary 2.5] proved that $a \in S^{\|d}$ if and only if $H_d a H_d = H_d$. We have an analogous result for $(b, c)$-inverses.

**Proposition 3.4.** *Let* $a, b, c, d, y \in S$*. If* $a\mathcal{D}c$ *and* $b\mathcal{D}d$*, then the following conditions are equivalent:*

(1) $y \in S^{\|(b,c)}$;

(2) $H_{(a,c)}yH_{(b,d)} \subseteq H_{(a,d)}$;

(3) $H_{(a,c)}yH_{(b,d)} = H_{(a,d)}$.

*Proof.* (1) ⇔ (2). If $u \in H_{(a,c)}$ and $w \in H_{(b,d)}$, then $wS^1 = bS^1$ and $S^1u = S^1c$ by definition. It follows that $y \in S^{\|(b,c)}$ if and only if $y \in S^{\|(w,u)}$ with $y^{\|(b,c)} = y^{\|(w,u)}$. By [11, Theorem 2.2], we know that $y \in S^{\|(w,u)}$ if and only if $uyw \in H_{(u,w)}$. And $H_{(u,w)} = H_{(a,d)}$ because $uS^1 = aS^1$ and $S^1w = S^1d$. Thus, $y \in S^{\|(b,c)}$ if and only if $H_{(a,c)}yH_{(b,d)} \subseteq H_{(a,d)}$.

(2) ⇒ (3). If $u \in H_{(a,c)}$ and $w \in H_{(b,d)}$ such that $uyw \in H_{(a,d)}$, then we know that $y \in S^{\|(w,u)}$ from the proof above. It follows that $uyw$ is regular by [23, Proposition 3.3]. So we have $h = h(uyw)^{-}uyw$ for any $h \in H_{(a,d)}$ by Lemma 2.1. Noting that $(uyw)^{-} \in S^{\|(a,d)}$ from Lemma 2.2, it can be proved that $h(uyw)^{-}u \in H_{(a,c)}$ by a similar discussion as (1) ⇔ (2). Thus, $H_{(a,d)} \subseteq H_{(a,c)}yH_{(b,d)}$.

(3) ⇒ (2). Obviously. □

**Remark 3.5.** *If* $u \in H_{(a,c)}$ *and* $w \in H_{(b,d)}$ *such that* $uyw \in H_{(a,d)}$*, then* $y \in S^{\|(w,u)} = S^{\|(b,c)}$ *with* $y^{\|(w,u)} = y^{\|(b,c)}$ *by the proof above. According to [24, Theorem 2.7], we have* $y^{\|(b,c)} = y^{\|(w,u)} = w(uyw)^{-}u$ *for any* $(uyw)^{-} \in (uyw)\{1\}$.

In a ring $R$, if $x, y \in R^{\|(b,c)}$, then $x^{\|(b,c)}xy^{\|(b,c)} = y^{\|(b,c)}$ and $x^{\|(b,c)}yy^{\|(b,c)} = x^{\|(b,c)}$ by Lemma 2.1. It follows that

$$x^{\|(b,c)}(x + y)y^{\|(b,c)} = x^{\|(b,c)} + y^{\|(b,c)}.$$

Combining this fact with Proposition 3.4, we obtain the following additive property.

**Corollary 3.6.** *Let* $b, c \in R$*. If* $x, y \in R^{\|(b,c)}$*, then* $x + y \in R^{\|(b,c)}$ *if and only if* $x^{\|(b,c)} + y^{\|(b,c)} \in H_{(b,c)}$*. In this case,*

$$(x + y)^{\|(b,c)} = y^{\|(b,c)}(x^{\|(b,c)} + y^{\|(b,c)})^{-}x^{\|(b,c)},$$

*for any* $(x^{\|(b,c)} + y^{\|(b,c)})^{-} \in (x^{\|(b,c)} + y^{\|(b,c)})\{1\}$.

Taking $a = b$ and $c = d$ in Theorem 3.1 and Proposition 3.4, we have the following result.

**Corollary 3.7.** *Let* $b, c \in S$.

(1) *If* $x, y, z \in S^{\|(b,c)}$*, then* $zy^{\|(b,c)}x \in S^{\|(b,c)}$.

(2) *If* $u, w \in H_{(b,c)}$ *and* $y \in S^{\|(b,c)}$*, then* $uyw \in H_{(b,c)}$.

As an application of Corollary 3.7, we can construct many completely regular subsemigroups of $S$.

**Proposition 3.8.** *Let* $b, c \in S$*. If* $\emptyset \neq Y \subseteq S^{\|(b,c)}$ *and* $\emptyset \neq X \subseteq H_{(b,c)}$*, then* $XS^{\|(b,c)}$*,* $S^{\|(b,c)}X$*,* $YH_{(b,c)}$ *and* $H_{(b,c)}Y$ *are completely regular subsemigroups of* $S$.

*Proof.* According to Corollary 3.7, it is easy to see that $XS^{\|(b,c)}XS^{\|(b,c)} \subseteq XS^{\|(b,c)}$ and $S^{\|(b,c)}XS^{\|(b,c)}X \subseteq S^{\|(b,c)}X$. Thus, $XS^{\|(b,c)}$ and $S^{\|(b,c)}X$ are subsemigroups of $S$.

Similarly, $YH_{(b,c)}YH_{(b,c)} \subseteq YH_{(b,c)}$ and $H_{(b,c)}YH_{(b,c)}Y \subseteq H_{(b,c)}Y$ imply that $YH_{(b,c)}$ and $H_{(b,c)}Y$ are subsemigroups of $S$ by Corollary 3.7.

Suppose that $x \in H_{(b,c)}$ and $a \in S^{\|(b,c)}$. According to Lemma 2.7, we have $a \in S^{\|x}$, which follows that $ax, xa \in S^{\#}$ by [15, Theorem 7]. This proves that all above semigroups are completely regular. □

Note that $XS^{\|(b,c)} = eS^{\|(b,c)}$ and $S^{\|(b,c)}X = S^{\|(b,c)}e$ when $H_{(b,c)}$ is a group with the identity element $e$ by Theorem 2.11.

We know that a completely regular semigroup can be expressed as the union of all its (maximal) subgroups. And it is clear that

$$YH_{(b,c)} = \bigcup_{a\in Y} aH_{(b,c)}.$$

An interesting fact is that $aH_{(b,c)}$ is indeed a subgroup of $YH_{(b,c)}$ by the following proposition.

**Proposition 3.9.** *Let $b, c \in S$. If $a \in S^{\|(b,c)}$, then*

$$aH_{(b,c)} = H_{aa^{\|(b,c)}} \text{ and } H_{(b,c)}a = H_{a^{\|(b,c)}a}.$$

*Proof.* We only prove that $aH_{(b,c)} = H_{aa^{\|(b,c)}}$ here, and the proof of $H_{(b,c)}a = H_{a^{\|(b,c)}a}$ is similar.

By [12, Lemma], we know that $\lambda_a : x \mapsto ax$ is a $\mathcal{L}$-class preserving bijection from $R_{a^{\|(b,c)}}$ to $R_{aa^{\|(b,c)}}$. It follows that

$$aH_{(b,c)} = a(R_{a^{\|(b,c)}} \cap L_{aa^{\|(b,c)}}) = \lambda_a(R_{a^{\|(b,c)}} \cap L_{aa^{\|(b,c)}}) = R_{aa^{\|(b,c)}} \cap L_{aa^{\|(b,c)}} = H_{aa^{\|(b,c)}}.$$

□

Meanwhile, according to [20, Theorem 3], we have

$$a^{\|(b,c)}H_{aa^{\|(b,c)}} = H_{(b,c)} = H_{a^{\|(b,c)}a}a^{\|(b,c)}.$$

By Proposition 3.9, we have the following corollary.

**Corollary 3.10.** *Let $b, c, e, f \in S$. If $a \in S^{\|(b,c)}$ and $d \in S^{\|(e,f)}$, then*

(1) *$aa^{\|(b,c)} = d^{\|(e,f)}d$ if and only if $aH_{(b,c)} = H_{(e,f)}d$;*

(2) *$aa^{\|(b,c)} = dd^{\|(e,f)}$ if and only if $aH_{(b,c)} = dH_{(e,f)}$.*

*Proof.* (1) If $aa^{\|(b,c)} = d^{\|(e,f)}d$, then $aH_{(b,c)} = H_{aa^{\|(b,c)}} = H_{d^{\|(e,f)}d} = H_{(e,f)}d$ by Proposition 3.9.

Conversely, if $aH_{(b,c)} = H_{(e,f)}d$, then $H_{aa^{\|(b,c)}} = aH_{(b,c)} = H_{(e,f)}d = H_{d^{\|(e,f)}d}$. So we have $aa^{\|(b,c)}\mathcal{H}d^{\|(e,f)}d$, which implies that $aa^{\|(b,c)} = d^{\|(e,f)}d$.

(2) can be proved similarly. □

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