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# Some notes on integrable Teichmüller space on the real line

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**Abstract.** We will introduce and discuss various models of the integrable Teichmüller space $T_p$ in the real line case, extending some known results on the Weil-Petersson Teichmüller space $T_2$ to the general space $T_p$ for $p > 1$.

## 1. Introduction and statement of main results

We first fix some basic notations. Let $\mathbb{U} = \{z = x + iy : y > 0\}$ and $\mathbb{U}^* = \{z = x + iy : y < 0\}$ denote the upper and lower half plane in the complex plane $\mathbb{C}$, respectively. $\mathbb{R} = \partial\mathbb{U} = \partial\mathbb{U}^*$ is the real line, and $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the extended real line in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\Delta = \{z : |z| < 1\}$ denote the unit disk. $\Delta^* = \hat{\mathbb{C}} - \overline{\Delta}$ is the exterior of $\Delta$, and $S^1 = \partial\Delta = \partial\Delta^*$ is the unit circle. $\mathbb{D}$ will always denote the unit disk $\Delta$ or the upper half plane $\mathbb{U}$ so that $\mathbb{S} = \partial\mathbb{D}$ is the unit circle $S^1$ or the real line $\mathbb{R}$. Similarly, $\mathbb{D}^*$ will always denote the exterior $\Delta^*$ of the unit disk or the lower half plane $\mathbb{U}^*$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a positive constant $C$ independent of $A$ and $B$ such that $A \leq CB$ ($A \geq CB$), while $A \asymp B$ means both $A \lesssim B$ and $A \gtrsim B$.

One of the models of the universal Teichmüller space $T$ can be defined as the right coset space $T = \mathrm{QS}(\mathbb{S})/\mathrm{M\ddot{o}b}(\mathbb{S})$. Here, $\mathrm{QS}(\mathbb{S})$ denotes the group of all quasisymmetric homeomorphisms of $\mathbb{S}$ onto itself, and $\mathrm{M\ddot{o}b}(\mathbb{S})$ the subgroup of $\mathrm{QS}(\mathbb{S})$ which consists of Möbius transformations keeping $\mathbb{S}$ fixed. Recall that a sense preserving self-homeomorphism $h$ of $\mathbb{S}$ is quasisymmetric if there exists a (least) positive constant $C(h)$, called the quasisymmetric constant of $h$, such that $|h(I_1)| \leq C(h)|h(I_2)|$ for all pairs of adjacent arcs $I_1$ and $I_2$ on $\mathbb{S}$ with the same arc-length $|I_1| = |I_2| (\leq |\mathbb{S}|/2)$. Beurling-Ahlfors [3] proved that a self-homeomorphism $h$ of $\mathbb{R}$ is quasisymmetric if and only if there exists some quasiconformal homeomorphism of $\mathbb{U}$ onto itself which has boundary values $h$. Later Douady-Earle [12] gave a quasiconformal extension of a quasisymmetric homeomorphism of $S^1$ to the unit disk which is conformally invariant.

Let $p > 1$ be a fixed number. A quasisymmetric homeomorphism $h \in \mathrm{QS}(\mathbb{S})$ is said to be a $p$-integrable asymptotic affine homeomorphism if it has a quasiconformal extension $f$ to $\mathbb{D}$ whose Beltrami coefficient $\mu$ is $p$-integrable in the Poincaré metric $\lambda_{\mathbb{D}}$, namely,

$$\iint_{\mathbb{D}} |\mu(z)|^p \lambda_{\mathbb{D}}^2(z)\,dxdy < \infty.$$

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Let $QS_p(\mathbb{S})$ denote the set of $p$-integrable asymptotic affine homeomorphisms of $\mathbb{S}$. The right coset space $T_p = QS_p(\mathbb{S})/\text{Möb}(\mathbb{S})$ is called the $p$-integrable Teichmüller space. The class $QS_2(S^1)$ was first introduced by Cui [10] and was much investigated in recent years (see [17], [32], [33], [35], [36], [40], [48]), and nowadays $T_2$ is usually called the Weil-Petersson Teichmüller space. For a general $p \geq 2$, $QS_p(S^1)$ was first introduced and investigated by Guo [18] (see also [26], [41], [42], [47]).

The first goal of the paper is to give the following intrinsic characterization of a quasisymmetric homeomorphism in the class $QS_p(\mathbb{R})$ ($p \geq 2$) without using quasiconformal extensions. Recall that the Besov space $\mathcal{B}_p(\mathbb{S})$ is the collection of locally integrable functions $u$ on $\mathbb{S}$ such that

$$\|u\|^p_{\mathcal{B}_p(\mathbb{S})} \doteq \frac{1}{4\pi^2}\int_{\mathbb{S}}\int_{\mathbb{S}} \frac{|u(\zeta) - u(\eta)|^p}{|\zeta - \eta|^2}|d\zeta||d\eta| < +\infty.$$

We denote by $\mathcal{B}_{p,\mathbb{R}}(\mathbb{S})$ the real-valued functions in $\mathcal{B}_p(\mathbb{S})$.

**Theorem 1.1.** *Let $p \geq 2$ be a fixed number and $h$ be an increasing homeomorphism on the real line $\mathbb{R}$. Then $h$ is a $p$-integrable asymptotic affine homeomorphism if and only if $h$ is locally absolutely continuous and $\log h'$ belongs to the Besov class $\mathcal{B}_p(\mathbb{R})$.*

It is known that an analogous result is true in the unit circle setting, namely, a sense-preserving homeomorphism $h$ on the unit circle is a $p$-integrable asymptotic affine homeomorphism if and only if $h$ is absolutely continuous and $\log h'$ belongs to the Besov class $\mathcal{B}_p(S^1)$. For $p = 2$, this was first proved Shen [35], answering a question explicitly proposed by Takhtajan-Teo (see Remark II.1.2 of [40]), and reproved later by Wu-Hu-Shen [46]. Very recently, Bishop ([4], [5]) gave a more geometric approach for $p = 2$. For a general $p \geq 2$, this was proved by Tang-Shen [42].

It should be pointed out that Theorem 1.1 can not be deduced directly from the unit circle case since the pre-logarithmic derivative is not invariant under a Möbius transformation. On the other hand, Theorem 1.1 has been proved in the special case $p = 2$. Actually, when $p = 2$, the if part of Theorem 1.1 was proved by Shen-Tang ([36], [37]) by means of a construction due to Semmes (see [34]), while the only if part was proved by Shen-Tang-Wu [38] by considering the pre-logarithmic derivative models of the little and Weil-Petersson Teichmüller spaces of the half plane. Theorem 1.1 generalizes the corresponding results in these two papers. After this research was completed in a previous version of this paper, the authors got to know that Wei-Matsuzaki ([44], [45]) gave a proof of Theorem 1.1 independently. Instead of using Semmes' construction, Wei-Matsuzaki used a variant of the Beurling-Ahlfors extension by the heat kernel introduced by Fefferman-Kenig-Pipher [13].

It is an open problem to determine whether Theorem 1.1 remains true when $1 < p < 2$. We will prove a result which holds for any $p > 1$ and is precisely Theorem 1.1 when $p \geq 2$. To make this precise, we recall the notion of strong $p$-integrable asymptotic affine homeomorphism, which was discussed on the unit circle in our companion paper [24] (see also [20]). A quasisymmetric homeomorphism $h \in QS(\mathbb{S})$ is said to be a strong $p$-integrable asymptotic affine homeomorphism if it has a quasiconformal extension $f$ to $\mathbb{D}$ such that $f$ is quasi-isometric under the Poincaré metric, that is,

$$\lambda_{\mathbb{D}}(f(z))|df(z)| \asymp C(f)\lambda_{\mathbb{D}}(z)|dz|, \, z \in \mathbb{D},$$

and has Beltrami coefficient $\mu$ being $p$-integrable in the Poincaré metric. By means of a recent result in [24] (see Theorem 2.1 there), we conclude that a quasisymmetric homeomorphism $h \in QS(\mathbb{S})$ is a strong $p$-integrable asymptotic affine homeomorphism if and only if it has a quasiconformal extension $f$ to $\mathbb{D}$ such that both the Beltrami coefficients of $f$ and the inverse mapping $f^{-1}$ are $p$-integrable in the Poincaré metric. Let $SQS_p(\mathbb{S})$ denote the set of strong $p$-integrable asymptotic affine homeomorphisms of $\mathbb{S}$. Clearly, $SQS_p(\mathbb{S}) \subset QS_p(\mathbb{S})$, and $SQS_p(\mathbb{S}) = QS_p(\mathbb{S})$ when $p \geq 2$ (see [10], [41]). It is also clear that $SQS_p(\mathbb{S})$ is a subgroup of $QS(\mathbb{S})$. The right coset space $T^s_p = SQS_p(\mathbb{S})/\text{Möb}(\mathbb{S})$ is called the strong $p$-integrable Teichmüller space.

**Theorem 1.2.** *Let $p > 1$ be a fixed number and $h$ be an increasing homeomorphism on the real line $\mathbb{R}$. Then $h$ is a strong $p$-integrable asymptotic affine homeomorphism if and only if $h$ is locally absolutely continuous and $\log h'$ belongs to the Besov class $\mathcal{B}_p(\mathbb{R})$.*

As stated above, Theoem 1.2 contains Theorem 1.1 since $\mathrm{SQS}_p(\mathbb{S}) = \mathrm{QS}_p(\mathbb{S})$ when $p \geq 2$. The proof of Theorem 1.2 is based on the investigation in our two papers [36] and [38], where the case $p = 2$ was considered. For completeness we will repeat the details here. In particular, we will discuss the pre-logarithmic derivative model and the Schwarzian derivative model of the strong $p$-integrable Teichmüller space $T_p^s$. Let $\Gamma$ be a closed Jordan curve through the point at infinity with complementary domains $\Omega$ and $\Omega^*$. Then there exists a pair of conformal mappings $f : \mathbb{U} \to \Omega$ and $g : \mathbb{U}^* \to \Omega^*$ with $f(\infty) = g(\infty) = \infty$, which can be continuously extended to $\mathbb{R}$ and thus determine an increasing homeomorphism $h \doteq g^{-1} \circ f : \mathbb{R} \to \mathbb{R}$, known as a conformal sewing mapping for $\Gamma$. It is known that $h$ is quasisymmetric if and only if $\Gamma$ is a quasicircle (see [1]).

Theorem 1.2 can be expanded to the following result.

**Theorem 1.3.** *Let $p > 1$ and $h = g^{-1} \circ f$ be a quasisymmetric conformal sewing for a quasicircle $\Gamma$ through $\infty$. Then the following statements are equivalent:*

*(1) $h$ is a strong $p$-integrable asymptotic affine homeomorphism;*

*(2) The Schwarzian derivative $S_f$ belongs to the Bergman space $B_p(\mathbb{U})$, namely,*

$$\iint_{\mathbb{U}} |S_f(z)|^p y^{2p-2} dxdy < \infty.$$

*(3) The pre-logarithmic derivative $\log f'$ belongs to the Besov space $\mathcal{B}_p(\mathbb{U})$, that is,*

$$\iint_{\mathbb{U}} |(\log f')'(z)|^p y^{p-2} dxdy < \infty.$$

*(4) $h$ is locally absolutely continuous and $\log h'$ belongs to the Besov class $\mathcal{B}_p(\mathbb{R})$.*

In order to prove Theorem 1.3, we also need to consider the quasicircle model of the strong $p$-integrable Teichmüller space $T_p^s$. A quasicircle $\Gamma$ is said to be a $p$-integrable quasicircle if a conformal mapping $f$ which maps $\mathbb{D}$ onto the left domain bounded by $\Gamma$ satisfies the condition $\log f' \in \mathcal{B}_p(\mathbb{D})$. A 2-integrable quasicircle is usually called a Weil-Petersson quasicircle (see [4], [5], [39]). A natural question is to give a geometric characterization of a $p$-integrable quasicircle without using the Riemann mapping. This question was explicitly proposed by Takhtajan-Teo for (bounded) Weil-Petersson quasicircles (see Remark II.1.2 of [40]). By means of a result of Pommerenke [30], it can be shown that a $p$-integrable quasicircle must be a chord-arc curve (see Lemma 6.1 below). Recall that a locally rectifiable closed Jordan curve $\Gamma$ is called a chord-arc curve with constant $k$ if $\mathrm{length}(\widetilde{\zeta z}) \leq (1 + k)|\zeta - z|$ for the smaller (i.e., with less length) subarc $\widetilde{\zeta z}$ of $\Gamma$ joining any finite two points $z$ and $\zeta$ of $\Gamma$ (see [22], [30], [31]). Recently, Bishop ([4], [5]) gave various characterizations for bounded Weil-Petersson curves from the points of harmonic analysis, geometric measure theory and hyperbolic geometry. A question was invited to extend those characterizations to other curve families, say, $p$-integrable quasicircles (see [4-6]). When dealing with the dependence of the Riemann mapping $f$ on a curve $\Gamma$, we gave a geometric characterization for unbounded Weil-Petersson quasicircles (see [39]). We now extend this characterization to general unbounded $p$-integrable quasicircles.

**Theorem 1.4.** *Let $p > 1$ and $h = g^{-1} \circ f$ be a quasisymmetric conformal sewing for a quasicircle $\Gamma$ through $\infty$. Then the following statements are equivalent:*

*(1) $\Gamma$ is a $p$-integrable quasicircle, that is, $\log f' \in \mathcal{B}_p(\mathbb{U})$;*

*(2) $\log g' \in \mathcal{B}_p(\mathbb{U}^*)$;*

*(3) $\Gamma$ is a chord-arc curve and an arclength parameterization $z : \mathbb{R} \to \Gamma$ satisfies the condition $z'(s) = e^{ib(s)}$ for some $b \in \mathcal{B}_{p,\mathbb{R}}(\mathbb{R})$;*

*(4) $\Gamma$ is a chord-arc curve and the unit tangent direction $\tau$ to $\Gamma$ satisfies the condition $\tau(z) = e^{iu(z)}$ for some real-valued function $u \in \mathcal{B}_p(\Gamma)$, namely,*

$$\int_{\Gamma} \int_{\Gamma} \frac{|u(z) - u(w)|^p}{|z - w|^2} |dz||dw| < \infty.$$

## 2. Preliminaries

In this section, we give some basic definitions and results on the universal Teichmüller space $T$ and its two subspaces, the little Teichmüller space $T_0$ and the integrable Teichmüller space $T_p$. In particular, we will recall the Schwarzian derivative models of these Teichmüller spaces.

We begin with the standard theory of the universal Teichmüller space (see [1], [14], [23] and [28] for more details). Let $M(\mathbb{D}^*)$ denote the open unit ball of the Banach space $L^\infty(\mathbb{D}^*)$ of essentially bounded measurable functions on $\mathbb{D}^*$. For $\mu \in M(\mathbb{D}^*)$, let $f_\mu$ be the quasiconformal mapping on the extended plane $\hat{\mathbb{C}}$ with complex dilatation equal to $\mu$ in $\mathbb{D}^*$, conformal in $\mathbb{D}$, normalized by $f_\mu(0) = 0$, $f_\mu(1) = 1$ and $f_\mu(\infty) = \infty$. Two elements $\mu$ and $\nu$ in $M(\mathbb{D}^*)$ are said to be equivalent, denoted by $\mu \sim \nu$, if $f_\mu|_{\mathbb{D}} = f_\nu|_{\mathbb{D}}$. Then $T = M(\mathbb{D}^*)/_\sim$ is the Bers model of the universal Teichmüller space. We let $\Phi$ denote the natural projection from $M(\mathbb{D}^*)$ onto $T$ so that $\Phi(\mu)$ is the equivalence class $[\mu]$. $[0]$ is called the base point of $T$.

It is known that the universal Teichmüller space $T$ is an infinite dimensional complex Banach manifold. To make this precise, we first recall some important Banach spaces. Let $\Omega$ be an arbitrary simply connected domain in the extended complex plane $\hat{\mathbb{C}}$ which is conformally equivalent to the upper half plane. Then the hyperbolic metric $\lambda_\Omega$ (with curvature constantly equal to $-4$) in $\Omega$ can be defined by

$$\lambda_\Omega(f(z))|f'(z)| = \frac{1}{2y}, \quad z = x + iy \in \mathbb{U},$$

where $f : \mathbb{U} \to \Omega$ is any conformal mapping. Let $B(\Omega)$ denote the Bers space of functions $\phi$ holomorphic in $\Omega$ with finite norm

$$\|\phi\|_{B(\Omega)} \doteq \sup_{z\in\Omega} |\phi(z)|\lambda_\Omega^{-2}(z),$$

and $B_0(\Omega)$ the closed subspace of $B(\Omega)$ which consists of those functions $\phi$ such that

$$\inf\left\{\sup_{z\in\Omega\setminus K} |\phi(z)|\lambda_\Omega^{-2}(z) : K \subset \Omega \text{ compact}\right\} = 0.$$

We also denote by $B_p(\Omega)$ the Bergman space of functions $\phi$ holomorphic in $\Omega$ with finite norm

$$\|\phi\|_{B_p(\Omega)} \doteq \left(\frac{1}{\pi}\iint_\Omega |\phi(z)|^p \lambda_\Omega^{2-2p} dxdy\right)^{\frac{1}{p}}.$$

It is easy to see that a conformal mapping $g : \Omega_1 \to \Omega_2$ induces a map $g^* : \phi \mapsto (\phi \circ g)(g')^2$, which are isometric isomorphisms from $B(\Omega_2)$ onto $B(\Omega_1)$, from $B_0(\Omega_2)$ onto $B_0(\Omega_1)$, and from $B_p(\Omega_2)$ onto $B_p(\Omega_1)$. Therefore, $B_p(\Omega) \subset B_0(\Omega)$, and the inclusion map is continuous (see [49]).

Now we consider the map $S : M(\mathbb{D}^*) \to B(\mathbb{D})$ which sends $\mu$ to the Schwarzian derivative of $f_\mu|_{\mathbb{D}}$. Recall that for any locally univalent function $f$, its Schwarzian derivative $S_f$ is defined by

$$S_f \doteq N_f' - \frac{1}{2}N_f^2, \quad N_f \doteq (\log f')'.$$

$S$ is a holomorphic split submersion onto its image, which descends down to a map $\beta : T \to B(\mathbb{D})$ known as the Bers embedding. Via the Bers embedding, $T$ carries a natural complex Banach manifold structure so that $\Phi$ is a holomorphic split submersion.

Let $L_0(\mathbb{D}^*)$ be the closed subspace of $L^\infty(\mathbb{D}^*)$ which consists of those functions $\mu$ such that

$$\inf\{\|\mu|_{\mathbb{D}^*\setminus K}\|_\infty : K \subset \mathbb{D}^* \text{ compact}\} = 0.$$

Set $M_0(\mathbb{D}^*) = M(\mathbb{D}^*) \cap L_0(\mathbb{D}^*)$. Then $T_0 = M_0(\mathbb{D}^*)/_\sim$ is called the little Teichmüller space. Under the Bers projection $S : M(\mathbb{D}^*) \to B(\mathbb{D})$, $S(M_0(\mathbb{D}^*) = S(M(\mathbb{D}^*)) \cap B_0(\mathbb{D})$ (see [14], [15], [31]).

We proceed to consider the integrable Teichmüller space $T_p$. We denote by $L_p(\mathbb{D}^*)$ the Banach space of all essentially bounded measurable functions $\mu$ on $\mathbb{D}^*$ with norm

$$\|\mu\|_{QS_p(\mathbb{S})} \doteq \|\mu\|_\infty + \left(\frac{1}{\pi}\iint_{\mathbb{D}^*} |\mu(z)|^p \lambda^2_{\mathbb{D}^*}(z)dxdy\right)^{\frac{1}{p}}.$$

Set $M_p(\mathbb{D}^*) = M(\mathbb{D}^*) \cap L_p(\mathbb{D}^*)$. Then $T_p = M_p(\mathbb{D}^*)/_\sim$ is the $p$-integrable Teichmüller space. Under the Bers projection $S : M(\mathbb{D}^*) \to B(\mathbb{D})$, $S(M_p(\mathbb{D}^*)) = S(M(\mathbb{D}^*)) \cap B_p(\mathbb{D})$ for $p \geq 2$ (see [10], [18], [40]). Finally, we denote by $M_p^s(\mathbb{D}^*)$ the subset of all $\mu$ in $M_p(\mathbb{D}^*)$ such that $f_\mu|_{\mathbb{D}^*}$ is quasi-isometric under the Poincaré metric, that is,

$$\lambda_{f_\mu(\mathbb{D}^*)}(f_\mu(z))|df_\mu(z)| \asymp C(f_\mu)\lambda_{\mathbb{D}^*}(z)|dz|,\ z \in \mathbb{D}^*.$$

Then $T_p^s = M_p^s(\mathbb{D}^*)/_\sim$ is the strong $p$-integrable Teichmüller space. Under the Bers projection $S : M(\mathbb{D}^*) \to B(\mathbb{D})$, $S(M_p^s(\mathbb{D}^*)) \subset S(M(\mathbb{D}^*)) \cap B_p(\mathbb{D})$ for each $p > 1$. This result was proved very recently in our companion paper [24] for $\mathbb{D} = \Delta$, which implies the case for $\mathbb{D} = \mathbb{U}$ by Möbius invariance. Recall that $S(M_p(\mathbb{D}^*)) = S(M_p^s(\mathbb{D}^*))$ when $p \geq 2$.

## 3. Pre-logarithmic derivative models of Teichmüller spaces

In this section, we will prove the part (2) ⇔ (3) in Theorem 1.3 (i.e., Theorem 3.3 below), which will be used to prove Theorem 1.2. We will follow the lines in our paper [38], where $p = 2$ was considered.

We first recall the pre-logarithmic derivative model of the universal Teichmüller space (see [2], [50]). Contrary to the Schwarzian derivative model, the pre-logarithmic derivative is not invariant under a Möbius transformation. Therefore, we need to treat pre-logarithmic derivative models of subspaces of the little Teichmüller space separately in the unit circle case and real line case (see [38]). In this section, we will deal with the pre-logarithmic derivative model of the (strong) integrable Teichmüller space in the half plane case.

Let $\mathcal{B}(\Omega)$ denote the Bloch space of functions $\phi$ holomorphic in $\Omega$ with semi-norm

$$\|\phi\|_{\mathcal{B}(\Omega)} \doteq \sup_{z\in\Omega} |\phi'(z)|\lambda^{-1}_{\mathbb{D}}(z),$$

and $\mathcal{B}_0(\Omega)$ the subspace of $\mathcal{B}(\Omega)$ which consists of those functions $\phi$ such that

$$\inf\left\{\sup_{z\in\Omega\setminus K} |\phi'(z)|\lambda^{-1}_{\Omega}(z) : K \subset \Omega \text{ compact}\right\} = 0.$$

We also denote by $\mathcal{B}_p(\Omega)$ the Besov space of functions $\phi$ holomorphic in $\Omega$ with semi-norm

$$\|\phi\|_{\mathcal{B}_p(\Omega)} \doteq \left(\frac{1}{\pi}\iint_{\Omega} |\phi'(z)|^p \lambda^{2-p}_{\Omega} dxdy\right)^{\frac{1}{p}}.$$

It is known that $\mathcal{B}_p(\Omega) \subset \mathcal{B}_0(\Omega)$, and the inclusion map is continuous (see [49]). It is also known that for each holomorphic function $\phi$ on $\Omega$, $\phi'' \in B(\Omega)$ if $\phi \in \mathcal{B}(\Omega)$, $\phi'' \in B_0(\Omega)$ if $\phi \in \mathcal{B}_0(\Omega)$, and $\phi'' \in B_p(\Omega)$ if $\phi \in \mathcal{B}_p(\Omega)$ (see [49]). The converse is also true, with some normalized conditions at $\infty$ whenever $\Omega$ is not a bounded domain (see [36], [38]).

Koebe distortion theorem implies that $\log f'_\mu|_{\mathbb{D}} \in \mathcal{B}(\mathbb{D})$ for $\mu \in M(\mathbb{D}^*)$. Furthermore, the map $L$ induced by the correspondence $\mu \mapsto \log f'_\mu|_{\mathbb{D}}$ is a continuous map from $M(\mathbb{D}^*)$ into $\mathcal{B}(\mathbb{D})$ (see [23]). Actually, $L : M(\mathbb{D}^*) \to \mathcal{B}(\mathbb{D})$ is even holomorphic (see [19]). It is known that $L(M_0(\mathbb{D}^*) = L(M(\mathbb{D}^*)) \cap \mathcal{B}_0(\mathbb{D})$ (see [14], [15], [31], [38]), and $L(M_p(\Delta^*)) = L(M(\Delta^*)) \cap \mathcal{B}_p(\Delta)$ for $p \geq 2$ (see [10], [18], [40]). Theorem 3.3 below implies that the latter result also holds in the half plane case.

**Lemma 3.1.** ([21]) *Let $1 < p \le q < \infty$, and $u(s), v(s)$ be two positive measurable functions in the interval $(a, b)$. If*

$$A \doteq \sup_{a<x<b} \left( \int_a^x u(s)ds \right)^{1/q} \left( \int_x^b v(s)^{1-p'} ds \right)^{1/p'} < \infty,$$

*where $p' = p/(p-1)$. Then there is constant $C(p, q) > 0$ such that for all positive measurable functions $f$ in the interval $(a, b)$, the following inequality holds*

$$\left( \int_a^b \left( \int_x^b f(s)ds \right)^q u(x)dx \right)^{1/q} \le C(p, q)A \left( \int_a^b f(x)^p v(x)dx \right)^{1/p}.$$

**Lemma 3.2.** *Let $\phi$ be a holomorphic function on the upper half plane $\mathbb{U}$ such that $\phi' \in B(\mathbb{U})$ and $\lim_{y\to\infty} \phi(x+iy) = 0$ uniformly for $x \in \mathbb{R}$. Then there exists some constant $C(p) > 0$ such that*

$$\iint_{\mathbb{U}_t} |\phi(x+iy)|^p y^{p-2} dxdy \le C(p) \left( \iint_{\mathbb{U}_t} |\phi'(x+iy)|^p y^{2p-2} dxdy + \|\phi'\|^p_{B(\mathbb{U})} \right),$$

*where*

$$\mathbb{U}(t) = \{x + iy : -1/t < x < 1/t,\ t < y < 1/t\}, \quad 0 < t < 1.$$

*Proof.* By assumption we have

$$\phi(x+iy) = -i \int_y^\infty \phi'(x+iv)dv,$$

which implies that

$$|\phi(x+iy)| \le \int_y^\infty |\phi'(x+iv)|dv \le \int_y^{1/t} |\phi'(x+iv)|dv + \frac{\|\phi'\|_{B(\mathbb{U})}}{4} t.$$

Noting that $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for positive $a, b$, we conclude that

$$\int_t^{1/t} |\phi(x+iy)|^p y^{p-2} dy \le 2^{p-1} \int_t^{1/t} \left( \int_y^{1/t} |\phi'(x+iv)|dv \right)^p y^{p-2} dy + \frac{\|\phi'\|^p_{B(\mathbb{U})}}{2^{p+1}(p-1)} t.$$

By Lemma 3.1 with $u(s) = s^{p-2}$, $v(s) = s^{2p-2}$, and $q = p$, we conclude that

$$\int_t^{1/t} \left( \int_y^{1/t} |\phi'(x+iv)|dv \right)^p y^{p-2} dy \le C^p(p, p) \sup_{t<x<1/t} A(x) \int_t^{1/t} |\phi'(x+iy)|^p y^{2p-2} dy,$$

where

$$A(x) = \int_t^x s^{p-2} ds \left( \int_x^{\frac{1}{t}} s^{(2p-2)(1-p')} ds \right)^{p-1} = \frac{1}{p-1}(x^{p-1} - t^{p-1}) \left( \frac{1}{x} - t \right)^{p-1}.$$

A direct computation shows that the unique critical point of $A(x)$ in $(t, 1/t)$ is $x_0 = t^{1-2/p}$. Thus

$$\sup_{t<x<1/t} A(x) = A(x_0) = \frac{1}{p-1} \left( 1 - t^{2-\frac{2}{p}} \right)^p \le \frac{1}{p-1}.$$

Consequently, there exists some constant $C(p)$ such that

$$\int_t^{1/t} |\phi(x+iy)|^p y^{p-2} dy \le C(p) \left( \int_t^{1/t} |\phi'(x+iy)|^p y^{2p-2} dy + \|\phi'\|^p_{B(\mathbb{U})} t \right).$$

Integrating both sides of the above inequality with respect to $x$ from $-1/t$ to $1/t$, we get

$$\iint_{\mathbb{U}_t} |\phi(x+iy)|^p y^{p-2} dxdy \le C(p) \left( \iint_{\mathbb{U}_t} |\phi'(x+iy)|^p y^{2p-2} dxdy + \|\phi'\|^p_{B(\mathbb{U})} \right).$$

This completes the proof of the lemma. □

Now we can prove the main result ((2) $\Leftrightarrow$ (3) in Theorem 1.3) of this section. It is known that the same result holds on the disk case (see [18]).

**Theorem 3.3.** *Let $p > 1$ and $\mu \in M(\mathbb{U}^*)$ be given. Then $L(\mu) \in \mathcal{B}_p(\mathbb{U})$ if and only if $S(\mu) \in B_p(\mathbb{U})$.*

*Proof.* Recall that $\mathcal{B}_p(\mathbb{U}) \subset \mathcal{B}_0(\mathbb{U})$, and for each holomorphic function $\phi$ on $\mathbb{U}$, $\phi'' \in B_p(\mathbb{U})$ if $\phi \in \mathcal{B}_p(\mathbb{U})$. Thus the only if part follows immediately from

$$S(\mu) = L''(\mu) - \frac{1}{2}(L'(\mu))^2,$$

where $L'(\mu)$ and $L''(\mu)$ are respectively the first and second order derivatives of $L(\mu)$. Precisely,

$$\|S(\mu)\|^p_{B_p(\mathbb{U})} \le 2^{p-1}\|L''(\mu)\|^p_{B_p(\mathbb{U})} + \frac{1}{2}\|L(\mu)\|^p_{\mathcal{B}(\mathbb{U})}\|L(\mu)\|^p_{\mathcal{B}_p(\mathbb{U})} < +\infty.$$

To prove the if part, we assume that $S(\mu) \in B_p(\mathbb{U})$ so that $S(\mu) \in B_0(\mathbb{U})$, which implies that $L(\mu) \in \mathcal{B}_0(\mathbb{U})$. Fix some $\epsilon > 0$ so small such that $\epsilon < 1/C(p)$, where $C(p) > 0$ is the constant in Lemma 3.2. Then there is a positive constant $t_0 < 1$ such that for all $z = x + iy \in \mathbb{U}\backslash\mathbb{U}(t_0)$,

$$y^p|L'(\mu)(x+iy)|^p < \epsilon.$$

By Lemma 3.2 we have for $0 < t < t_0$ that

$$\begin{aligned}
&\frac{1}{C(p)}\iint_{\mathbb{U}(t)} |L'(\mu)(x+iy)|^p y^{p-2}dxdy \\
&\le \iint_{\mathbb{U}(t)} |L''(\mu)(x+iy)|^p y^{2p-2}dxdy + \|L''(\mu)\|^p_{B(\mathbb{U})} \\
&\le 2^{p-1}\iint_{\mathbb{U}(t)} |S(\mu)(x+iy)|^p y^{2p-2}dxdy + \frac{1}{2}\iint_{\mathbb{U}(t)} |L'(\mu)(x+iy)|^{2p} y^{2p-2}dxdy + \|L''(\mu)\|^p_{B(\mathbb{U})} \\
&\le \frac{\pi}{2^{p+1}}\|S(\mu)\|^p_{B_p(\mathbb{U})} + \frac{1}{2}\iint_{\mathbb{U}(t_0)} |L'(\mu)(x+iy)|^{2p} y^{2p-2}dxdy \\
&\quad + \frac{1}{2}\iint_{\mathbb{U}(t)\backslash\mathbb{U}(t_0)} |L'(\mu)(x+iy)|^{2p} y^{2p-2}dxdy + \|L''(\mu)\|^p_{B(\mathbb{U})} \\
&\le \frac{1}{2}\iint_{\mathbb{U}(t_0)} |L'(\mu)(x+iy)|^{2p} y^{2p-2}dxdy + \frac{\epsilon}{2}\iint_{\mathbb{U}(t)} |L'(\mu)(x+iy)|^p y^{p-2}dxdy \\
&\quad + \frac{\pi}{2^{p+1}}\|S(\mu)\|^p_{B_p(\mathbb{U})} + \|L''(\mu)\|^p_{B(\mathbb{U})},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left(\frac{1}{C(p)} - \frac{\epsilon}{2}\right)\iint_{\mathbb{U}(t)} |L'(\mu)(x+iy)|^p y^{p-2}dxdy \\
&\le \frac{1}{2}\iint_{\mathbb{U}(t_0)} |L'(\mu)(x+iy)|^{2p} y^{2p-2}dxdy + \frac{\pi}{2^{p+1}}\|S(\mu)\|^p_{B_p(\mathbb{U})} + \|L''(\mu)\|^p_{B(\mathbb{U})} < \infty.
\end{aligned}$$

Letting $t \to 0$ we obtain $L(\mu) \in \mathcal{B}_p(\mathbb{U})$ as desired. □

## 4. BMO functions and Semmes' construction revisited

In order to prove (the if part of) Theorem 1.2, we need a construction concerning quasiconformal extensions of strongly quasisymmetric homeomorphisms introduced by Semmes [34], which was used to prove the if part of Theorem 1.2 for $p = 2$ in our paper [36].

A locally integrable function $u \in L^1_{loc}(\mathbb{S})$ is said to have bounded mean oscillation and belongs to the space BMO($\mathbb{S}$) if

$$\|u\|_{\mathrm{BMO}} \doteq \sup \frac{1}{|I|} \int_I |u(t) - u_I||dt| < +\infty,$$

where the supremum is taken over all finite sub-intervals $I$ of $\mathbb{S}$, while $u_I$ is the average of $u$ on the interval $I$, namely,

$$u_I = \frac{1}{|I|} \int_I u(t)|dt|.$$

If $u$ also satisfies the condition

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_I |u(t) - u_I||dt| = 0,$$

we say $u$ has vanishing mean oscillation and belongs to the space VMO($\mathbb{S}$). In the following, we denote by $\mathrm{BMO}_{\mathbb{R}}(\mathbb{S})$ and $\mathrm{VMO}_{\mathbb{R}}(\mathbb{S})$ the set of all real-valued BMO and VMO functions, respectively. By the well-known theorem of John-Nirenberg for BMO functions (see [16]), it is known that

$$\frac{1}{|I|} \int_I e^{|u-u_I|} dt \lesssim \|u\|_{\mathrm{BMO}} \tag{1}$$

when $\|u\|_{\mathrm{BMO}}$ is small. It is also known that $\mathcal{B}_p(\mathbb{S}) \subset \mathrm{VMO}(\mathbb{S})$, and the inclusion map is continuous (see [24] for a proof).

We next recall a basic result of Coifman-Meyer [9]. For $u \in \mathrm{BMO}(\mathbb{R})$, set

$$\gamma_u(x) = \frac{\int_0^x e^{u(t)} dt}{\int_0^1 e^{u(t)} dt}, \quad x \in \mathbb{R}.$$

Coifman-Meyer [9] showed that $\gamma_u$ is a strongly quasisymmetric homeomorphism from the real line $\mathbb{R}$ onto a chord-arc curve $\Gamma_u = \gamma_u(\mathbb{R})$ when $\|u\|_{\mathrm{BMO}}$ is small. Recall that a sense preserving homeomorphism $h$ on $\mathbb{R}$ is strongly quasisymmetric if it is locally absolutely continuous so that $|h'|$ is an $A^\infty$ weight introduced by Muckenhoupt [27] (see also [16]) and it maps $\mathbb{R}$ onto a chord-arc curve (see [34]). Clearly, a strongly quasisymmetric homeomorphism from the real line onto itself is quasisymmetric.

In an important paper [34], Semmes showed that, when $\|u\|_{\mathrm{BMO}}$ is small, $\gamma_u$ can be extended to a quasiconformal mapping to the whole plane whose Beltrami coefficient satisfies certain Carleson measure condition. To be precise, let $\varphi$ and $\psi$ be two $C^\infty$ real-valued function on the real line supported on $[-1, 1]$ such that $\varphi$ is even, $\psi$ is odd and $\int_{\mathbb{R}} \varphi(x)dx = 1$, $\int_{\mathbb{R}} \psi(x)xdx = -1$. Define

$$\rho(x, y) = \rho_u(x, y) = \varphi_y * \gamma_u(x) + i\psi_y * \gamma_u(x), \quad z = x + iy \in \mathbb{U}, \tag{2}$$

where $\varphi_y$, $y > 0$, is defined by $\varphi_y(x) = y^{-1}\varphi(y^{-1}x)$. $\psi_y$ is defined by the same way. Semmes proved that $\rho$ is a quasiconformal mapping from the upper half plane $\mathbb{U}$ onto the left domain bounded by $\Gamma_u$ when $\|u\|_{\mathrm{BMO}}$ is small. Furthermore, when $u$ is real-valued, $\rho$ is a quasiconformal mapping of $\mathbb{U}$ onto itself and is quasi-isometric under the Poincaré metric $|dz|/y$. By the standard BMO estimates, Semmes [34] (see also [36]) showed that the Beltrami coefficient $\mu = \bar{\partial}\rho/\partial\rho$ satisfies $\|\mu\|_\infty \lesssim \|u\|_{\mathrm{BMO}}$ if $\|u\|_{\mathrm{BMO}}$ is small.

## 5. Proof of Theorem 1.2

We first recall the following result due to Bourdaud [7] (see also [8], [43]).

**Proposition 5.1.** ([7]) *Let $p > 1$ and $h$ be a quasisymmetric homeomorphism on $\mathbb{S}$. Then the pull-back operator $P_h$ defined by $P_h u = u \circ h$ is a bounded operator from $\mathcal{B}_p(\mathbb{S})$ into itself.*

**Proof of Theorem 1.2 (only if part)** Let $h$ be an increasing homeomorphism from the real line $\mathbb{R}$ onto itself such that $h \in \mathrm{SQS}_p(\mathbb{R})$. Then $h$ can be extended to a quasiconformal mapping of the lower half plane onto itself with Beltrami coefficient $\mu \in M_p^s(\mathbb{U}^*)$. Without loss of generality, we may assume that $h(0) = 0$, $h(1) = 1$. Then there exists a conformal mapping $g$ on the lower half plane such that $g \circ h = f_\mu$ on the real line. Notice that $J \circ f_\mu \circ J = J \circ g \circ J \circ h$ on the real line, where $J(z) = \bar{z}$ is the standard conformal reflection. Since $\mathrm{SQS}_p(\mathbb{R})$ is a group, there exists some $\nu \in M_p^s(\mathbb{U}^*)$ such that $J \circ g \circ J = f_\nu$ on the upper half plane. Noting that $S(M_p^s(\mathbb{U}^*)) \subset S(M(\mathbb{U}^*)) \cap B_p(\mathbb{U})$, we conclude by Theorem 3.3 that $\log f'_\mu \in \mathcal{B}_p(\mathbb{U})$, and $\log(J \circ g \circ J)' \in \mathcal{B}_p(\mathbb{U})$, or equivalently, $\log g' \in \mathcal{B}_p(\mathbb{U}^*)$. Consequently, each of $h$, $f_\mu$ and $g$ is locally absolutely continuous on the real line.

On the other hand, it is well known that each element $\phi \in \mathcal{B}_p(\mathbb{U})$ has boundary values almost everywhere on the real line, and the boundary function $\phi|_{\mathbb{R}}$ belongs to the Sobolev class $\mathcal{B}_p(\mathbb{R})$ (see [49]). We use $\log f'_\mu$ to denote the boundary function of $\log f'_\mu|_{\mathbb{U}}$. Then $\log f'_\mu \in \mathcal{B}_p(\mathbb{R})$. Similarly, $\log g'$ has boundary value function on the real line, denoted by $\log g'$, also being in the Sobolev class $\mathcal{B}_p(\mathbb{R})$.

Now from $g \circ h = f_\mu$ we obtain

$$\log h' = \log f'_\mu - \log g' \circ h,$$

which implies by Proposition 5.1 that $\log h' \in \mathcal{B}_p(\mathbb{R})$ as required. □

The proof of if part will be given by repeating the reasoning from our papers ([36], [37]), where $p = 2$ was considered again. We first prove the following result.

**Lemma 5.2.** *There exists some universal constant $\delta > 0$ such that, for any $u \in \mathcal{B}_p(\mathbb{R})$ with $\|u\|_{\mathcal{B}_p(\mathbb{R})} < \delta$, the mapping $\rho = \rho_u$ defined by (2) is quasiconformal whose Beltrami coefficient $\mu$ satisfies $\|\mu\|_{QS_p(\mathbb{R})} \lesssim \|u\|_{\mathcal{B}_p(\mathbb{R})}$ and thus belongs to the class $M_p(\mathbb{U})$*[1)].

*Proof.* By the continuity of the inclusion $\mathcal{B}_p(\mathbb{R}) \to \mathrm{BMO}(\mathbb{R})$, we conclude that there exists some universal constant $\delta > 0$ such that, for any $u \in \mathcal{B}_p(\mathbb{R})$ with $\|u\|_{\mathcal{B}_p(\mathbb{R})} < \delta$, the mapping $\rho = \rho_u$ defined by (2) is quasiconformal. It remains to show that $\mu \in M_p(\mathbb{U})$.

For $z = x + iy \in \mathbb{U}$, set $I = [x - y, x + y]$ so that

$$u_I = \frac{1}{2y}\int_{x-y}^{x+y} u(t)dt.$$

Then we have (see [36], [37])

$$|\mu(z)| \lesssim \frac{1}{|I|}\int_I |u(t) - u_I| e^{|u(t)-u_I|}dt.$$

By Hölder's inequality, we conclude by (1) that

$$\begin{aligned}
|\mu(z)|^p &\lesssim \frac{1}{|I|^p}\int_I |u(t) - u_I|^p dt \left(\int_I e^{p'|u(t)-u_I|}dt\right)^{\frac{p}{p'}} \\
&\lesssim \frac{1}{|I|}\int_I |u(t) - u_I|^p dt \\
&\lesssim \frac{1}{|I|}\int_I |u(t) - u(x)|^p dt + |u(x) - u_I|^p.
\end{aligned}$$

On the other hand, we conclude by Hölder's inequality again that

$$\begin{aligned}
|u(x) - u_I|^p &= \left|\frac{1}{|I|}\int_I u(t)dt - u(x)\right|^p \\
&= \left|\frac{1}{|I|}\int_I (u(t) - u(x))dt\right|^p \\
&\lesssim \frac{1}{|I|}\int_I |u(t) - u(x)|^p dt.
\end{aligned}$$

[1)] $M_p(\mathbb{U})$ can be defined in the same manner as $M_p(\mathbb{U}^*)$.

Consequently,

$$|\mu(z)|^p \lesssim \frac{1}{|I|}\int_I |u(t)-u(x)|^p dt \asymp \frac{1}{y}\int_{-y}^{y}|u(t+x)-u(x)|^p dt.$$

Thus, we have

$$\begin{aligned}\iint_{\mathbb{U}} \frac{|\mu(z)|^p}{y^2}dxdy &\lesssim \iint_{\mathbb{U}}\int_{-y}^{y}\frac{|u(t+x)-u(x)|^p}{y^3}dtdxdy\\ &= \int_{-\infty}^{+\infty}dx\int_0^{+\infty}\frac{dy}{y^3}\int_{-y}^{y}|u(t+x)-u(x)|^p dt\\ &= \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}|u(x+t)-u(x)|^p dt\int_{|t|}^{+\infty}\frac{dy}{y^3}\\ &= \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}\frac{|u(x+t)-u(x)|^p}{2t^2}dt\\ &\asymp \|u\|^p_{\mathcal{B}_p(\mathbb{R})}.\end{aligned}$$

□

**Corollary 5.3.** *Let $h$ be an increasing and locally absolutely continuous homeomorphism from the real line onto itself such that $\|\log h'\|_{\mathcal{B}_p(\mathbb{R})} < \delta$. Then $h$ can be extended to a quasiconformal mapping to the upper half plane which is quasi-isometric under the Poincaré metric $|dz|/y$ and has Beltrami coefficient in $M_p(\mathbb{U})$. In particular, $h$ belongs the class $SQS_p(\mathbb{R})$.*

To prove (the if part of) Theorem 1.2, we will decompose a homeomorphism $h$ with finite $\|\log h'\|_{\mathcal{B}_p(\mathbb{R})}$ into homeomorphisms $h_j$ with small norms $\|\log h_j'\|_{\mathcal{B}_p(\mathbb{R})}$. We need

**Lemma 5.4.** *Let $h$ be an increasing and locally absolutely continuous homeomorphism from the real line onto itself such that $\|\log h'\|_{\mathcal{B}_p(\mathbb{R})} < \infty$. Then $\log h'$ is in the closure of $L^\infty(\mathbb{R})$ under the BMO norm. In particular, $h$ is strongly quasisymmetric.*

*Proof.* Consider the Cayley transformation $\gamma(z) = \frac{z-i}{z+i}$ from the upper half plane $\mathbb{U}$ onto the unit disk $\Delta$. Since $\log h' \in \mathcal{B}_p(\mathbb{R})$, $\log h' \circ \gamma^{-1} \in \mathcal{B}_p(S^1) \subset \mathrm{VMO}(S^1)$, which implies that $\log h' \circ \gamma^{-1}$ can be approximated by a sequence of bounded functions $(u_n)$ on the unit circle under the BMO norm (see [16]). Thus, $\log h'$ can be approximated by the bounded functions $u_n \circ \gamma$ on the real line under the BMO norm. The second statement follows immediately from Lemma 1.4 in [29]. □

**Proof of Theorem 1.2 (if part)** Let $h$ be an increasing and locally absolutely continuous homeomorphism from the real line onto itself such that $\log h'$ belongs to the Sobolev class $\mathcal{B}_p(\mathbb{R})$. Without loss of generality, we assume $h(0) = 0$. For each real number $t \in [0,1]$, set

$$h_t(x) = \int_0^x (h'(s))^t ds,\ x \in \mathbb{R}.$$

Then $h_t$ is an increasing and locally absolutely continuous homeomorphism from the real line onto itself with $h_0 = \mathrm{id}$, $h_1 = h$, and $\log h_t' = t\log h'$, which implies by Lemma 5.4 that $h_t$ is strongly quasisymmetric. Noting that for any fixed $t \in [0,1]$,

$$\|\log(h_s \circ h_t^{-1})'\|_{\mathcal{B}_p(\mathbb{R})} = \|(\log h_s' - \log h_t') \circ h_t^{-1}\|_{\mathcal{B}_p(\mathbb{R})} = |s-t|\|P_{h_t}^{-1}\log h'\|_{\mathcal{B}_p(\mathbb{R})},$$

we conclude by Proposition 5.1 that there exists a neighbourhood $I_t$ such that $\|\log(h_s \circ h_t^{-1})'\|_{\mathcal{B}_p(\mathbb{R})} < \delta$ when $s \in I_t$. By compactness, we conclude that there exists a sequence of finite numbers $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$ such that $\|\log(h_{t_j} \circ h_{t_{j+1}}^{-1})'\|_{\mathcal{B}_p(\mathbb{R})} < \delta$ for $j = 0,1,2,\cdots,n-1,n$. Since $\mathrm{SQS}_p(\mathbb{R})$ is a group, and

$$h^{-1} = (h_{t_0} \circ h_{t_1}^{-1}) \circ (h_{t_1} \circ h_{t_2}^{-1}) \circ \cdots \circ (h_{t_n} \circ h_{t_{n+1}}^{-1}),$$

We conclude by Corollary 5.3 that $h \in \mathrm{SQS}_p(\mathbb{R})$. □

**6. Proof of Theorems 1.3 and 1.4**

We first prove the following result which was stated in section 1.

**Lemma 6.1.** *A p-integrable quasicircle must be a chord-arc curve.*

*Proof.* Let $\Gamma$ be a $p$-integrable quasicircle. Choose a conformal mapping $f$ from $\mathbb{D}$ onto the left domain $\Omega$ bounded by $\Gamma$. Then $\log f' \in \mathcal{B}_p(\mathbb{D})$ so that its boundary function $\log f' \in \mathcal{B}_p(\mathbb{S}) \subset \mathrm{VMO}(\mathbb{S})$.

If $\Gamma$ is a bounded curve, we let $\mathbb{D} = \Delta$ and then conclude by a Pommerenke's result (see [30]) that $\Gamma$ is asymptotically smooth, which means that $\Gamma$ is rectifiable, and

$$\lim_{|\zeta - z| \to 0} \frac{\mathrm{length}(\widetilde{\zeta z})}{|\zeta - z|} = 1$$

for any two points $z$ and $\zeta$ of $\Gamma$. Since $\Gamma$ is a quasicircle, this already implies that $\Gamma$ is a chord-arc curve.

If $\Gamma$ passes through $\infty$, we let $\mathbb{D} = \mathbb{U}$ and consider again the Cayley transformation $\gamma(z) = \frac{z-i}{z+i}$ from the upper half plane $\mathbb{U}$ onto the unit disk $\Delta$. Choose a point $z_0 \in \Omega$ and set $\tilde{\gamma}(z) = \frac{1}{z - z_0}$. Then $\tilde{f} \doteq \tilde{\gamma} \circ f \circ \gamma^{-1}$ is a conformal mapping from $\Delta$ onto a bounded domain with boundary the quasicircle $\tilde{\Gamma} = \tilde{\gamma}(\Gamma)$. Since $\log f' \in \mathcal{B}_p(\mathbb{U})$, Theorem 3.3 implies that $S(f) \in B_p(\mathbb{U})$, which implies that $S(f) \circ (\gamma)^{-1} (\gamma^{-1})'^2 \in B_p(\Delta)$, that is, $S(\tilde{f}) \in B_p(\Delta)$, which in turn implies that $\log \tilde{f}' \in \mathcal{B}_p(\Delta)$. Then $\tilde{\Gamma}$ is a chord-arc curve. By the Möbius invariance of chord-arc curves (see [25]), we conclude that $\Gamma$ is a chord-arc curve. □

Now we begin to prove Theorems 1.3 and 1.4. Let $\Gamma$ be a chord-arc curve passing through $\infty$ and $z = z(s)$ be an arc-length parametrization of $\Gamma$. Let $f$ map the upper half plane $\mathbb{U}$ conformally onto the left domain $\Omega$ bounded by $\Gamma$ with $f(\infty) = \infty$. Set $h_1 : \mathbb{R} \to \mathbb{R}$ by $z \circ h_1 = f$. Then we have

**Lemma 6.2.** *Under the above notations, the following statements are equivalent*:
(1) $\Gamma$ *is a p-integrable quasicircle*;
(2) $h_1 \in SQS_p(\mathbb{R})$;
(3) $\arg z' \in \mathcal{B}_p(\mathbb{R})$.

*Proof.* From $z \circ h_1 = f$ we obtain $f' = (z' \circ h_1) h_1'$, which implies that

$$\Re \log f' = \log h_1', \ \Im \log f' = \arg z' \circ h_1 \tag{3}$$

on the real line. Since $\Gamma$ is a chord-arc curve, $\Gamma$ is a $p$-integrable quasicircle if and only if

$$\log f' \in \mathcal{B}_p(\mathbb{U}) \Leftrightarrow \Re \log f' \in \mathcal{B}_p(\mathbb{R}) \Leftrightarrow \Im \log f' \in \mathcal{B}_p(\mathbb{R}).$$

By (3) and Theorem 1.2 we obtain that (1) $\Leftrightarrow$ (2). On the other hand, since $\Gamma$ is a chord-arc curve, a classical result of Lavrentiev [22] implies that $h_1$ is locally absolutely continuous so that $h_1'$ belongs to the class of weights $A^\infty$, or equivalently, $h_1$ is a strongly quasisymmetric homeomorphism and consequently quasisymmetric. By (3) and Proposition 5.1, we conclude that (1) $\Leftrightarrow$ (3). □

**Proof of Theorem 1.4** (1) $\Rightarrow$ (3) Let $\Gamma$ be a $p$-integrable quasicircle passing through $\infty$. Lemma 6.1 implies that $\Gamma$ is a chord-arc curve. We conclude by David's result (see [11]) that there exists a real-valued BMO function $b \in \mathrm{BMO}_{\mathbb{R}}(\mathbb{R})$ such that an arc-length parametrization $z = z(s)$ of $\Gamma$ satisfies the condition $z'(s) = e^{ib(s)}$. Now Lemma 6.2 implies that $b = \arg z' \in \mathcal{B}_{p,\mathbb{R}}(\mathbb{R})$.

(3) $\Rightarrow$ (1) Suppose $\Gamma$ is a chord-arc curve and an arclength parameterization $z : \mathbb{R} \to \Gamma$ satisfies the condition $z'(s) = e^{ib(s)}$ for some $b \in \mathcal{B}_{p,\mathbb{R}}(\mathbb{R})$. Since $\arg z' = b \in \mathcal{B}_p(\mathbb{R})$, we conclude by Lemma 6.2 again that $\Gamma$ is a $p$-integrable quasicircle.

(2) $\Leftrightarrow$ (3) Since $h = g^{-1} \circ f$ is the quasisymmetric conformal sewing for $\Gamma$, $h^{-1} = f^{-1} \circ g = (J \circ f \circ J)^{-1} \circ (J \circ g \circ J)$ is the conformal sewing for $J(\Gamma)$. Clearly, $J(z(s))$ is the arclength parameterization for $J(\Gamma)$ and

$(J(z))'(s) = e^{-ib(s)}$. By (1) ⇔ (3), we conclude that $b \in \mathcal{B}_p(\mathbb{R})$ if and only if $\log(J \circ g \circ J)' \in \mathcal{B}_p(\mathbb{U})$, which is equivalent to $\log g' \in \mathcal{B}_p(\mathbb{U}^*)$.

(3) ⇔ (4) $\tau(z)$ and $z(s)$ are related by $\tau \circ z = z'$, that is, $u \circ z = b$. Since $z : \mathbb{R} \to \Gamma$ is an arc-length parameterization of the chord-arc curve $\Gamma$, $z : \mathbb{R} \to \Gamma$ is bi-Lipschitz, that is, $|z(t) - z(s)| \le |t - s| \le C|z(t) - z(s)|$ for some $C \ge 1$. Thus $b \in \mathcal{B}_p(\mathbb{R})$ if and only if $u \in \mathcal{B}_p(\Gamma)$. □

**Proof of Theorem 1.3** (1) ⇔ (4) follows from Theorem 1.2, while (2) ⇔ (3) from Theorem 3.3. (1) ⇒ (2) follows from the relation $S(M_p^s(\mathbb{U}^*)) \subset S(M(\mathbb{U}^*)) \cap B_p(\mathbb{U})$, which has been used in the proof of the only if part of Theorem 1.2. The same reasoning can be used to prove (3) ⇒ (4). In fact, (3) implies $\log g' \in \mathcal{B}_p(\mathbb{U}^*)$ by Theorem 1.4. Then from $g \circ h = f$, we obtain $\log h' = \log f' - \log g' \circ h$, which implies by Proposition 5.1 that $\log h' \in \mathcal{B}_p(\mathbb{R})$. This completes the proof of Theorem 1.3. □

## 7. Concluding remarks and questions

When $p \ge 2$, $T_p$ has a unique complex Banach manifold structure (via the Bers embedding $\beta : T_p \to B_p(\mathbb{U})$) such that the natural projection $\Phi : M_p(\mathbb{U}^*) \to T_p$ is a holomorphic split submersion (see [42]). In [36] we proved that the correspondence $h \mapsto \log h'$ induces a real analytic map from (the quasisymmetric homeomorphism model of) $T_2$ onto $\mathcal{B}_{2,\mathbb{R}}(\mathbb{R})/\mathbb{R}$ whose inverse is also real analytic. By the same reasoning we can show that this result still holds for a general $p > 2$, namely, the correspondence $h \mapsto \log h'$ induces a real analytic map from (the quasisymmetric homeomorphism model of) $T_p$ onto $\mathcal{B}_{p,\mathbb{R}}(\mathbb{R})/\mathbb{R}$ whose inverse is also real analytic. A more general result can be found in Wei-Matsuzaki ([44], [45]).

A $p$-integrable quasicircle $\Gamma$ is said to be normalized if it passes through 0 and $\infty$, and the unique arclength parameterization $z : \mathbb{R} \to \Gamma$ with $z(0) = 0$ satisfies the condition $z(1) > 0$. Then the quasicircle model of $T_p$ is precisely the set of all normalized $p$-integrable quasicircles. For each normalized $p$-integrable quasicircle $\Gamma$, Theorem 1.4 says that there exists some $b \in \mathcal{B}_{p,\mathbb{R}}(\mathbb{R})/\mathbb{R}$ such that $z'(s) = e^{ib(s)}$. It is easy to see that for each $p > 1$ the set $\hat{T}_p$ of all these functions $b$ is open in $\mathcal{B}_{p,\mathbb{R}}(\mathbb{R})/\mathbb{R}$. In [39] we proved that the correspondence $\Gamma \mapsto b$ induces a homeomorphism from (the quasicircle model of) $T_2$ onto $\hat{T}_2$. Wei-Matsuzaki ([44], [45]) showed that this result remains true when $p > 2$.

The situation seems to be complicated when $p < 2$. We first need to endow the strong $p$-integrable Teichmüller space $T_p^s$ with a complex Banach manifold structure. Since $\beta(T_p^s)$ is an open subset of $B_p(\mathbb{U})$, a natural complex Banach manifold structure is endowed by declaring $\beta$ to be a biholomorphic isomorphism from $T_p^s$ onto $\beta(T_p^s)$. $T_p^s$ can also be endowed with two real Banach manifold structures, one is by the correspondence $h \mapsto \log h'$ from (the quasisymmetric homeomorphism model of) $T_p^s$ onto $\mathcal{B}_{p,\mathbb{R}}(\mathbb{R})/\mathbb{R}$, the other is by the correspondence $\Gamma \mapsto b$ from (the quasicircle model of) $T_p^s$ onto $\hat{T}_p$. It is not clear whether these manifold structures are well compatible with each other. Finally, it remains open whether a $p$-integrable asymptotic affine homeomorphism is strong $p$-integrable asymptotic affine when $p < 2$.

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