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Exponential stability of second-order fractional stochastic integro-differential equations

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Abstract. In this paper studies the exponential stability result is derived for the second-order fractional stochastic integro-differential equations (FSIDEs) driven by sub-fractional Brownian motion (sub-fBm). By constructing a successive approximation method, we present pth moment exponential stability result of second-order FSIDEs using stochastic analysis techniques and fractional calculus (FC). At last, an example is demonstrated to illustrate the obtained theoretical result.

1. Introduction

FC was introduced around the nineteenth century by great mathematicians Riemann and Liouville. The theory of FC is a generalization of the integer order calculus specified by Leibnitz and L'Hospital in 1695. It has become expeditiously burgeoning area in optics and signal processing [16], electrical networks [31] and fluid flow [48]. In recent years, there has been a significant development in FC. Hence, there is a growing need to find the qualitative behaviors of the fractional differential equations (FDEs), for more details interested readers may refer the monographs [18, 20, 25, 28, 34, 40], articles [1–4, 6, 7, 9, 13, 30, 36, 38, 39, 41–43, 45, 51] and references cited therein. The concept of semigroups of bounded linear operators is taken as an important concept for dealing differential and integro-differential equations in infinite dimensional spaces (see, [1–3, 6]). On the other hand, in numerous mathematical models of real world or man made phenomena, we are led to dynamical systems which involve some inherent randomness. These systems are called stochastic systems. Stochastic differential equations (SDEs) have attracted much attention and have played an important role in many ways such as option pricing, forecast of the growth of population, etc. The modeling of most problems in real situations is described by stochastic differential equations rather than deterministic equations. Thus, it is of great importance to design stochastic effects in the study of fractional-order dynamical systems.

The focus on second-order equations is to study them directly rather than make them become first-order system, see [15, 44]. In many cases, it is advantageous to treat the second-order stochastic differential equations directly rather than converting them to first-order systems. A variety of problems arising in mechanics, elasticity theory, molecular dynamics and quantum mechanics can be described in general by second-order nonlinear differential equations. The second-order differential equations involving randomness are seem to be more accurate model in continuous time to account for integrated processes that can

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be made stationary. Due to this reason, focus on second-order differential equations are emerged in recent years.

In general, the fBm is a generalization of standard Bm that exhibits self-similarity, long-range dependence, and stationary increments. Many researchers studied stochastic differential equations with fBm, see [11, 17, 26, 29, 38] and devoted to study the mild solution of the second-order neutral stochastic differential equation with infinite delay driven by fBm of the following form

$$\begin{aligned} {}^{C}D^{\alpha}_{0^+}[y'(t) - f(t, y_t)] &= [\mathbb{A}(t)y(t) + g(t, y_t)]dt + h(t, y_t)dw(t) + \sigma(t)dB^H_Q(t),\ t \in J = [0, T] \\ y(0) &= \phi \in \mathcal{B}, \\ y'(0) &= \xi, \end{aligned} \tag{1}$$

where ${}^{C}D^{\alpha}_{0^+}$ refers the Caputo derivative of order $0 < \alpha < 1$ and $B^H_Q(t)$ is a fBm with Hurst index $H \in (0, 1)$. The above fBm model (1) as a self-similar Gaussian process and random field has exciting applications in many areas, including science, turbulence, and the financial market, see [5, 26]. Also, the Rosenblatt process appears as a stationary series of long-range dependent and self-similar processes, but it is not a Gaussian process see [13]. However, when the Gaussianity is plausible for the model in concrete situations, one can use the fBm. However, a huge number of literature for fBm and Rosenblatt processes exist. However, in 2004, Bojdecki et al. [8] proposed another improvement of the Brownian motion, which has all the properties of the fBm except stationary increments, and is called sub-fractional Brownian motion (sub-fBm in shortly). Compared with the fBm, increments of sub-fBm are correlated weakly in non-overlapping intervals, and their covariance decays rapidly as the distance between intervals tends to infinity. Because of this feature, it is obvious that the sub-fBm is more appropriate than the fBm to model the financial markets problems. The sub-fBm $(\zeta^H_\delta)_{\delta \in R^+}$ satisfies the following properties:

1. **Self-similarity:** for any $a \geq 0$,
$$(\zeta^H_{a\delta})_{\delta \in R^+} = a^H(\zeta^H_\delta)_{\delta \in R^+}.$$
2. **Long-rang dependence:** for any $H \in (\frac{1}{2}, 1)$, if we let $\rho(n) = cov(\zeta^H_1, \zeta^H_{n+1} - \zeta^H_n)$, then $\sum_{n=1}^{\infty} \rho(n) = \infty$.
3. **Varience:** $\forall \delta \in R^+, \forall H \in (0, 1), Var(\zeta^H_\delta) = \mathbb{E}[(\zeta^H_\delta)^2] = (2 - 2^{2H-1})\delta^{2H}$.
4. If $H \neq \frac{1}{2}$, the process $(\zeta^H_\delta)_{\delta \in R^+}$, is neither a markov process nor a semi-martingale.
5. The second-order moment increments of the process $(\zeta^H_\delta)_{\delta \in R^+}$ are not stationary, in the sense that, supposed $\delta > s$, the second-order moment increments are
$$\mathbb{E}[(\zeta^H_\delta - \zeta^H_s)^2] = -2^{2H-1}(\delta^{2H} + s^{2H}) + (\delta + s)^{2H} + (\delta - s)^{2H}.$$
Then, there is
$$\begin{aligned} &(\delta - s)^{2H} \leq \mathbb{E}[(\zeta^H_\delta - \zeta^H_s)^2] \leq (2 - 2^{2H-1})(\delta - s)^{2H}, H \in (0, \frac{1}{2}) \\ &(2 - 2^{2H-1})(\delta - s)^{2H} \leq \mathbb{E}[(\zeta^H_\delta - \zeta^H_s)^2] \leq (\delta - s)^{2H}, H \in (\frac{1}{2}, 1). \end{aligned}$$

One can write the real-time model of stock price process $\{S_\delta, \delta \geq 0\}$ involving sub-fBm as

$$dS_\delta = \mu S_\delta\, d\delta + \sigma S_\delta\, d\zeta^H_\delta$$

under the following assumptions:

1. The dynamic of the underlying stock prices follow the sub-fBm $(\zeta^H_\delta)_{\delta \in R^+}$.
2. Before the maturity time of the options, the risk-free interest rate r, the expected return of the stock μ, and the volatility of the stock price σ are all constants.
3. There are no transaction costs or taxes in buying or selling the stocks or options (i.e., the market is frictionless).

4. The dividends paid by the stocks before the maturity time of the options are zero (i.e., the stocks are dividend-free).
5. The option can be exercised only at the maturity time.

Therefore, some other generalizations of fBm and Rosenblatt process to be introduced. There has been a thorough investigation of self-similar Gaussian processes. The major reason for the complexity of self-similar Gaussian processes is that they do not have a stationary increment process. Henceforth, a mechanism of sub-fBm has been introduced, which is an intermediate between Brownian motion and fBm. This process arises from occupation on the branching particle time fluctuations with Poisson initial condition; for convenience, the reader may refer [8, 33, 47]. We provide comparisons with fBm and sub-fBm in the following table.

S. No	**fBm**	**sub-fBm**
1.	Long-memory if $H < 1$	Short memory
2.	The mixed processes = sums of independent Bm and fBm	The mixed processes = independent Bm and sub-fBm
3.	Is Dirichlet if $H < 1$	Quasi-Dirichlet if $H < 1$
4.	mean zero and covariance function $\min(s, t)$	Similar.

Due to the exciting property of sub-fBm, few results have been analyzed SDEs with sub-fBm, for more details readers may refer [8, 10, 14, 24, 33, 35, 47] and references therein. In the last few years, some significant works have been done on the stability analysis of FSDEs driven by fBm; for details, see [7, 9, 12, 13, 19, 21–23, 38, 39, 49, 50]. Inspired by the above works, we study the p^{th} moment exponential stability of FSIDEs driven by sub-fBm. Best of authors knowledge, so far only a very few works exist in the literature related to the exponential stability of FSIDEs driven by sub-fBm.

The major contributions of this manuscript are listed as below:

- The well-posedness of FSIDEs is proved in stochastic settings.
- A contemporary integral inequality technique is used together with the successive approximation method.
- The exponential stability result is obtained in the p^{th} moment norm for the second-order FSIDEs driven by sub-fBm.
- An example is provided to validate the obtained results.

This article is summarized as follows: Section 2 deals with problem formation and fundamental theories which will be used in the sequel. The exponential stability of FSIDEs driven by sub-fBm is studied in section 3. An example is provided in section 4 to validate the efficiency of the obtained theoretical result. Finally, the conclusion is drawn in section 5.

2. Model Description

In this section, we focus on the following FSIDEs driven by sub-fBm,

$$
\begin{aligned}
{}^{C}D^{\alpha}_{0^+}x(\delta) &= \mathbb{A}x(\delta) + f(\delta, x_\delta) + \int_0^\delta g(\tau, x_\tau)dw(\tau) + \sigma(\delta)dS^H_Q(\delta),\ \delta \in J := [0, T],\ T > 0 \\
x(\delta) &= \phi_0,\ x'(\delta) = \phi_1, \quad -r \le \delta \le 0,
\end{aligned}
\tag{2}
$$

where $x(\delta)$ denotes the state variable takes values in a Hilbert space $\mathcal{H}$ with the inner product $<\cdot,\cdot>_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$, ${}^{C}D^{\alpha}_{0^+}$ denotes the Caputo derivative of order $1 < \alpha < 2$; $\mathbb{A} : \mathbb{D}(\mathbb{A}) \subset \mathcal{H} \to \mathcal{H}$ generates the infinitesimal generator of an cosine families $\{\mathbb{S}_\alpha(\delta)\}_{\delta\geq 0}$ and related sine families $\{\mathbb{T}_\alpha(\delta)\}_{\delta\geq 0}$ of operators on the Hilbert space $\mathcal{H}$. Let $\mathbb{K}$ be the separable Hilbert space with inner product $<\cdot,\cdot>_{\mathbb{K}}$ and the norm $\|\cdot\|_{\mathbb{K}}$. Let $\mathbb{C} := C([-r,0],\mathcal{H})$ be the space of all continuous functions $x : [-r,0] \to \mathcal{H}$ with norm defined by $\|x(s)\|^p = \sup_{s\in[-r,0]} \|x(s)\|^p$. Also, for $x \in C([-r,T],\mathcal{H})$, we have $x_\delta \in \mathbb{C}$ for $\delta \in J$, $x_\delta(s) = x(\delta+s)$ for $s \in [-r,0]$. Here, $S^H_Q(\delta)$ denotes an Q-sub-fBm with Hurst index $H \in (\frac{1}{2},1)$ and $\{w(\delta) : \delta \in J\}$ is a standard Wiener process. ϕ_0 and ϕ_1 are $\mathfrak{F}_0$-measurable $\mathcal{H}$-valued random processes. The nonlinear maps $f : J\times\mathbb{C} \to \mathcal{H}$, $g : J\times\mathbb{C} \to L^0_Q(\mathbb{K},\mathcal{H})$ and $\sigma : J \to L^0_Q(\mathbb{K},\mathcal{H})$ are the appropriate continuous functions. Here, $L^0_Q(\mathbb{K},\mathcal{H})$ denote the Q-Hilbert Schmidt operators from $\mathbb{K} \to \mathcal{H}$. Let $\mathcal{B} = \hat{C}(J, L_p(\tilde{\Omega},\mathcal{H}))$ be the space of all continuous maps from J into $L_p(\tilde{\Omega},\mathcal{H})$ which is also a Banach space with norm $\|x(\delta)\| = \Big[\sup_{\delta\in J}\|x(\delta)\|^p\Big]^{\frac{1}{p}}$.

3. Preliminaries

In this section, we summarily recollect some elementary definitions of Riemann-Liouville (R-L) fractional derivative & integral and Caputo derivative. Also, the basic lemmas, and semi-group theory are highlighted, which are used in the sequel. Let $(\tilde{\Omega},\mathfrak{F},\mathcal{P})$ be a complete filtered probability space equipped with complete family of right continuous increasing sub σ-algebras $\{\mathfrak{F}_\delta, \delta \in J\}$ satisfying $\mathfrak{F}_\delta \subset \mathfrak{F}$, a $\mathcal{H}$-valued random variable of $\mathfrak{F}$-measurable function $x(\delta) : \tilde{\Omega} \to \mathcal{H}$. Let $\mathcal{S} = \{x(\delta,\omega) : \tilde{\Omega} \to \mathcal{H} : \delta \in J\}$ be a collection of random variables known as stochastic process and usually represented by $x(\delta)$ by suppressing $\omega \in \tilde{\Omega}$. Let $\{\gamma_n(\delta)\}_{n=1}^\infty$ be a real valued one-dimensional standard Bm independent of $(\tilde{\Omega},\mathfrak{F},\mathcal{P})$. Let $w(\delta) = \sum_{n=1}^\infty \sqrt{\lambda_n}\gamma_n(\delta)\zeta_n(\delta), \delta \geq 0$, where, $\lambda_n \geq 0$ are non-negative real numbers and $\{\zeta_n\}(n = 1,2,\dots)$ is complete orthonormal basis in $\mathbb{K}$. Let $Q \in L(\mathbb{K},\mathcal{H})$ be defined by $Q\zeta_n = \lambda_n\zeta_n$ with finite $Tr(Q) = \sum_{n=1}^\infty \lambda_n < \infty$. From the above, the $\mathbb{K}$-valued stochastic process $w(\delta)$ is the Q-Wiener process. Let $\Psi \in L^0_Q(\mathbb{K},\mathcal{H})$,

$$\|\Psi\|^2_Q = Tr(\Psi Q\Psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\Psi\zeta_n\|^2.$$

If $\|\Psi\|_Q < \infty$, then Ψ is known as Q-Hilbert Schmidt operator. For more details on concepts and theories on SDEs, one can refer the articles [27, 29, 33, 36, 37, 41] and references therein.

Definition 3.1. *[18] The R-L fractional integral of order q for a continuous function $f : J \to \mathbb{R}$ is given by,*

$$I^q_{0^+}f(\delta) = \frac{1}{\Gamma(q)}\int_0^\delta (\delta-s)^{q-1}f(s)ds, \quad \delta > 0,\ n-1 < q < n,$$

provided that the RHS is pointwise defined on J.

Definition 3.2. *[20] The R-L fractional derivative of order q, of a function $f : J \to \mathbb{R}$ is defined by,*

$$D^q_{0^+}f(\delta) = \frac{1}{\Gamma(n-q)}\Big(\frac{d}{d\delta}\Big)^n\int_0^\delta (\delta-s)^{n-q-1}f(s)ds, \quad \delta > 0,\ n-1 < q < n,$$

provided that the RHS is pointwise defined on J, where $n = [q]+1$, $[q]$ denotes the integral part of number q, and Γ is the usual Gamma function.

Definition 3.3. *[22] The Caputo fractional derivative of order q for a function $f : J \to \mathbb{R}$ is defined as,*

$${}^C D^q_{0^+}f(\delta) = \frac{1}{\Gamma(n-q)}\int_0^\delta (\delta-s)^{n-q-1}f^{(n)}(s)ds,\ n-1 < q < n,$$

where (n) denotes the n^{th} derivative, furnished that the RHS is pointwise defined on J.

Definition 3.4. *[23, 32, 46] The solution operator $\{\mathbb{S}_\alpha(\delta)\}_{\delta\geq 0}$ and $\mathbb{A}$ is the infinitesimal generator, if*

1. $\mathbb{S}_\alpha(0) = I$, *$\mathbb{S}_\alpha(\delta)$ is strongly continuous for $\delta \geq 0$;*
2. $\mathbb{A}\mathbb{S}_\alpha(\delta)\tau = \mathbb{S}_\alpha(\delta)\mathbb{A}\tau$ *and* $\mathbb{S}_\alpha(\delta)D(\mathbb{A}) \subset D(\mathbb{A})$ *and* $\forall\ \tau \in D(\mathbb{A})$, $\delta \geq 0$;
3. *The solution of system* (2) *is* $\mathbb{S}_\alpha(\delta)\tau$, $\forall\ \tau \in D(\mathbb{A})$.

Definition 3.5. *The sub-fBm is a continuous centered Gaussian process $(\zeta^H_\delta)_{\delta\in R^+}$, starting from zero, and with the covariance given by*

$$Cov(\zeta^H_\delta, \zeta^H_s) = \delta^{2H} + s^{2H} - \frac{1}{2}((\delta+s)^{2H} + |\delta - s|^{2H}), \forall \delta \in R^+, s \in R^+.$$

where Hurst index $H \in (0, 1)$.

Definition 3.6. *A $\{\mathfrak{F}_\delta\}_{\delta\geq 0}$-adapted X-valued stochastic process $x(\delta)$ $(\delta \in J)$ with cadlag path is known as mild solution of* (2) *provided the following integral equation holds*

$$x(\delta) = \begin{cases} \phi(\delta), \qquad \delta \in [-r, 0] \\ \mathbb{S}_\alpha(\delta)\phi_0 + \mathbb{T}_\alpha(\delta)\phi_1 + \int\limits_0^\delta \mathbb{T}_\alpha(\delta - s) f(s, x_s) ds + \int\limits_0^\delta \mathbb{T}_\alpha(\delta - s) \\ \times\Big(\int\limits_0^\tau g(\theta, x_\theta) dw(\theta)\Big) ds + \int\limits_0^\delta \mathbb{T}_\alpha(\delta - s)\sigma(s) dS^H_Q(s),\ \delta \in J. \end{cases}$$

Definition 3.7. *[13] The mild solution $x(\delta)$ of the given Cauchy problem* (2) *is called p^{th} moment exponentially stable $(p \geq 2)$, if $\exists$ +ve constants $\mu > 0$, $\hat{\oplus} \geq 1$*

$$\mathbb{E}\|x(\delta)\|^p \leq \hat{\oplus} e^{-\mu\delta}, \qquad \delta \geq 0,\ p \geq 2.$$

Lemma 3.8. *[27]* ***(Burkholder-Davis-Gundy inequality)*** *If $p \geq 2$, then the $L^0_Q(\mathbb{K}, \mathcal{H})$-valued predictable process $g(s)$ satisfies*

$$\sup_{s\in J} \mathbb{E}\Big\| \int\limits_0^\delta g(s) dw(s) \Big\|^p \leq C_p \left(\int\limits_0^\delta (\mathbb{E}\|g(s)\|^p_{L^0_Q})^{\frac{2}{p}} ds \right)^{\frac{p}{2}},\ \delta \in J,$$

where $C_p = \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}$, and $\mathbb{E}$ denotes the mathematical expectation.

Lemma 3.9. *[24] If $\sigma : J \to L^0_Q(\mathbb{K}, \mathbb{H})$ and satisfies $\int\limits_0^\delta \|\sigma(s)\|^p_{L^0_Q} ds < \infty$ for any $\delta \in [0, T]$, then*

$$\mathbb{E}\Big\| \int\limits_0^\delta \sigma(s) dS^H_Q(s) \Big\|^p_{\mathbb{H}} \leq CH\delta^{pH-1} \int\limits_0^\delta \mathbb{E}\|\sigma(s)\|^p_{L^0_Q} ds.$$

Lemma 3.10. *[9] Suppose that for $h > 0, \eta_1, \eta_2 \in (0, h]$, there exist constants $\xi_i > 0$ $(i = 1, 2, 3, 4)$ and a function $\psi : [-\tau, \infty) \to [0, \infty)$ s.t*

$$\psi(\delta) \leq \begin{cases} \xi_1 e^{-\eta_1\delta} + \xi_2 e^{-\eta_2\delta}, \quad \delta \in [-r, 0] \\ \xi_1 e^{-\eta_1\delta} + \xi_2 e^{-\eta_2\delta} + \xi_3 \int\limits_0^\delta e^{-\eta_2(\delta - s)} \sup\limits_{\theta\in[-r,0]} \psi(s+\theta) ds \\ +\xi_4 \int\limits_0^\delta e^{-\eta_2(\delta-s)} \sup\limits_{\theta\in[-r,0]} \psi(s+\theta) ds, \quad \delta \geq 0, \end{cases} \tag{3}$$

and if

$$\xi_4 \frac{e^{-\mu\delta_1}}{\eta_2 - \mu} - \xi_3 \frac{e^{-\eta_2\delta_1}}{\eta_2 - \mu} < 1 \tag{4}$$

then, we have

$$\psi(t) \leq N_\epsilon e^{-\mu\delta} \text{ for } \delta \geq -\tau, \tag{5}$$

where $\mu \in (0, \eta_1 \Lambda\ \eta_2)$ is a positive root of the equation $\xi_4 \frac{e^{-\mu\delta_1}}{\eta_2-\mu} - \xi_3 \frac{e^{-\eta_2\delta_1}}{\eta_2-\mu} = 1$ and

$$N_\epsilon = max\Big\{\xi_1 + \xi_2, \frac{(\eta_2 - \mu)\xi_2}{\xi_4 e^{\mu\delta_1}}\Big\} > 0.$$

In order to prove our main result, we enforce the following hypotheses.

(A_1) $\mathbb{A}$ generates cosine families of bounded linear operators $\mathbb{S}_\alpha(\delta)$ and related sine families of operators $\mathbb{T}_\alpha(\delta),\ \delta \geq 0$ on $\mathcal{H}$, and thus there exist non-negative constants a_1, a_2 s.t

$$\sup_{\delta\geq 0} \|\mathbb{S}_\alpha(\delta)\| \leq a_1$$

$$\sup_{\delta\geq 0} \|\mathbb{T}_\alpha(\delta)\| \leq a_2$$

(A_2) The nonlinear continuous functions $f : J \times \mathbb{C} \to \mathcal{H}$ and $g : J \times \mathbb{C} \to L_Q^0(\mathbb{K}, \mathcal{H})$ satisfy the Lipschitz condition, for all $\delta \in J,\ x_1, x_2 \in \mathcal{H}$, s.t

$$\mathbb{E}\|f(\delta, x_1(\delta)) - f(\delta, x_2(\delta))\|^p \bigvee \mathbb{E}\|g(s, x_1(s)) - g(s, x_2(s))\|^p \leq \rho(\delta)(\mathbb{E}\|x_1 - x_2\|^p)$$

where, $\rho(\cdot)$ is a concave non-decreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ s.t $\rho(0) = 0, \rho(\nu) > 0$ for $\nu > 0$ and $\int_{0^+} \frac{d\nu}{\rho(\nu)} = \infty$.

(A_3) $\forall\ \delta \in J$, $\exists$ a +ve constant N_0 s.t

(i) $\int_0^\delta \mathbb{E}\|f(s, 0)\|^p ds\ \bigvee \int_0^\tau \mathbb{E}\|g(s, 0)\|^p ds \leq N_0,$

(A_4) The mapping $\sigma : J \to L_Q^0(\mathbb{K}, \mathcal{H})$ satisfies

$$\int_0^\delta \mathbb{E}\|\sigma(s)\|^p ds < \infty.$$

(A_5) For strongly continuous cosine families $\mathbb{S}_\alpha(\delta)$ and sine families $\mathbb{T}_\alpha(\delta), \delta \geq 0$, $\exists$ +ve constants μ_1 and μ_2 with $\tilde{M}_1, \tilde{M}_2 > 1$ such that

$$\sup_{\delta\geq 0} \|\mathbb{S}_\alpha(\delta)\| \leq \tilde{M}_1 e^{-\mu_1\delta}$$
$$\sup_{\delta\geq 0} \|\mathbb{T}_\alpha(\delta)\| \leq \tilde{M}_2 e^{-\mu_2\delta}.$$

4. Main result

In order to prove the existence of mild solution for a given system (2), we construct the sequence of successive approximation as follows

$$x^0(\delta) = \mathbb{S}_\alpha(\delta)\phi_0 + \mathbb{T}_\alpha(\delta)\phi_1,\ \delta \in J$$
$$x^n(\delta) = \phi(\delta),\ \delta \in [-r, 0]$$
$$x^n(\delta) = \mathbb{S}_\alpha(\delta)\phi_0 + \mathbb{T}_\alpha(\delta)\phi_1 + \int_0^\delta \mathbb{T}_\alpha(\delta - s) f(s, x_s^{n-1})ds + \int_0^\delta \mathbb{T}_\alpha(\delta - s)\Big(\int_0^\tau g(\theta, x_\theta^{n-1})dw(\theta)\Big)ds$$
$$+ \int_0^\delta \mathbb{T}_\alpha(\delta - s)\sigma(s)dS_Q^H(s),\ \delta \in J,\ n \geq 1. \tag{6}$$

Theorem 4.1. *Assume that the hypotheses $(A_1) - (A_4)$ hold, then the given Cauchy problem* (2) *has a unique mild solution.*

Proof: For better readability, the proof is given by splitting into the following three steps
Step 1:
For all $\delta \in J$, $\{x^n(\delta)\}_{n=1}^{\infty}$, $n \geq 1$ is bounded. It is apparently $x^0(\delta) \in \mathcal{B}$. Let x^0 be a initial approximation and from (6), we have

$$\mathbb{E}\|x^n(\delta)\|^p = 5^{p-1}\Big\{\mathbb{E}\Big\|\mathbb{S}_\alpha(\delta)\phi_0 + \mathbb{T}_\alpha(\delta)\phi_1 + \int_0^\delta \mathbb{T}_\alpha(\delta - s) f(s, x_s^{n-1})ds$$
$$+ \int_0^\delta \mathbb{T}_\alpha(\delta - s)\Big(\int_0^\tau g(\theta, x_\theta^{n-1})dw(\theta)\Big)ds + \int_0^\delta \mathbb{T}_\alpha(\delta - s)\sigma(s)dS_Q^H(s)\Big\|^p\Big\}$$
$$\leq 5^{p-1}\sum_{i=1}^{5} I_i. \tag{7}$$

Now, we estimate each term on the RHS of the above inequality (7). By using assumption (A_1), we have

$$I_1 \leq a_1^p\, \mathbb{E}\|\phi_0\|^p,$$
$$I_2 \leq a_2^p\, \mathbb{E}\|\phi_1\|^p.$$

By assumptions $(A_1) - (A_3)$ and the Hölder's inequality, we get the following estimate for I_3

$$I_3 = \mathbb{E}\Big\|\int_0^\delta \mathbb{T}_\alpha(\delta - s) f(s, x_s^{n-1})ds\Big\|^p$$
$$\leq a_2^p \int_0^\delta \mathbb{E}\Big\|f(s, x_s^{n-1}) - f(s, 0) + f(s, 0)\Big\|^p ds$$
$$\leq 2^{p-1}a_2^p\delta^{p-1}\Big[\int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds + N_0\Big].$$

We estimate I_4 by using Lemma 3.8 and assumptions $(A_1)-(A_3)$ as follows

$$\begin{aligned}
I_4 = &\ \mathbb{E}\Big\|\int_0^\delta \mathbb{T}_\alpha(\delta-s)\Big(\int_0^\tau g(\theta,x_\theta^{n-1})dw(\theta)\Big)ds\Big\|^p \\
\leq &\ a_2^p C_p\, T^{\frac{p}{2}}\Big[\int_0^\tau \mathbb{E}\|g(\theta,x_\theta^{n-1})\|^p d\theta\Big] \\
\leq &\ 2^{p-1}a_2^p C_p T^{\frac{p}{2}}\Big[\int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds + N_0\Big].
\end{aligned}$$

Lemma 3.9 is used to derive the following estimate

$$\begin{aligned}
I_5 = &\ \mathbb{E}\Big\|\int_0^\delta \mathbb{T}_\alpha(\delta-s)\sigma(s)dS_Q^H(s)\Big\|^p \\
\leq &\ a_2^p CH\delta^{pH-1}\int_0^\delta \mathbb{E}\|\sigma(s)\|^p ds \\
\leq &\ a_2^p CH\delta^{pH-1}L.
\end{aligned}$$

These estimates together with (7) yields

$$\begin{aligned}
\mathbb{E}\|x^n(\delta)\|^p \leq &\ 5^{p-1}\Big\{a_1^p\,\mathbb{E}\|\phi_0\|^p + a_2^p\,\mathbb{E}\|\phi_1\|^p + 2^{p-1}a_2^p\delta^{p-1}\Big[\int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds + N_0\Big] \\
&+ 2^{p-1}a_2^p C_p T^{\frac{p}{2}}\Big[\int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds + N_0\Big] + a_2^p CH\delta^{pH-1}L\Big\}. \\
\leq &\ 5^{p-1}[a_1^p\mathbb{E}\|\phi_0\|^p + a_2^p\,\mathbb{E}\|\phi_1\|^p + a_2^p CH\delta^{pH-1}L] \\
&+ 10^{p-1}a_2^p\delta^{p-1}\int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds + 10^{p-1}a_2^p\delta^{p-1}N_0 \\
&+ 10^{p-1}a_2^p C_p T^{\frac{p}{2}}\int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds + 10^{p-1}a_2^p C_p T^{\frac{p}{2}}N_0 \\
\leq &\ R_1 + 10^{p-1}a_2^p\delta^{p-1}\int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds + 10^{p-1}a_2^p C_p T^{\frac{p}{2}}\int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds,
\end{aligned}$$

where

$$R_1 = 5^{p-1}[a_1^p\mathbb{E}\|\phi_0\|^p + a_2^p\,\mathbb{E}\|\phi_1\|^p + a_2^p CH\delta^{pH-1}L] + 10^{p-1}a_2^p\delta^{p-1} + 10^{p-1}a_2^p C_p T^{\frac{p}{2}}N_0.$$

Here, $\rho(\cdot)$ is concave and $\rho(0)=0$, and one can find a pair of +ve constants β_1 and β_2 s.t $\rho(\delta)\leq\beta_1+\beta_2\delta$ for

$\delta \geq 0$, now

$$\mathbb{E}\|x^n(\delta)\|^p \leq R_1 + 10^{p-1}\beta_1 a_2^p(\delta^{p-1} + C_p T^{\frac{p}{2}}) + 10^{p-1}\beta_2 a_2^p(\delta^{p-1} + C_p T^{\frac{p}{2}}) \int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds$$

$$\leq R_2 + 10^{p-1}\beta_2 a_2^p(\delta^{p-1} + C_p\, T^{\frac{p}{2}}) \int_0^\delta \rho(\mathbb{E}\|x_s^{n-1}\|^p)ds,$$

where $R_2 = R_1 + 10^{p-1}\beta_1 a_2^p\,(\delta^{p-1} + C_p\, T^{\frac{p}{2}})$.
For any $k \geq 1$

$$\max_{1\leq n\leq k} \mathbb{E} \sup_{0\leq s\leq\delta} \|x^{n-1}(s)\|^p \leq \mathbb{E}\|x^0(s)\|^p + \max_{1\leq n\leq k} \sup_{0\leq s\leq\delta} \mathbb{E}\|x^n(s)\|^p$$

$$\max_{1\leq n\leq k} \sup_{0\leq s\leq\delta} \mathbb{E}\|x^n(s)\|^p \leq R_2 + 20^{p-1}\beta_2\, a_2^p\,(\delta^{p-1} + C_p T^{\frac{p}{2}})\rho\,\delta\,\mathbb{E}\|x^0(s)\|^p$$

$$+ 20^{p-1}\,\beta_2\,\rho\, a_2^p(\delta^{p-1} + C_p T^{\frac{p}{2}}) \int_0^\delta \Big(\max_{1\leq n\leq k} \sup_{0\leq s\leq\delta} \mathbb{E}\|x^n(s)\|^p\Big)ds$$

$$\leq R_3 + 20^{p-1}\,\beta_2\,\rho\, a_2^p(\delta^{p-1} + C_p T^{\frac{p}{2}}) \int_0^\delta \Big(\max_{1\leq n\leq k} \sup_{0\leq s\leq\delta} \mathbb{E}\|x^n(s)\|^p\Big)ds,$$

where $R_3 = R_2 + 20^{p-1}\beta_2\, a_2^p\,(\delta^{p-1} + C_p T^{\frac{p}{2}})\rho\,\delta\,\mathbb{E}\|x^0(s)\|^p$.
Hence,

$$\max_{1\leq n\leq k} \sup_{0\leq s\leq\delta} \mathbb{E}\|x^n(s)\|^p \leq R_3 + 20^{p-1}\,\beta_2\,\rho\, a_2^p\,(\delta^{p-1} + C_p\, T^{\frac{p}{2}}) \int_0^\delta \Big(\max_{1\leq n\leq k} \sup_{0\leq s\leq\delta} \mathbb{E}\|x^n(s)\|^p\Big)ds. \tag{8}$$

Using the Gronwall inequality, the above inequality (8) becomes

$$\max_{1\leq n\leq k} \sup_{0\leq s\leq\delta} \mathbb{E}\|x^n(s)\|^p \leq R_3\, e^{20^{p-1}\beta_2\,\rho\, a_2^p\,(\delta^{p-1}+C_p T^{\frac{p}{2}})\delta}.$$

Hence,

$$\mathbb{E}\|x^n(\delta)\|^p \leq a_1^p \mathbb{E}\|\phi\|^p + b\, R_3\, e^{20^{p-1}\beta_2\,\rho\, a_2^p\,(\delta^{p-1}+C_p T^{\frac{p}{2}})\delta} \leq K,$$

for $n \geq 1$, $\delta \in J$ which shows that $\{x^n(\delta)\}_{n=1}^\infty$ is bounded in $\mathcal{B}$.
Step 2 :
Now we prove $\{x^n(\delta)\}_{n=1}^\infty$, $n \geq 1$ is a Cauchy sequence.
Consider the $\{x^n(\delta)\}_{n=1}^\infty$ defined in (6) and define the sequence $\{x^{n+1}(\delta)\}_{n=1}^\infty$ as

$$x^{n+1}(\delta) = \mathbb{S}_\alpha(\delta)\phi_0 + \mathbb{T}_\alpha(\delta)\phi_1 + \int_0^\delta \mathbb{T}_\alpha(\delta - s) f(s, x_s^n)ds + \int_0^\delta \mathbb{T}_\alpha(\delta - s)\Big(\int_0^\tau g(\theta, x_\theta^n)dw(\theta)\Big)ds$$

$$+ \int_0^\delta \mathbb{T}_\alpha(\delta - s)\sigma(s)dS_Q^H(s),\ \delta \in J,\ n \geq 1.$$

then, we have

$$\begin{aligned}\mathbb{E}\|x^{n+1}(\delta)-x^{n}(\delta)\|^{p} \leq\ & 2^{p-1}\Big\{\mathbb{E}\Big\|\int_{0}^{\delta}\mathbb{T}_{\alpha}(\delta-s)[f(s,x_{s}^{n})-f(s,x_{s}^{n-1})]ds\Big\|^{p}\\ &+\mathbb{E}\Big\|\int_{0}^{\delta}\mathbb{T}_{\alpha}(\delta-s)\Big(\int_{0}^{\tau}[g(\theta,x_{\theta}^{n})-g(\theta,x_{\theta}^{n-1})]dw(\theta)\Big)ds\Big\|^{p}\Big\}\\ \leq\ & 2^{p-1}\sum_{i=1}^{2}J_{i}. \end{aligned} \tag{9}$$

By using assumptions $(A_1)-(A_2)$ we estimate J_1 as,

$$\begin{aligned} J_1 =\ & \mathbb{E}\Big\|\int_{0}^{\delta}\mathbb{T}_{\alpha}(\delta-s)[f(s,x_{s}^{n})-f(s,x_{s}^{n-1})]ds\Big\|^{p}\\ \leq\ & a_2^p\,\delta^{p-1}\int_{0}^{\delta}\mathbb{E}\|f(s,x_{s}^{n})-f(s,x_{s}^{n-1})\|^{p}ds\\ \leq\ & a_2^p\,\delta^{p-1}\int_{0}^{\delta}\rho(\mathbb{E}\|x_{s}^{n}-x_{s}^{n-1}\|^{p})ds. \end{aligned}$$

From Lemma 3.8 and assumptions $(A_1)-(A_2)$ we estimate

$$\begin{aligned} J_2 =\ & \mathbb{E}\Big\|\int_{0}^{\delta}\mathbb{T}_{\alpha}(\delta-s)\Big(\int_{0}^{\tau}[g(\theta,x_{\theta}^{n})-g(\theta,x_{\theta}^{n-1})]dw(\theta)\Big)ds\Big\|^{p}\\ \leq\ & a_2^p\,T^{\frac{p}{2}}\,C_p\Big[\int_{0}^{\tau}\mathbb{E}\|g(\theta,x_{\theta}^{n})-g(\theta,x_{\theta}^{n-1})\|^{p}d\theta\Big]\\ \leq\ & a_2^p\,T^{\frac{p}{2}}\,C_p\int_{0}^{\delta}\rho(\mathbb{E}\|x_{s}^{n}-x_{s}^{n-1}\|^{p})ds. \end{aligned}$$

Using the estimates J_1 and J_2, (9) can be written as,

$$\mathbb{E}\|x^{n+1}(\delta)-x^{n}(\delta)\|^{p} \leq 2^{p-1}\,a_2^p\Big(\delta^{p-1}+T^{\frac{p}{2}}\,C_p\Big)\int_{0}^{\delta}\rho(\mathbb{E}\|x_{s}^{n}-x_{s}^{n-1}\|^{p})ds.$$

Let $\Phi_n(\delta) = \sup_{\delta \in J} \mathbb{E}\|x^{n+1}(\delta) - x^n(\delta)\|^p$. Thus the above inequality becomes

$$\begin{aligned}
\Phi_n(\delta) = 2^{p-1}\, a_2^p\Big(\delta^{p-1} + T^{\frac{p}{2}}\, C_p\Big) \int_0^\delta \rho(\mathbb{E}\|x^n(s) - x^{n-1}(s)\|^p) ds \\
\leq\; 2^{p-1}\, a_2^p\Big(\delta^{p-1} + T^{\frac{p}{2}}\, C_p\Big) \int_0^\delta \rho(\Phi_{n-1}(s)) ds \\
:=\; R_4 \int_0^\delta \rho(\Phi_{n-1}(s)) ds,
\end{aligned} \tag{10}$$

where $R_4 = 2^{p-1}\, a_2^p\Big(\delta^{p-1} + T^{\frac{p}{2}}\, C_p\Big)$.
Moreover, for $n = 1$ in (10),

$$\begin{aligned}
\Phi_1(\delta) &\leq\; R_4 \int_0^\delta \rho(\mathbb{E}\|x^1(s) - x^0(s)\|^p) ds \\
&\leq\; R_4 \int_0^\delta \rho(\mathbb{E}\|x^0(s)\|^p) ds \quad \text{where, } k_1 = \rho\; \mathbb{E}\|x^0(s)\|^p \\
&\leq\; R_4\, k_1\, \delta,
\end{aligned}$$

where $k_1 := \rho\; \mathbb{E}\|x^0(s)\|^p$. Also, for $n = 2$ in (10), we have

$$\begin{aligned}
\Phi_2(\delta) &\leq\; R_4 \int_0^\delta \rho(\mathbb{E}\|x^2(s) - x^1(s)\|^p) ds \\
&\leq\; R_4 \int_0^\delta \rho(\mathbb{E}\|x^1(s)\|^p) ds \\
&\leq\; R_4 \int_0^\delta \rho\; (\Phi_1(s)) ds \\
&\leq\; (R_4)^2\, k_1\, \frac{\delta^2}{2}.
\end{aligned}$$

By applying mathematical induction from (10), we have

$$\Phi_n(\delta) \leq\; (R_4)^n\, k_1\, \frac{\delta^n}{n!}, \quad n \geq 1,\; \delta \in J.$$

If $m \geq n \geq 0$,

$$\begin{aligned}
\sup_{\delta \in J} \mathbb{E}\|x^m(\delta) - x^n(\delta)\|^p &\leq\; \sum_{r=n}^{\infty} \mathbb{E}\|x^{r+1}(\delta) - x^{r(\delta)}\|^p \\
&\leq\; \sum_{r=n}^{\infty} (R_4)^r\, k_1\, \frac{\delta^r}{r!} \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Hence, we conclude that $\{x^n(\delta)\}_{n=1}^{\infty}$, $n \geq 1$, is a Cauchy sequence.

Step 3 :

We have to show that the existence and uniqueness of system (2).

Now, we prove the existence of solution.

By using Step 2, (i.e) $\{x^n(\delta)\}_{n=1}^{\infty}$, $n \geq 1$, is a Cauchy sequence, which is convergence. Then by using Lemma (Borel-Cantelli) taking the limits on both sides of the equation (6), we get

$$x(\delta) = \mathbb{S}_\alpha(\delta)\phi_0 + \mathbb{T}_\alpha(\delta)\phi_1 + \int_0^\delta \mathbb{T}_\alpha(\delta - s) f(s, x_s)ds + \int_0^\delta \mathbb{T}_\alpha(\delta - s)\Big(\int_0^\tau g(\theta, x_\theta)dw(\theta)\Big)ds + \int_0^\delta \mathbb{T}_\alpha(\delta - s)\sigma(s)dS_Q^H(s).$$

Hence, we obtain that $x(\delta)$ is a solution to the given Cauchy problem (2).

Now, we prove the Uniqueness of solution.

Now, we prove the uniqueness of solution. Let $x(\delta), y(\delta) \in \mathcal{B}$ be two solutions on $\delta \in J$, we have

$$\mathbb{E}\|x(\delta) - y(\delta)\|^p \leq 2^{p-1}\, a_2^p\Big(\delta^{p-1} + T^{\frac{p}{2}}\, C_p\Big) \int_0^\delta \rho(\mathbb{E}\|x(s) - y(s)\|^p)ds. \tag{11}$$

By Bihari inequality, the above (11)

$$\mathbb{E}\|x(\delta) - y(\delta)\|^p = 0.$$

Therefore, $x(\delta) = y(\delta), \quad \forall\, \delta \in J$.

Hence, the existence and uniqueness of solution of (2) on J is obtained. Accordingly, all the conditions are satisfied by the iteration technique which implies that the system (2) has a unique mild solution.

5. Exponential Stable

In this section, the sufficient criteria of the mild solution of p^{th} moment exponential stability for FSIDEs driven by sub-fBm are established.

Theorem 5.1. *Suppose that assumptions $(A_2)-(A_5)$ hold, then the mild solution $x(\delta)$ described in (2) is exponentially stable in the p^{th} moment sense on J provided that*

$$5^{p-1}\{2^{p-1}\tilde{M}_2^p b^{p-1} + 2^{p-1}\tilde{M}_2^p b^{p-1}N_0 + 2^{p-1}\tilde{M}_2^p C_p\Big(\frac{\mu_2(p-1)}{(p-2)}\Big)^{1-\frac{p}{2}}(1 + N_0)\} < 1. \tag{12}$$

Proof: Let $x(\delta)$ be the solution of the given Cauchy problem (2) described by

$$x(\delta) = \mathbb{S}_\alpha(\delta)\phi_0 + \mathbb{T}_\alpha(\delta)\phi_1 + \int_0^\delta \mathbb{T}_\alpha(\delta - s) f(s, x_s)ds + \int_0^\delta \mathbb{T}_\alpha(\delta - s)\Big(\int_0^\tau g(\theta, x_\theta)dw(\theta)\Big)ds + \int_0^\delta \mathbb{T}_\alpha(\delta - s)\sigma(s)dS_Q^H(s).$$

Then

$$\mathbb{E}\|x(\delta)\|^p = 5^{p-1}\Big\{\mathbb{E}\Big\|\mathbb{S}_\alpha(\delta)\phi_0 + \mathbb{T}_\alpha(\delta)\phi_1 + \int_0^\delta \mathbb{T}_\alpha(\delta - s) f(s, x_s)ds$$

$$+ \int_0^\delta \mathbb{T}_\alpha(\delta - s)\Big(\int_0^\tau g(\theta, x_\theta)dw(\theta)\Big)ds + \int_0^\delta \mathbb{T}_\alpha(\delta - s)\sigma(s)dS_Q^H(s)\Big\|^p\Big\}$$

$$\mathbb{E}\|x(\delta)\Big\|^p \leq 5^{p-1}\sum_{i=1}^5 B_i. \tag{13}$$

Here, it is easy to estimate each term of the RHS of the above inequality (13) as below.

$$B_1 \leq \tilde{M}_1^p e^{-p\mu_1(\delta)}\mathbb{E}\|\phi_0\|^p$$
$$B_2 \leq \tilde{M}_2^p e^{-p\mu_2(\delta)}\mathbb{E}\|\phi_1\|^p.$$

By using assumptions $(A_2) - (A_4)$, we estimate B_3 as

$$\begin{aligned} B_3 = {} & \mathbb{E}\Big\|\int_0^\delta \mathbb{T}_\alpha(\delta - s)f(s, x_s)ds\Big\|^p \\ \leq {} & \tilde{M}_2^p b^{p-1}\Big(\int_0^\delta e^{-p\mu_2(\delta-s)}\mathbb{E}\|f(s, x_s) - f(s, 0) + f(s, 0)\|^p ds\Big) \\ \leq {} & 2^{p-1}\tilde{M}_2^p b^{p-1}[\int_0^\delta e^{-p\mu_2(\delta-s)}\rho(\mathbb{E}\|x_s\|^p)ds + N_0]. \end{aligned}$$

By using assumptions $(A_2) - (A_4)$ and Lemma 3.8, we estimate B_4 as

$$\begin{aligned} B_4 = {} & \mathbb{E}\Big\|\int_0^\delta \mathbb{T}_\alpha(\delta - s)\Big(\int_0^\tau g(\theta, x_\theta)dw(\theta)\Big)ds\Big\|^p \\ \leq {} & 2^{p-1}\tilde{M}_2^p C_p\Big(\frac{\mu_2(p-1)}{(p-2)}\Big)^{1-\frac{p}{2}}\Big\{\int_0^\delta e^{-p\mu_2(\delta-s)}\Big(\int_0^\tau \mathbb{E}\|g(\theta, x_\theta) - g(\theta, 0)\|^p d\theta \\ & + \int_0^\tau \mathbb{E}\|g(\theta, 0)\|^p d\theta\Big)ds\Big\} \\ \leq {} & 2^{p-1}\tilde{M}_2^p C_p\Big(\frac{\mu_2(p-1)}{(p-2)}\Big)^{1-\frac{p}{2}}\Big\{\int_0^\delta e^{-p\mu_2(\delta-s)}\Big(\int_0^\tau \rho(\mathbb{E}\|x_\theta)\|^p d\theta\Big)ds + N_0\Big\}. \end{aligned}$$

Using assumption (A_4) and Lemma 3.10 the estimate B_5 is given by

$$\begin{aligned} B_5 = {} & \mathbb{E}\Big\|\int_0^\delta \mathbb{T}_\alpha(\delta - s)\sigma(s)dS_Q^H(s)\Big\|^p \\ \leq {} & \tilde{M}_2^p\int_0^\delta e^{-p\mu_2(\delta-s)}\mathbb{E}\|\sigma(s)\|^p ds \\ \leq {} & \tilde{M}_2^p L^p C H\delta^{pH-1}\int_0^\delta e^{-p\mu_2(\delta-s)}ds. \end{aligned}$$

From the above estimates B_i $(i = 1, 2, \ldots, 5)$, the inequality (13) becomes

$$\begin{aligned}\mathbb{E}\|x(\delta)\|^p \leq\ & 5^{p-1}\Big\{\tilde{M}_1^p e^{-p\mu_1(\delta)}\mathbb{E}\|\phi_0\|^p + \tilde{M}_2^p e^{-p\mu_2(\delta)}\mathbb{E}\|\phi_1\|^p \\ & + 2^{p-1}\tilde{M}_2^p b^{p-1}[\int_0^\delta e^{-p\mu_2(\delta-s)}\rho(\mathbb{E}\|x(s)\|^p)ds + N_0] \\ & + 2^{p-1}\tilde{M}_2^p C_p\Big(\frac{\mu_2(p-1)}{(p-2)}\Big)^{1-\frac{p}{2}}\Big\{\int_0^\delta e^{-p\mu_2(\delta-s)}\Big(\int_0^\tau \rho(\mathbb{E}\|x(\theta))\|^p d\theta\Big)ds + N_0\Big\} \\ & + \tilde{M}_2^p L^p CH\delta^{pH-1}\int_0^\delta e^{-p\mu_2(\delta-s)}ds\Big\}. \end{aligned} \tag{14}$$

By using the inequality (12) in the above inequality, it is equivalently estimated as

$$\begin{aligned}\mathbb{E}\|x(\delta)\|^p \leq\ & 5^{p-1}[\tilde{M}_1^p e^{-p\mu_1(\delta)}\mathbb{E}\|\phi_0\|^p + \tilde{M}_2^p e^{-p\mu_2(\delta)}\mathbb{E}\|\phi_1\|^p + \tilde{M}_2^p L^p CH\delta^{pH-1}] \\ \leq\ & \hat{\oplus}_1 e^{-\mu_1(\delta)} + \hat{\oplus}_2 e^{-\mu_2(\delta)}, \quad \mu_1, \mu_2 \geq 0,\end{aligned}$$

where

$$\begin{aligned}\hat{\oplus}_1 &= 5^{p-1}\tilde{M}_1^p e^{-p\mu_1(\delta)}\mathbb{E}\|\phi_0\|^p \\ \hat{\oplus}_2 &= 5^{p-1}\tilde{M}_2^p e^{-p\mu_2(\delta)}\mathbb{E}\|\phi_1\|^p + \tilde{M}_2^p L^p CH\delta^{pH-1}.\end{aligned}$$

By Lemma 3.10 and equation (14) we have, $\mathbb{E}\|x(\delta)\|^p \leq \hat{\oplus}\, e^{-\eta\delta},\ \delta \geq -r,\ \eta \in (0, \eta_1\Lambda\eta_2)$, where

$$\hat{\oplus} = \max\Big\{\hat{\oplus}_1 + \hat{\oplus}_2,\ \hat{\oplus}_3 := 2^{p-1}\tilde{M}^p b^{p-1} + 2^{p-1}\tilde{M}^p b^{p-1}N_0$$

$$\hat{\oplus}_4 := 2^{p-1}\tilde{M}^p C_p\Big(\frac{\mu_2(p-1)}{(p-2)}\Big)^{1-\frac{p}{2}}(1+N_0)\Big\}$$

η is a +ve root of the equation, we have $\hat{\oplus}_1 e^{-\mu_1\delta_1} + \hat{\oplus}_2 e^{-\mu_2\delta_1} + \hat{\oplus}_3\frac{e^{-\mu_2\delta_1}}{\eta_2-\mu} = 1$.
Here, $\xi_1 = \hat{\oplus}_1,\ \xi_2 = \hat{\oplus}_2,\ \hat{\oplus}_3,\ \hat{\oplus}_4$ as defined above. We conclude that the mild solution of the Cauchy problem (2) is p^{th} moment exponentially stable.

Remark 5.2. *In the article [44], the asymptotic stability and mean square stability have been analysed for the second-order stochastic differential equations of fractional order with variable delay in the state by using the Banach fixed point theorem by imposing the Lipschitz condition on nonlinearity and estimated parameters of the solution is less than* 1*. Authors in [15] studied stability analysis of fractional stochastic Clarke's subdifferential type with Poisson jumps by using the multi-valued fixed point theorem in mean square estimation. Different from the above two papers by using the successive approximation the exponential stability is established for the fast convergence of the stochastic integro-differential sub-fBm instead of Bm with relaxed restrictive conditions in* p^{th} *moment norm through the new integral inequality.*

6. Illustration

In this section, an example is provided to verify the obtained theoretical result. Consider the following fractional stochastic partial integro-differential equation driven by sub-fBm of the form

$$\begin{aligned}&{}^{C}D_{0^+}^{\frac{5}{4}}y(\delta,x) = \frac{\partial^2}{\partial x^2}\,y(\delta,x) + \frac{e^{-3\delta}y(\delta,x)}{(1+e^{2\delta}) + (1+y(\delta,x))} + \int_0^\delta e^{-4\delta}\sin\pi y(\delta,x)dw(s) + e^{-5\delta}y(x)dS_Q^H(s), \\ &y(0,\delta) = y(\pi,\delta) = 0, \quad y'(0,\delta) = y'(\pi,\delta) = \phi_1,\ \delta \geq 0\end{aligned} \tag{15}$$

Here, ${}^{C}D_{0^+}^{\frac{5}{4}}$ denotes Caputo fractional partial derivative of order $\alpha = \frac{5}{4}$. Let $w(\delta)$ refers the standard Wiener process and $dS_Q^H(s)$ is the sub-fBm.

Consider the operator $\mathbb{A} : \mathcal{D}(\mathbb{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by $\mathbb{A}\mathcal{Z} = \mathcal{Z}''$ with the domain $\mathcal{D}(\mathbb{A}) = \{\mathcal{Z} \in \mathcal{H},\ \mathcal{Z}, \mathcal{Z}'$ absolutely continuous, $\mathcal{Z}'' \in \mathcal{H},\ \mathcal{Z}(0) = \mathcal{Z}(\pi) = 0\}$. Then

$$\mathbb{A}\mathcal{Z} = \sum_{n=1}^{\infty} -n^2(\mathcal{Z}, \mathcal{Z}_n),$$

here, $\mathcal{Z}_n(x) = \sqrt{\frac{2}{\pi}}\ \sin(nx)$ be an orthonormal set of eigenvalue of $\mathbb{A}$. $\mathbb{A}$ generates a compact semigroup $(\mathbb{T}_\alpha(\delta))_{\delta\geq 0}$ in $\mathcal{H}$ is

$$\mathbb{T}_\alpha(\delta)\mathcal{Z} = \sum_{n=1}^{\infty} e^{-n^2\delta}(\mathcal{Z}, \mathcal{Z}_n),\ \delta \geq 0,\ \mathcal{Z} \in \mathcal{H}.$$

Now, define the non-linear continuous functions $f : J \times \mathcal{H} \to \mathcal{H},\ g : J \times \mathcal{H} \to L_Q^0(\mathbb{K}, \mathcal{H})$ and $\sigma : J \times \mathcal{H} \to L_Q^0(\mathbb{K}, \mathcal{H})$ as described by

$$\begin{aligned} f(\delta, y) &= \frac{e^{-3\delta} y(\delta, x)}{(1 + e^{2\delta}) + (1 + y(\delta, x))}, \\ g(\delta, y) &= \int_0^{\delta} e^{-4\delta} \sin \pi y(\delta, x) dw(s), \\ \sigma(\delta) &= e^{-5\delta}, \quad \delta \in J. \end{aligned}$$

Put $\tilde{M}_2^p = 0.5, \delta = \frac{1}{3}, p = 1, \mu_2 = 0.2, C_p = 1, b = 0.1, N_0 = 0.25.$

$$\begin{aligned} &5^{p-1}\{2^{p-1}\tilde{M}_2^p b^{p-1}(1 + N_0) + 2^{p-1}\tilde{M}_2^p C_p\Big(\frac{\mu_2(p-1)}{(p-2)}\Big)^{1-\frac{p}{2}}(1 + N_0) + \tilde{M}_2^p L^p C H \delta^{pH-1}\} < 1 \\ &5^{p-1}\{2^{p-1}\tilde{M}_2^p b^{p-1}(1 + N_0)\Big(\frac{e^{-3\delta}}{1 + e^{2\delta}}\Big) + 2^{p-1}\tilde{M}_2^p C_p\Big(\frac{\mu_2(p-1)}{(p-2)}\Big)^{1-\frac{p}{2}} \\ &\times (1 + N_0)e^{-4\delta} sin\pi + \tilde{M}_2^p e^{-5\delta}\} < 1 \\ &0.87525 < 1. \end{aligned}$$

It can be effortlessly proved, all conditions $(A_2) - (A_5)$ of Theorem 5.1 are satisfied. Hence, we can conclude that the system (15) is p^{th} moment exponentially stable.

Remark 6.1. *The exponentially stability does not hold for* $p = 2$ *based on the Theorem 5.1.*

Remark 6.2. *Stability is a critical property of the dynamical systems for investigation in various domains. In fractional order systems, there are many challenging and unsolved problems related to stability theory. The stability analysis has been performed by the convergence of solutions for fractional order differential and trajectories of dynamical systems under small perturbations of the initial condition. Recently, different types of stability such as Mittag-Leffler stability, generalized Mittag-Leffler stability, Ulam stability, and Ulam-Hyers stability have been discussed. The exponential stability cannot be used to characterize the asymptotic stability of fractional order systems.*

7. Conclusion

This manuscript addressed the wellposedness of mild solution and stability analysis for FSIDEs driven by sub-fBm. Sufficient conditions have been derived for the existence and uniqueness of the mild solution for FSIDEs with sub-fBm by using the successive approximation technique. Also, the p^{th} moment exponential stability result has been attained. Finally, a numerical example has been validated to prove the efficiency of the obtained theoretical results.

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