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On inequalities of Simpson type for co-ordinated convex functions via generalized fractional integrals

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Abstract. In this study, we prove equality for twice partially differentiable mappings involving the double generalized fractional integral. Using the established identity, we offer some Simpson's type inequalities for differentiable co-ordinated convex functions in a rectangle from the plane $\mathbb{R}^2$.

1. Introduction

Simpson's inequality plays an essential role role in many areas of mathematics. The classical Simpson's inequality is expressed as follows for four times continuously differentiable functions:

Theorem 1.1. *For a mapping $\mathbb{F} : [\tau_1, \tau_2] \to \mathbb{R}$ which is four times continuously differentiable on (τ_1, τ_2), and let $\left\|\mathbb{F}^{(4)}\right\|_\infty = \sup_{\kappa_1 \in (\tau_1,\tau_2)} \left|\mathbb{F}^{(4)}(\kappa_1)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\mathbb{F}(\tau_1) + \mathbb{F}(\tau_2)}{2} + 2\mathbb{F}\left(\frac{\tau_1 + \tau_2}{2}\right)\right] - \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \mathbb{F}(\kappa_1)d\kappa_1\right| \le \frac{1}{2880}\left\|\mathbb{F}^{(4)}\right\|_\infty (\tau_2 - \tau_1)^4.$$

Over the years, many variations of this inequality have been studied for various function classes, such as convex functions, bounded functions, functions of bounded variation, etc. Specifically, since convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics, many papers have been dedicated to Simpson inequality for convex functions. For example, some authors proved several Simpson type inequalities for differentiable and twice differentiable convex functions [5, 36–38]. In [10, 19], the authors study on obtaining some new Simpson inequalities for Riemann-Liouville fractional integrals. What's more, a number of papers are devoted to Simpson type inequalities and important other types of inequalities for several kinds of fractional integrals or for functions belong to other convex classes [2–4, 6, 11–14, 17, 18, 20, 23–25, 28, 29, 31, 33]. On the other hand, Ozdemir et al. extended the Simpson inequality for co-ordinated convex mappings in [30]. For some of the other papers on Simpson inequalities functions of two variables, one can see [1, 8, 32, 40, 41].

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This paper aims to obtain some Simpson type inequalities for co-ordinated convex functions involving generalized fractional integrals. The general structure of the study consists of five chapters including an introduction. The remaining part of the paper proceeds as follows: In Section 2, we give the definitions and theorems to be used in the main section. In Section 3, an identity involving generalized fractional integrals are presented for partial differentiable functions. Then we establish several Simpson type inequalities for mappings whose partial derivatives in absolute value are co-ordinated convex in Section 4. At the end of the paper, some conclusions and further directions of research are discussed in Section 5.

2. Preliminaries

In this section, we present some definitions and results which will be used in our main section. First of all, we will need the following definition:

Definition 2.1. *Let* $\Delta =: [\tau_1, \tau_2] \times [\tau_3, \tau_4]$ *in* $\mathbb{R}^2$ *with* $\tau_1 < \tau_2$ *and* $\tau_3 < \tau_4$. *A mapping* $\mathbb{F} : \Delta \to \mathbb{R}$ *is said to be convex in the bidimensional interval* Δ, *if the following inequality holds:*

$$\begin{aligned} & \mathbb{F}\left(t\kappa_1 + (1-t)\kappa_2, s\kappa_3 + (1-s)\kappa_4\right) \\ \leq \;& ts\mathbb{F}(\kappa_1, \kappa_3) + s(1-t)\mathbb{F}(\kappa_2, \kappa_3) + t(1-s)\mathbb{F}(\kappa_1, \kappa_4) + (1-t)(1-s)\mathbb{F}(\kappa_2, \kappa_4) \end{aligned} \tag{1}$$

for all $(\kappa_1, \kappa_2), (\kappa_3, \kappa_4) \in \Delta$ *and* $t, s \in [0, 1]$.

In this section we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [35].

Let's define a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions :

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{\tau_1+}I_{\varphi}\mathbb{F}(\kappa_1) = \int_{\tau_1}^{\kappa_1} \frac{\varphi(\kappa_1 - t)}{\kappa_1 - t}\mathbb{F}(t)dt, \quad \kappa_1 > \tau_1, \tag{2}$$

$$_{\tau_2-}I_{\varphi}\mathbb{F}(\kappa_1) = \int_{\kappa_1}^{\tau_2} \frac{\varphi(t - \kappa_1)}{t - \kappa_1}\mathbb{F}(t)dt, \quad \kappa_1 < \tau_2. \tag{3}$$

Some forms of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc are generalized as the most significant feature of generalized fractional integrals. These important special cases of the integral operators (2) and (3) are mentioned below.

i) If we take $\varphi(t) = t$, the operators (2) and (3) reduce to the Riemann integral as follows:

$$I_{\tau_1+}\mathbb{F}(\kappa_1) = \int_{\tau_1}^{\kappa_1} \mathbb{F}(t)dt, \quad \kappa_1 > \tau_1,$$

$$I_{\tau_2-}\mathbb{F}(\kappa_1) = \int_{\kappa_1}^{\tau_2} \mathbb{F}(t)dt, \quad \kappa_1 < \tau_2.$$

ii) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\alpha > 0$, the operators (2) and (3) reduce to the Riemann-Liouville fractional integral as follows:

$$J_{\tau_1^+}^{\alpha}\mathbb{F}(\kappa_1) = \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\kappa_1} (\kappa_1 - t)^{\alpha - 1}\mathbb{F}(t)dt, \quad \kappa_1 > \tau_1,$$

$$J^{\alpha}_{\tau_2-}\mathbb{F}(\kappa_1) = \frac{1}{\Gamma(\alpha)}\int_{\kappa_1}^{\tau_2} (t-\kappa_1)^{\alpha-1}\mathbb{F}(t)dt, \ \ \kappa_1 < \tau_2.$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$, $\alpha, k > 0$, the operators (2) and (3) reduce to the k-Riemann-Liouville fractional integral as follows:

$$J^{\alpha}_{\tau_1^+,k}\mathbb{F}(\kappa_1) = \frac{1}{k\Gamma_k(\alpha)}\int_{\tau_1}^{\kappa_1} (\kappa_1-t)^{\frac{\alpha}{k}-1}\mathbb{F}(t)dt, \ \ \kappa_1 > \tau_1,$$

$$J^{\alpha}_{\tau_2-,k}\mathbb{F}(\kappa_1) = \frac{1}{k\Gamma_k(\alpha)}\int_{\kappa_1}^{\tau_2} (t-\kappa_1)^{\frac{\alpha}{k}-1}\mathbb{F}(t)dt, \ \ \kappa_1 < \tau_2$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-\frac{t^k}{k}}dt, \ \ \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \ \ \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [26].

There are numerous articles in the literature on inequalities via generalized fractional integrals. Please refer to any of them [7, 9, 14, 16, 17, 21, 22, 27, 42].

Inspired by this definition, Turkay et al. [39] give the following definitions:

Definition 2.2. *Let* $\mathbb{F} \in L_1([\tau_1,\tau_2]\times[\tau_3,\tau_4])$. *The Generalized double fractional integrals* ${}_{\tau_1+,\tau_3+}I_{\varphi,\psi}$, ${}_{\tau_1+,\tau_4-}I_{\varphi,\psi}$, ${}_{\tau_2-,\tau_3+}I_{\varphi,\psi}$, ${}_{\tau_2-,\tau_4-}I_{\varphi,\psi}$ *are defined by*

$${}_{\tau_1+,\tau_3+}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_2) = \int_{\tau_1}^{\kappa_1}\int_{\tau_3}^{\kappa_2} \frac{\varphi(\kappa_1-t)}{\kappa_1-t}\frac{\psi(\kappa_2-s)}{\kappa_2-s}\mathbb{F}(t,s)dsdt, \ \kappa_1 > \tau_1, \ \kappa_2 > \tau_3, \tag{4}$$

$${}_{\tau_1+,\tau_4-}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_2) = \int_{\tau_1}^{\kappa_1}\int_{\kappa_2}^{\tau_4} \frac{\varphi(\kappa_1-t)}{\kappa_1-t}\frac{\psi(s-\kappa_2)}{s-\kappa_2}\mathbb{F}(t,s)dsdt, \ \kappa_1 > \tau_1, \ \kappa_2 < \tau_4, \tag{5}$$

$${}_{\tau_2-,\tau_3+}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_2) = \int_{\kappa_1}^{\tau_2}\int_{\tau_3}^{\kappa_2} \frac{\varphi(t-\kappa_1)}{t-\kappa_1}\frac{\psi(\kappa_2-s)}{\kappa_2-s}\mathbb{F}(t,s)dsdt, \ \kappa_1 < \tau_2, \ \kappa_2 > \tau_3, \tag{6}$$

and

$${}_{\tau_2-,\tau_4-}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_2) = \int_{\kappa_1}^{\tau_2}\int_{\kappa_2}^{\tau_4} \frac{\varphi(t-\kappa_1)}{t-\kappa_1}\frac{\psi(s-\kappa_2)}{s-\kappa_2}\mathbb{F}(t,s)dsdt, \ \kappa_1 < \tau_2, \ \kappa_2 < \tau_4, \tag{7}$$

where $\varphi,\psi : [0,\infty) \to [0,\infty)$ *functions which satisfy* $\int_0^1 \frac{\varphi(t)}{t}dt < \infty$ *and* $\int_0^1 \frac{\psi(s)}{s}ds < \infty$, *respectively.*

In this definition, known fractional integrals can be obtained by some special choices. For example;

i) If we take $\varphi(t) = t$ and $\psi(s) = s$, then the operators (4), (5), (6) and (7) transform into the the Riemann integrals on two coordinates respectively as the following

$$I_{\tau_1+,\tau_3+}\mathbb{F}(\kappa_1,\kappa_2) = \int_{\tau_1}^{\kappa_1}\int_{\tau_3}^{\kappa_2}\mathbb{F}(t,s)dsdt, \ \kappa_1 > \tau_1, \ \kappa_2 > \tau_3,$$

$$I_{\tau_1+,\tau_4-}\mathbb{F}(\kappa_1,\kappa_2) = \int_{\tau_1}^{\kappa_1}\int_{\kappa_2}^{\tau_4}\mathbb{F}(t,s)dsdt, \ \kappa_1 > \tau_1, \ \kappa_2 < \tau_4,$$

$$I_{\tau_2-,\tau_3+}\mathbb{F}(\kappa_1,\kappa_2)=\int_{\kappa_1}^{\tau_2}\int_{\tau_3}^{\kappa_2}\mathbb{F}(t,s)\,dsdt,\ \kappa_1<\tau_2,\ \kappa_2>\tau_3,$$

and

$$I_{\tau_2-,\tau_4-}\mathbb{F}(\kappa_1,\kappa_2)=\int_{\kappa_1}^{\tau_2}\int_{\kappa_2}^{\tau_4}\mathbb{F}(t,s)\,dsdt,\ \kappa_1<\tau_2,\ \kappa_2<\tau_4.$$

ii) If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, $\psi(s)=\frac{s^{\beta}}{\Gamma(\beta)}$, then for $\alpha,\beta>0$ the operators (4), (5), (6) and (7) transform into the Riemann-Liouville integrals on two coordinates [34] respectively as the following

$$J_{\tau_1+,\tau_3+}^{\alpha,\beta}\mathbb{F}(\kappa_1,\kappa_2)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\tau_1}^{\kappa_1}\int_{\tau_3}^{\kappa_2}(\kappa_1-t)^{\alpha-1}(\kappa_2-s)^{\beta-1}\mathbb{F}(t,s)\,dsdt,\ \kappa_1>\tau_1,\ \kappa_2>\tau_3,$$

$$J_{\tau_1+,\tau_4-}^{\alpha,\beta}\mathbb{F}(\kappa_1,\kappa_2)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\tau_1}^{\kappa_1}\int_{\kappa_2}^{\tau_4}(\kappa_1-t)^{\alpha-1}(s-\kappa_2)^{\beta-1}\mathbb{F}(t,s)\,dsdt,\ \kappa_1>\tau_1,\ \kappa_2<\tau_4,$$

$$J_{\tau_2-,\tau_3+}^{\alpha,\beta}\mathbb{F}(\kappa_1,\kappa_2)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\tau_2}\int_{\tau_3}^{\kappa_2}(t-\kappa_1)^{\alpha-1}(\kappa_2-s)^{\beta-1}\mathbb{F}(t,s)\,dsdt,\ \kappa_1<\tau_2,\ \kappa_2>\tau_3,$$

and

$$J_{\tau_2-,\tau_4-}^{\alpha,\beta}\mathbb{F}(\kappa_1,\kappa_2)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\tau_2}\int_{\kappa_2}^{\tau_4}(t-\kappa_1)^{\alpha-1}(s-\kappa_2)^{\beta-1}\mathbb{F}(t,s)\,dsdt,\ \kappa_1<\tau_2,\ \kappa_2<\tau_4,$$

along with, Γ is the Gamma function.

iii) If we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, for $\alpha,\beta,k>0$ then the operators (4), (5), (6) and (7) transform into the Riemann-Liouville $k-$fractional integrals on two coordinates [15] respectively as the following

$$J_{\tau_1+,\tau_3+}^{\alpha,\beta,k}\mathbb{F}(\kappa_1,\kappa_2)=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\tau_1}^{\kappa_1}\int_{\tau_3}^{\kappa_2}(\kappa_1-t)^{\frac{\alpha}{k}-1}(\kappa_2-s)^{\frac{\beta}{k}-1}\mathbb{F}(t,s)\,dsdt,\ \kappa_1>\tau_1,\ \kappa_2>\tau_3,$$

$$J_{\tau_1+,\tau_4-}^{\alpha,\beta,k}\mathbb{F}(\kappa_1,\kappa_2)=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\tau_1}^{\kappa_1}\int_{\kappa_2}^{\tau_4}(\kappa_1-t)^{\frac{\alpha}{k}-1}(s-\kappa_2)^{\frac{\beta}{k}-1}\mathbb{F}(t,s)\,dsdt,\ \kappa_1>\tau_1,\ \kappa_2<\tau_4,$$

$$J_{\tau_2-,\tau_3+}^{\alpha,\beta,k}\mathbb{F}(\kappa_1,\kappa_2)=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\kappa_1}^{\tau_2}\int_{\tau_3}^{\kappa_2}(t-\kappa_1)^{\frac{\alpha}{k}-1}(\kappa_2-s)^{\frac{\beta}{k}-1}\mathbb{F}(t,s)\,dsdt,\ \kappa_1<\tau_2,\ \kappa_2>\tau_3,$$

and

$$J_{\tau_2-,\tau_4-}^{\alpha,\beta,k}\mathbb{F}(\kappa_1,\kappa_2)=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\kappa_1}^{\tau_2}\int_{\kappa_2}^{\tau_4}(t-\kappa_1)^{\frac{\alpha}{k}-1}(s-\kappa_2)^{\frac{\beta}{k}-1}\mathbb{F}(t,s)\,dsdt,\ \kappa_1<\tau_2,\ \kappa_2<\tau_4,$$

where Γ_k is the k-Gamma function.

3. An identity for Generalized double fractional integrals

Throughout this study, for brevity, we define

$$\Lambda_1(\kappa_1,t)=\int_0^t\frac{\varphi((\tau_2-\kappa_1)u)}{u}du,\quad \Delta_1(\kappa_1,t)=\int_0^t\frac{\varphi((\kappa_1-\tau_1)u)}{u}du, \tag{8}$$

and

$$\Lambda_2(\kappa_2, s) = \int_0^s \frac{\psi((\tau_4 - \kappa_2)u)}{u}du, \quad \Delta_2(\kappa_2, s) = \int_0^s \frac{\psi((\kappa_2 - \tau_3)u)}{u}du. \tag{9}$$

Particularly, if we choose $\kappa_1 = \frac{\tau_1+\tau_2}{2}$ and $\kappa_2 = \frac{\tau_3+\tau_4}{2}$, then we have

$$\Lambda_1\left(\frac{\tau_1+\tau_2}{2}, t\right) = \Delta_1\left(\frac{\tau_1+\tau_2}{2}, t\right) = \Upsilon_1(t) = \int_0^t \frac{\varphi\left(\left(\frac{\tau_2-\tau_1}{2}\right)u\right)}{u}du \tag{10}$$

and

$$\Lambda_2\left(\frac{\tau_1+\tau_2}{2}, s\right) = \Delta_2\left(\frac{\tau_1+\tau_2}{2}, s\right) = \Upsilon_2(s) = \int_0^s \frac{\psi((\kappa_2 - \tau_3)u)}{u}du. \tag{11}$$

Lemma 3.1. *Let $\mathbb{F} : \Delta := [\tau_1, \tau_2] \times [\tau_3, \tau_4] \to \mathbb{R}$ be an absolutely continuous function on Δ such that the partial derivative of order $\frac{\partial^2 \mathbb{F}(t,s)}{\partial t \partial s}$ exist for all $(t, s) \in \Delta$. Then, the following equality holds:*

$$\begin{aligned}
&\Omega(\tau_1, \tau_2, \kappa_1; \tau_3, \tau_4, \kappa_2) \\
= \ &\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)} \int_0^1\int_0^1 (\Lambda_1(\kappa_1,1) - 3\Lambda_1(\kappa_1,t))(\Lambda_2(\kappa_2,1) - 3\Lambda_2(\kappa_2,s)) \\
&\times \frac{\partial^2}{\partial t \partial s}\mathbb{F}(t\kappa_1 + (1-t)\tau_2, s\kappa_2 + (1-s)\tau_4)\,dsdt \\
&- \frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)} \int_0^1\int_0^1 (\Lambda_1(\kappa_1,1) - 3\Lambda_1(\kappa_1,t))(\Delta_2(\kappa_2,1) - 3\Delta_2(\kappa_2,s)) \\
&\times \frac{\partial^2}{\partial t \partial s}\mathbb{F}(t\kappa_1 + (1-t)\tau_2, s\kappa_2 + (1-s)\tau_3)\,dsdt \\
&- \frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)} \int_0^1\int_0^1 (\Delta_1(\kappa_1,1) - 3\Delta_1(\kappa_1,t))(\Lambda_2(\kappa_2,1) - 3\Lambda_2(\kappa_2,s)) \\
&\times \frac{\partial^2}{\partial t \partial s}\mathbb{F}(t\kappa_1 + (1-t)\tau_1, s\kappa_2 + (1-s)\tau_4)\,dsdt \\
&+ \frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)} \int_0^1\int_0^1 (\Delta_1(\kappa_1,1) - 3\Delta_1(\kappa_1,t))(\Delta_2(\kappa_2,1) - 3\Delta_2(\kappa_2,s)) \\
&\times \frac{\partial^2}{\partial t \partial s}\mathbb{F}(t\kappa_1 + (1-t)\tau_1, s\kappa_2 + (1-s)\tau_3)\,dsdt
\end{aligned}$$

where

$$\begin{aligned}
&\Omega(\tau_1, \tau_2, \kappa_1; \tau_3, \tau_4, \kappa_2) \\
= \ &\frac{4\mathbb{F}(\kappa_1,\kappa_2) + \mathbb{F}(\kappa_1,\tau_4) + \mathbb{F}(\kappa_1,\tau_3) + \mathbb{F}(\tau_2,\kappa_2) + \mathbb{F}(\tau_1,\kappa_2)}{9} \\
&+ \frac{\mathbb{F}(\tau_2,\tau_4) + \mathbb{F}(\tau_2,\tau_3) + \mathbb{F}(\tau_1,\tau_4) + \mathbb{F}(\tau_1,\tau_3)}{36} \\
&- \frac{1}{3\Lambda_1(\kappa_1,1)}\ {}_{\kappa_1+}I_{\varphi}\mathbb{F}(\tau_2,\kappa_2) - \frac{1}{3\Delta_1(\kappa_1,1)}\ {}_{\kappa_1-}I_{\varphi}\mathbb{F}(\tau_1,\kappa_2)
\end{aligned}$$

$$-\frac{1}{3\Lambda_2(\kappa_2,1)}\ {}_{\kappa_2+}I_{\psi}\mathbb{F}(\kappa_1,\tau_4)-\frac{1}{3\Delta_2(\kappa_2,1)}\ {}_{\kappa_2-}I_{\psi}\mathbb{F}(\kappa_1,\tau_3)$$
$$-\frac{1}{12\Lambda_1(\kappa_1,1)}\left[{}_{\kappa_1+}I_{\varphi}\mathbb{F}(\tau_2,\tau_4)+{}_{\kappa_1+}I_{\varphi}\mathbb{F}(\tau_2,\tau_3)\right]$$
$$-\frac{1}{12\Delta_1(\kappa_1,1)}\left[{}_{\kappa_1-}I_{\varphi}\mathbb{F}(\tau_1,\tau_4)+{}_{\kappa_1-}I_{\varphi}\mathbb{F}(\tau_1,\tau_3)\right]$$
$$-\frac{1}{12\Lambda_2(\kappa_2,1)}\left[{}_{\kappa_2+}I_{\psi}\mathbb{F}(\tau_2,\tau_4)+{}_{\kappa_2+}I_{\psi}\mathbb{F}(\tau_1,\tau_4)\right]$$
$$-\frac{1}{12\Delta_2(\kappa_2,1)}\left[{}_{\kappa_2-}I_{\psi}\mathbb{F}(\tau_2,\tau_3)+{}_{\kappa_2-}I_{\psi}\mathbb{F}(\tau_1,\tau_3)\right]$$
$$+\frac{1}{4}\left[\frac{1}{\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\ {}_{\kappa_1+,\kappa_2+}I_{\varphi,\psi}\mathbb{F}(\tau_2,\tau_4)+\frac{1}{\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\ {}_{\kappa_1+,\kappa_2-}I_{\varphi,\psi}\mathbb{F}(\tau_2,\tau_3)\right.$$
$$\left.+\frac{1}{\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\ {}_{\kappa_1-,\kappa_2+}I_{\varphi,\psi}\mathbb{F}(\tau_1,\tau_4)+\frac{1}{\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\ {}_{\kappa_1-,\kappa_2-}I_{\varphi,\psi}\mathbb{F}(\tau_1,\tau_3)\right].$$

Proof. By using integration by parts, we have

$$\begin{aligned}\mathcal{H}_1 &= \int_0^1\int_0^1 (\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t))(\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s))\\ &\quad\times\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_4)\,dsdt\\ &= \frac{\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}\left[4\mathbb{F}(\kappa_1,\kappa_2)+2\mathbb{F}(\kappa_1,\tau_4)+2\mathbb{F}(\tau_2,\kappa_2)+\mathbb{F}(\tau_2,\tau_4)\right]\\ &\quad-\frac{6\Lambda_2(\kappa_2,1)}{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}\ {}_{\kappa_1+}I_{\varphi}\mathbb{F}(\tau_2,\kappa_2)-\frac{3\Lambda_2(\kappa_2,1)}{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}\ {}_{\kappa_1+}I_{\varphi}\mathbb{F}(\tau_2,\tau_4)\\ &\quad-\frac{6\Lambda_1(\kappa_1,1)}{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}\ {}_{\kappa_2+}I_{\psi}\mathbb{F}(\kappa_1,\tau_4)-\frac{3\Lambda_1(\kappa_1,1)}{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}\ {}_{\kappa_2+}I_{\psi}\mathbb{F}(\tau_2,\tau_4)\\ &\quad+\frac{9}{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}\ {}_{\kappa_1+,\kappa_2+}I_{\varphi,\psi}\mathbb{F}(\tau_2,\tau_4),\end{aligned}\tag{12}$$

$$\begin{aligned}\mathcal{H}_2 &= \int_0^1\int_0^1 (\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t))(\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s))\\ &\quad\times\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_3)\,dsdt\\ &= -\frac{\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}\left[4\mathbb{F}(\kappa_1,\kappa_2)+2\mathbb{F}(\kappa_1,\tau_3)+2\mathbb{F}(\tau_2,\kappa_2)+\mathbb{F}(\tau_2,\tau_3)\right]\\ &\quad+\frac{6\Lambda_2(\kappa_2,1)}{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}\ {}_{\kappa_1+}I_{\varphi}\mathbb{F}(\tau_2,\kappa_2)+\frac{3\Lambda_2(\kappa_2,1)}{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}\ {}_{\kappa_1+}I_{\varphi}\mathbb{F}(\tau_2,\tau_3)\\ &\quad+\frac{6\Lambda_1(\kappa_1,1)}{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}\ {}_{\kappa_2-}I_{\psi}\mathbb{F}(\kappa_1,\tau_3)+\frac{3\Lambda_1(\kappa_1,1)}{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}\ {}_{\kappa_2-}I_{\psi}\mathbb{F}(\tau_2,\tau_3)\\ &\quad-\frac{9}{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}\ {}_{\kappa_1+,\kappa_2-}I_{\varphi,\psi}\mathbb{F}(\tau_2,\tau_3),\end{aligned}\tag{13}$$

$$\mathcal{H}_3 = \int_0^1\int_0^1 (\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t))(\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s))\tag{14}$$

$$
\begin{aligned}
&\times \frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(t\kappa_1 + (1-t)\tau_1, s\kappa_2 + (1-s)\tau_4\right) dsdt \\
= \; & -\frac{\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}{(\kappa_1-\tau_1)(\tau_4-\kappa_2)} \left[4\mathbb{F}(\kappa_1,\kappa_2) + 2\mathbb{F}(\kappa_1,\tau_4) + 2\mathbb{F}(\tau_1,\kappa_2) + \mathbb{F}(\tau_1,\tau_4)\right] \\
&+ \frac{6\Lambda_2(\kappa_2,1)}{(\kappa_1-\tau_1)(\tau_4-\kappa_2)} \, {}_{\kappa_1-}I_{\varphi}\mathbb{F}(\tau_1,\kappa_2) + \frac{3\Lambda_2(\kappa_2,1)}{(\kappa_1-\tau_1)(\tau_4-\kappa_2)} \, {}_{\kappa_1-}I_{\varphi}\mathbb{F}(\tau_1,\tau_4) \\
&+ \frac{6\Delta_1(\kappa_1,1)}{(\kappa_1-\tau_1)(\tau_4-\kappa_2)} \, {}_{\kappa_2+}I_{\psi}\mathbb{F}(\kappa_1,\tau_4) + \frac{3\Delta_1(\kappa_1,1)}{(\kappa_1-\tau_1)(\tau_4-\kappa_2)} \, {}_{\kappa_2+}I_{\psi}\mathbb{F}(\tau_1,\tau_4) \\
&- \frac{9}{(\kappa_1-\tau_1)(\tau_4-\kappa_2)} \, {}_{\kappa_1-,\kappa_2+}I_{\varphi,\psi}\mathbb{F}(\tau_1,\tau_4),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}_4 = \; & \int_0^1 \int_0^1 \left(\Delta_1(\kappa_1,1) - 3\Delta_1(\kappa_1,t)\right)\left(\Delta_2(\kappa_2,1) - 3\Delta_2(\kappa_2,s)\right) \qquad (15) \\
&\times \frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(t\kappa_1 + (1-t)\tau_1, s\kappa_2 + (1-s)\tau_3\right) dsdt \\
= \; & \frac{\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}{(\kappa_1-\tau_1)(\kappa_2-\tau_3)} \left[4\mathbb{F}(\kappa_1,\kappa_2) + 2\mathbb{F}(\kappa_1,\tau_3) + 2\mathbb{F}(\tau_1,\kappa_2) + \mathbb{F}(\tau_1,\tau_3)\right] \\
&- \frac{6\Delta_2(\kappa_2,1)}{(\kappa_1-\tau_1)(\kappa_2-\tau_3)} \, {}_{\kappa_1-}I_{\varphi}\mathbb{F}(\tau_1,\kappa_2) - \frac{3\Delta_2(\kappa_2,1)}{(\kappa_1-\tau_1)(\kappa_2-\tau_3)} \, {}_{\kappa_1-}I_{\varphi}\mathbb{F}(\tau_1,\tau_3) \\
&- \frac{6\Delta_1(\kappa_1,1)}{(\kappa_1-\tau_1)(\kappa_2-\tau_3)} \, {}_{\kappa_2-}I_{\psi}\mathbb{F}(\kappa_1,\tau_3) - \frac{3\Delta_1(\kappa_1,1)}{(\kappa_1-\tau_1)(\kappa_2-\tau_3)} \, {}_{\kappa_2-}I_{\psi}\mathbb{F}(\tau_1,\tau_3) \\
&+ \frac{9}{(\kappa_1-\tau_1)(\kappa_2-\tau_3)} \, {}_{\kappa_1-,\kappa_2-}I_{\varphi,\psi}\mathbb{F}(\tau_1,\tau_3).
\end{aligned}
$$

By the equalities (12)-(15), we have

$$
\begin{aligned}
&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\mathcal{H}_1 - \frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\mathcal{H}_2 \\
&- \frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\mathcal{H}_3 + \frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\mathcal{H}_4 \\
= \; & \Omega(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)
\end{aligned}
$$

which completes the proof. □

Corollary 3.2. *Under assumptions of Lemma 3.1 with* $\kappa_1 = \frac{\tau_1+\tau_2}{2}$ *and* $\kappa_2 = \frac{\tau_3+\tau_4}{2}$ *we have the equality*

$$
\begin{aligned}
&\mho(\tau_1,\tau_2;\tau_3,\tau_4) \\
= \; & \frac{(\tau_2-\tau_1)(\tau_4-\tau_3)}{144\Upsilon_1(1)\Upsilon_2(1)} \int_0^1 \int_0^1 \left(\Upsilon_1(1) - 3\Upsilon_1(t)\right)\left(\Upsilon_2(1) - 3\Upsilon_2(s)\right) \\
&\times \frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\frac{t}{2}\tau_1 + \frac{2-t}{2}\tau_2, \frac{s}{2}\tau_3 + \frac{2-s}{2}\tau_4\right) dsdt \\
&- \frac{(\tau_2-\tau_1)(\tau_4-\tau_3)}{144\Upsilon_1(1)\Upsilon_2(1)} \int_0^1 \int_0^1 \left(\Upsilon_1(1) - 3\Upsilon_1(t)\right)\left(\Upsilon_2(1) - 3\Upsilon_2(s)\right) \\
&\times \frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\frac{t}{2}\tau_1 + \frac{2-t}{2}\tau_2, \frac{s}{2}\tau_4 + \frac{2-s}{2}\tau_3\right) dsdt
\end{aligned}
$$

$$-\frac{(\tau_2-\tau_1)(\tau_4-\tau_3)}{144\Upsilon_1(1)\Upsilon_2(1)}\int_0^1\int_0^1(\Upsilon_1(1)-3\Upsilon_1(t))(\Upsilon_2(1)-3\Upsilon_2(s))$$
$$\times\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{t}{2}\tau_2+\frac{2-t}{2}\tau_1,\frac{s}{2}\tau_3+\frac{2-s}{2}\tau_4\right)dsdt$$
$$+\frac{(\tau_2-\tau_1)(\tau_4-\tau_3)}{144\Upsilon_1(1)\Upsilon_2(1)}\int_0^1\int_0^1(\Upsilon_1(1)-3\Upsilon_1(t))(\Upsilon_2(1)-3\Upsilon_2(s))$$
$$\times\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{t}{2}\tau_2+\frac{2-t}{2}\tau_1,\frac{s}{2}\tau_4+\frac{2-s}{2}\tau_3\right)dsdt$$

where

$$\begin{aligned}
&\mho(\tau_1,\tau_2;\tau_3,\tau_4)\\
=\;&\frac{\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_4\right)+\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_3\right)+4\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)+\mathbb{F}\left(\tau_2,\frac{\tau_3+\tau_4}{2}\right)+\mathbb{F}\left(\tau_1,\frac{\tau_3+\tau_4}{2}\right)}{9}\\
&+\frac{\mathbb{F}(\tau_2,\tau_4)+\mathbb{F}(\tau_2,\tau_3)+\mathbb{F}(\tau_1,\tau_4)+\mathbb{F}(\tau_1,\tau_3)}{36}\\
&-\frac{1}{3\Upsilon_1(1)}\left[{}_{\frac{\tau_1+\tau_2}{2}+}I_{\varphi}\mathbb{F}\left(\tau_2,\frac{\tau_3+\tau_4}{2}\right)+{}_{\frac{\tau_1+\tau_2}{2}-}I_{\varphi}\mathbb{F}\left(\tau_1,\frac{\tau_3+\tau_4}{2}\right)\right]\\
&-\frac{1}{3\Upsilon_2(1)}\left[{}_{\frac{\tau_3+\tau_4}{2}+}I_{\psi}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_4\right)+{}_{\frac{\tau_3+\tau_4}{2}-}I_{\psi}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_3\right)\right]\\
&-\frac{1}{12\Upsilon_1(1)}\left[{}_{\frac{\tau_1+\tau_2}{2}+}I_{\varphi}\mathbb{F}(\tau_2,\tau_4)+{}_{\frac{\tau_1+\tau_2}{2}+}I_{\varphi}\mathbb{F}(\tau_2,\tau_3)\right.\\
&\left.+{}_{\frac{\tau_1+\tau_2}{2}-}I_{\varphi}\mathbb{F}(\tau_1,\tau_4)+{}_{\frac{\tau_1+\tau_2}{2}-}I_{\varphi}\mathbb{F}(\tau_1,\tau_3)\right]\\
&-\frac{1}{12\Upsilon_2(1)}\left[{}_{\frac{\tau_3+\tau_4}{2}+}I_{\psi}\mathbb{F}(\tau_2,\tau_4)+{}_{\frac{\tau_3+\tau_4}{2}+}I_{\psi}\mathbb{F}(\tau_1,\tau_4)\right.\\
&\left.+{}_{\frac{\tau_3+\tau_4}{2}-}I_{\psi}\mathbb{F}(\tau_2,\tau_3)+{}_{\frac{\tau_3+\tau_4}{2}-}I_{\psi}\mathbb{F}(\tau_1,\tau_3)\right]\\
&+\frac{1}{4\Upsilon_1(1)\Upsilon_2(1)}\left[{}_{\frac{\tau_1+\tau_2}{2}+,\frac{\tau_3+\tau_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\tau_2,\tau_4)+{}_{\frac{\tau_1+\tau_2}{2}+,\frac{\tau_3+\tau_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\tau_2,\tau_3)\right.\\
&\left.+{}_{\frac{\tau_1+\tau_2}{2}-,\frac{\tau_3+\tau_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\tau_1,\tau_4)+{}_{\frac{\tau_1+\tau_2}{2}-,\frac{\tau_3+\tau_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\tau_1,\tau_3)\right].
\end{aligned}$$

Corollary 3.3. *In Lemma 3.1, if we choose $\varphi(t)=t$ and $\psi(s)=s$ for all $(t,s)\in\triangle$, then we obtain the equality*

$$\begin{aligned}
&\aleph(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)\\
=\;&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36}\int_0^1\int_0^1(1-3t)(1-3s)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_4)\,dsdt\\
&-\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36}\int_0^1\int_0^1(1-3t)(1-3s)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_3)\,dsdt\\
&-\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36}\int_0^1\int_0^1(1-3t)(1-3s)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1,s\kappa_2+(1-s)\tau_4)\,dsdt
\end{aligned}$$

$$+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36}\int_0^1\int_0^1(1-3t)(1-3s)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1,s\kappa_2+(1-s)\tau_3)\,dsdt$$

where

$$\begin{aligned}
&\aleph(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)\\
=\;&\frac{4\mathbb{F}(\kappa_1,\kappa_2)+\mathbb{F}(\kappa_1,\tau_4)+\mathbb{F}(\kappa_1,\tau_3)+\mathbb{F}(\tau_2,\kappa_2)+\mathbb{F}(\tau_1,\kappa_2)}{9}\\
&+\frac{\mathbb{F}(\tau_2,\tau_4)+\mathbb{F}(\tau_2,\tau_3)+\mathbb{F}(\tau_1,\tau_4)+\mathbb{F}(\tau_1,\tau_3)}{36}\\
&-\frac{1}{3(\tau_2-\kappa_1)}\int_{\kappa_1}^{\tau_2}\mathbb{F}(t,\kappa_2)\,dt-\frac{1}{3(\kappa_1-\tau_1)}\int_{\tau_1}^{\kappa_1}\mathbb{F}(t,\kappa_2)\,dt\\
&-\frac{1}{3(\tau_4-\kappa_2)}\int_{\kappa_2}^{\tau_4}\mathbb{F}(\kappa_1,s)\,ds-\frac{1}{3(\kappa_2-\tau_3)}\int_{\tau_3}^{\kappa_2}\mathbb{F}(\kappa_1,s)\,ds\\
&-\frac{1}{12(\tau_2-\kappa_1)}\int_{\kappa_1}^{\tau_2}[\mathbb{F}(t,\tau_4)+\mathbb{F}(t,\tau_3)]\,dt-\frac{1}{12(\kappa_1-\tau_1)}\int_{\tau_1}^{\kappa_1}[\mathbb{F}(t,\tau_4)+\mathbb{F}(t,\tau_3)]\,dt\\
&-\frac{1}{12(\tau_4-\kappa_2)}\int_{\kappa_2}^{\tau_4}[\mathbb{F}(\tau_2,s)+\mathbb{F}(\tau_1,s)]\,ds-\frac{1}{12(\kappa_2-\tau_3)}\int_{\tau_3}^{\kappa_2}[\mathbb{F}(\tau_2,s)+\mathbb{F}(\tau_1,s)]\,ds\\
&+\frac{1}{4}\Bigg[\frac{1}{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}\int_{\kappa_1}^{\tau_2}\int_{\kappa_2}^{\tau_4}\mathbb{F}(t,s)\,dsdt+\frac{1}{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}\int_{\kappa_1}^{\tau_2}\int_{\tau_3}^{\kappa_2}\mathbb{F}(t,s)\,dsdt\\
&+\frac{1}{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}\int_{\tau_1}^{\kappa_1}\int_{\kappa_2}^{\tau_4}\mathbb{F}(t,s)\,dsdt+\frac{1}{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}\int_{\tau_1}^{\kappa_1}\int_{\tau_3}^{\kappa_2}\mathbb{F}(t,s)\,dsdt\Bigg].
\end{aligned}$$

Corollary 3.4. *In Lemma 3.1, if we choose $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ for all $(t,s)\in\triangle$, then we obtain the equality*

$$\begin{aligned}
&\Phi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)\\
=\;&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36}\int_0^1\int_0^1(1-3t^\alpha)\left(1-3s^\beta\right)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_4)\,dsdt\\
&-\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36}\int_0^1\int_0^1(1-3t^\alpha)\left(1-3s^\beta\right)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_3)\,dsdt\\
&-\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36}\int_0^1\int_0^1(1-3t^\alpha)\left(1-3s^\beta\right)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1,s\kappa_2+(1-s)\tau_4)\,dsdt\\
&+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36}\int_0^1\int_0^1(1-3t^\alpha)\left(1-3s^\beta\right)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1,s\kappa_2+(1-s)\tau_3)\,dsdt
\end{aligned}$$

where

$$\Phi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$$

$$
\begin{aligned}
= & \frac{4\mathbb{F}(\kappa_1,\kappa_2)+\mathbb{F}(\kappa_1,\tau_4)+\mathbb{F}(\kappa_1,\tau_3)+\mathbb{F}(\tau_2,\kappa_2)+\mathbb{F}(\tau_1,\kappa_2)}{9} \\
& +\frac{\mathbb{F}(\tau_2,\tau_4)+\mathbb{F}(\tau_2,\tau_3)+\mathbb{F}(\tau_1,\tau_4)+\mathbb{F}(\tau_1,\tau_3)}{36} \\
& -\frac{\Gamma(\alpha+1)}{3(\tau_2-\kappa_1)^{\alpha}}J^{\alpha}_{\kappa_1+}\mathbb{F}(\tau_2,\kappa_2)-\frac{\Gamma(\alpha+1)}{3(\kappa_1-\tau_1)^{\alpha}}J^{\alpha}_{\kappa_1-}\mathbb{F}(\tau_1,\kappa_2) \\
& -\frac{\Gamma(\beta+1)}{3(\tau_4-\kappa_2)^{\beta}}J^{\beta}_{\kappa_2+}\mathbb{F}(\kappa_1,\tau_4)-\frac{\Gamma(\beta+1)}{3(\kappa_2-\tau_3)^{\beta}}J^{\beta}_{\kappa_2-}\mathbb{F}(\kappa_1,\tau_3) \\
& -\frac{\Gamma(\alpha+1)}{12(\tau_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1+}\mathbb{F}(\tau_2,\tau_4)+J^{\alpha}_{\kappa_1+}\mathbb{F}(\tau_2,\tau_3)\right] \\
& -\frac{\Gamma(\alpha+1)}{12(\kappa_1-\tau_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1-}\mathbb{F}(\tau_1,\tau_4)+J^{\alpha}_{\kappa_1-}\mathbb{F}(\tau_1,\tau_3)\right] \\
& -\frac{\Gamma(\beta+1)}{12(\tau_4-\kappa_2)^{\beta}}\left[J^{\beta}_{\kappa_2+}\mathbb{F}(\tau_2,\tau_4)+J^{\beta}_{\kappa_2+}\mathbb{F}(\tau_1,\tau_4)\right] \\
& -\frac{\Gamma(\beta+1)}{12(\kappa_2-\tau_3)^{\beta}}\left[J^{\beta}_{\kappa_2-}\mathbb{F}(\tau_2,\tau_3)+J^{\beta}_{\kappa_2-}\mathbb{F}(\tau_1,\tau_3)\right] \\
& +\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4} \\
& \times\left[\frac{1}{(\tau_2-\kappa_1)^{\alpha}(\tau_4-\kappa_2)^{\beta}}J^{\alpha,\beta}_{\kappa_1+,\kappa_2+}\mathbb{F}(\tau_2,\tau_4)+\frac{1}{(\tau_2-\kappa_1)^{\alpha}(\kappa_2-\tau_3)^{\beta}}J^{\alpha,\beta}_{\kappa_1+,\kappa_2-}\mathbb{F}(\tau_2,\tau_3)\right. \\
& \left.+\frac{1}{(\kappa_1-\tau_1)^{\alpha}(\tau_4-\kappa_2)^{\beta}}J^{\alpha,\beta}_{\kappa_1-,\kappa_2+}\mathbb{F}(\tau_1,\tau_4)+\frac{1}{(\kappa_1-\tau_1)^{\alpha}(\kappa_2-\tau_3)^{\beta}}J^{\alpha,\beta}_{\kappa_1-,\kappa_2-}\mathbb{F}(\tau_1,\tau_3)\right].
\end{aligned}
$$

Corollary 3.5. *In Lemma 3.1, if we choose* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ *for all* $(t,s)\in\triangle$*, then we obtain the equality*

$$
\begin{aligned}
& \Xi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2) \\
= & \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36}\int_0^1\int_0^1\left(1-3t^{\frac{\alpha}{k}}\right)\left(1-3s^{\frac{\beta}{k}}\right)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_4)\,dsdt \\
& -\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36}\int_0^1\int_0^1\left(1-3t^{\frac{\alpha}{k}}\right)\left(1-3s^{\frac{\beta}{k}}\right)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_3)\,dsdt \\
& -\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36}\int_0^1\int_0^1\left(1-3t^{\frac{\alpha}{k}}\right)\left(1-3s^{\frac{\beta}{k}}\right)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1,s\kappa_2+(1-s)\tau_4)\,dsdt \\
& +\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36}\int_0^1\int_0^1\left(1-3t^{\frac{\alpha}{k}}\right)\left(1-3s^{\frac{\beta}{k}}\right)\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1,s\kappa_2+(1-s)\tau_3)\,dsdt
\end{aligned}
$$

where

$$
\begin{aligned}
& \Xi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2) \\
= & \frac{4\mathbb{F}(\kappa_1,\kappa_2)+\mathbb{F}(\kappa_1,\tau_4)+\mathbb{F}(\kappa_1,\tau_3)+\mathbb{F}(\tau_2,\kappa_2)+\mathbb{F}(\tau_1,\kappa_2)}{9} \\
& +\frac{\mathbb{F}(\tau_2,\tau_4)+\mathbb{F}(\tau_2,\tau_3)+\mathbb{F}(\tau_1,\tau_4)+\mathbb{F}(\tau_1,\tau_3)}{36} \\
& -\frac{\Gamma_k(\alpha+k)}{3(\tau_2-\kappa_1)^{\frac{\alpha}{k}}}J^{\alpha}_{\kappa_1+,k}\mathbb{F}(\tau_2,\kappa_2)-\frac{\Gamma_k(\alpha+k)}{3(\kappa_1-\tau_1)^{\frac{\alpha}{k}}}J^{\alpha}_{\kappa_1-,k}\mathbb{F}(\tau_1,\kappa_2)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Gamma_k(\beta+k)}{3\left(\tau_4-\kappa_2\right)^{\frac{\beta}{k}}} J_{\kappa_2+,k}^{\beta} \mathbb{F}\left(\kappa_1, \tau_4\right)-\frac{\Gamma_k(\beta+k)}{3\left(\kappa_2-\tau_3\right)^{\frac{\beta}{k}}} J_{\kappa_2-,k}^{\beta} \mathbb{F}\left(\kappa_1, \tau_3\right) \\
&-\frac{\Gamma_k(\alpha+k)}{12\left(\tau_2-\kappa_1\right)^{\frac{\alpha}{k}}}\left[J_{\kappa_1+,k}^{\alpha} \mathbb{F}\left(\tau_2, \tau_4\right)+J_{\kappa_1+,k}^{\alpha} \mathbb{F}\left(\tau_2, \tau_3\right)\right] \\
&-\frac{\Gamma_k(\alpha+k)}{12\left(\kappa_1-\tau_1\right)^{\frac{\alpha}{k}}}\left[J_{\kappa_1-,k}^{\alpha} \mathbb{F}\left(\tau_1, \tau_4\right)+J_{\kappa_1-,k}^{\alpha} \mathbb{F}\left(\tau_1, \tau_3\right)\right] \\
&-\frac{\Gamma_k(\beta+k)}{12\left(\tau_4-\kappa_2\right)^{\frac{\beta}{k}}}\left[J_{\kappa_2+,k}^{\beta} \mathbb{F}\left(\tau_2, \tau_4\right)+J_{\kappa_2+,k}^{\beta} \mathbb{F}\left(\tau_1, \tau_4\right)\right] \\
&-\frac{\Gamma_k(\beta+k)}{12\left(\kappa_2-\tau_3\right)^{\frac{\beta}{k}}}\left[J_{\kappa_2-,k}^{\beta} \mathbb{F}\left(\tau_2, \tau_3\right)+J_{\kappa_2-,k}^{\beta} \mathbb{F}\left(\tau_1, \tau_3\right)\right] \\
&+\frac{\Gamma_k(\alpha+k) \Gamma_k(\beta+k)}{4} \\
&\times\left[\frac{1}{\left(\tau_2-\kappa_1\right)^{\frac{\alpha}{k}}\left(\tau_4-\kappa_2\right)^{\frac{\beta}{k}}} J_{\kappa_1+, \kappa_2+}^{\alpha, \beta, k} \mathbb{F}\left(\tau_2, \tau_4\right)+\frac{1}{\left(\tau_2-\kappa_1\right)^{\frac{\alpha}{k}}\left(\kappa_2-\tau_3\right)^{\frac{\beta}{k}}} J_{\kappa_1+, \kappa_2-}^{\alpha, \beta, k} \mathbb{F}\left(\tau_2, \tau_3\right)\right. \\
&\left.+\frac{1}{\left(\kappa_1-\tau_1\right)^{\frac{\alpha}{k}}\left(\tau_4-\kappa_2\right)^{\frac{\beta}{k}}} J_{\kappa_1-, \kappa_2+}^{\alpha, \beta, k} \mathbb{F}\left(\tau_1, \tau_4\right)+\frac{1}{\left(\kappa_1-\tau_1\right)^{\frac{\alpha}{k}}\left(\kappa_2-\tau_3\right)^{\frac{\beta}{k}}} J_{\kappa_1-, \kappa_2-}^{\alpha, \beta, k} \mathbb{F}\left(\tau_1, \tau_3\right)\right].
\end{aligned}
$$

4. Simpson type inequalities for Generalized double Fractional Integrals

Theorem 4.1. *Suppose that the assumptions of Lemma 3.1 hold. If the mapping* $\left|\frac{\partial^2 \mathbb{F}}{\partial t \partial s}\right|$ *is co-ordinated convex on* Δ*, then we have the following inequality for generalized fractional integrals*

$$
\begin{aligned}
&\left|\Omega\left(\tau_1, \tau_2, \kappa_1 ; \tau_3, \tau_4, \kappa_2\right)\right| \\
\leq\ & \frac{\left(\tau_2-\kappa_1\right)\left(\tau_4-\kappa_2\right)}{36 \Lambda_1\left(\kappa_1, 1\right) \Lambda_2\left(\kappa_2, 1\right)}\left[\mathcal{A}_1 \mathcal{B}_1\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\kappa_1, \kappa_2\right)\right|+\mathcal{A}_1 \mathcal{B}_2\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\kappa_1, \tau_4\right)\right|\right. \\
&\left.+\mathcal{A}_2 \mathcal{B}_1\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\tau_2, \kappa_2\right)\right|+\mathcal{A}_2 \mathcal{B}_2\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\tau_2, \tau_4\right)\right|\right] \\
&+\frac{\left(\tau_2-\kappa_1\right)\left(\kappa_2-\tau_3\right)}{36 \Lambda_1\left(\kappa_1, 1\right) \Delta_2\left(\kappa_2, 1\right)}\left[\mathcal{A}_1 \mathcal{B}_4\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\kappa_1, \kappa_2\right)\right|+\mathcal{A}_1 \mathcal{B}_3\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\kappa_1, \tau_3\right)\right|\right. \\
&\left.+\mathcal{A}_2 \mathcal{B}_4\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\tau_2, \kappa_2\right)\right|+\mathcal{A}_2 \mathcal{B}_3\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\tau_2, \tau_3\right)\right|\right] \\
&+\frac{\left(\kappa_1-\tau_1\right)\left(\tau_4-\kappa_2\right)}{36 \Delta_1\left(\kappa_1, 1\right) \Lambda_2\left(\kappa_2, 1\right)}\left[\mathcal{A}_4 \mathcal{B}_1\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\kappa_1, \kappa_2\right)\right|+\mathcal{A}_4 \mathcal{B}_2\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\kappa_1, \tau_4\right)\right|\right. \\
&\left.+\mathcal{A}_3 \mathcal{B}_1\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\tau_1, \kappa_2\right)\right|+\mathcal{A}_3 \mathcal{B}_2\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\tau_1, \tau_4\right)\right|\right] \\
&+\frac{\left(\kappa_1-\tau_1\right)\left(\kappa_2-\tau_3\right)}{36 \Delta_1\left(\kappa_1, 1\right) \Delta_2\left(\kappa_2, 1\right)}\left[\mathcal{A}_4 \mathcal{B}_4\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\kappa_1, \kappa_2\right)\right|+\mathcal{A}_4 \mathcal{B}_3\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\kappa_1, \tau_3\right)\right|\right. \\
&\left.+\mathcal{A}_3 \mathcal{B}_4\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\tau_1, \kappa_2\right)\right|+\mathcal{A}_3 \mathcal{B}_3\left|\frac{\partial^2}{\partial t \partial s} \mathbb{F}\left(\tau_1, \tau_3\right)\right|\right]
\end{aligned}
$$

where

$$
\mathcal{A}_1 = \int_0^1 t\left|\Lambda_1\left(\kappa_1, 1\right)-3 \Lambda_1\left(\kappa_1, t\right)\right| dt, \tag{16}
$$

$$\mathcal{A}_2 = \int_0^1 (1-t)\left|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)\right| dt,$$

$$\mathcal{A}_3 = \int_0^1 (1-t)\left|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)\right| dt,$$

$$\mathcal{A}_4 = \int_0^1 t\left|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)\right| dt$$

and

$$\mathcal{B}_1 = \int_0^1 s\left|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)\right| ds, \tag{17}$$

$$\mathcal{B}_2 = \int_0^1 (1-s)\left|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)\right| ds,$$

$$\mathcal{B}_3 = \int_0^1 (1-s)\left|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)\right| ds,$$

$$\mathcal{B}_4 = \int_0^1 s\left|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)\right| ds.$$

Proof. By taking modulus in Lemma 3.1, we have

$$\begin{aligned}
&\left|\Omega(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)\right| \\
\leq\ & \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\int_0^1\int_0^1 \left|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)\right|\left|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)\right| \\
&\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\tau_2, s\kappa_2+(1-s)\tau_4\right)\right| dsdt \\
&+\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\int_0^1\int_0^1 \left|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)\right|\left|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)\right| \\
&\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\tau_2, s\kappa_2+(1-s)\tau_3\right)\right| dsdt \\
&+\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\int_0^1\int_0^1 \left|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)\right|\left|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)\right| \\
&\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\tau_1, s\kappa_2+(1-s)\tau_4\right)\right| dsdt \\
&+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\int_0^1\int_0^1 \left|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)\right|\left|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)\right|
\end{aligned} \tag{18}$$

$$\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\,\tau_1,s\kappa_2+(1-s)\,\tau_3\right)\right|dsdt.$$

Since $\left|\frac{\partial^2\mathbb{F}}{\partial t\partial s}\right|$ is co-ordinated convex, we have

$$\begin{aligned}
&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)| \qquad (19)\\
&\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\,\tau_2,s\kappa_2+(1-s)\,\tau_4\right)\right|dsdt\\
\leq\ &\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\\
&\times\left(ts\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|+t(1-s)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|\right.\\
&\left.+(1-t)s\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|+(1-t)(1-s)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|\right)dsdt\\
=\ &\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\\
&\times\int_0^1\int_0^1\left(|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,ts\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|\right.\\
&+|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,t(1-s)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|\\
&+|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,(1-t)s\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|\\
&\left.+|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,(1-t)(1-s)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|\right)dsdt\\
=\ &\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\\
&\times\left[\mathcal{A}_1\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|+\mathcal{A}_1\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|\right.\\
&\left.+\mathcal{A}_2\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|+\mathcal{A}_2\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|\right].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)| \qquad (20)\\
&\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\,\tau_2,s\kappa_2+(1-s)\,\tau_3\right)\right|dsdt\\
\leq\ &\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\left[\mathcal{A}_1\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|+\mathcal{A}_1\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|\right.
\end{aligned}$$

$$+ \mathcal{A}_2\mathcal{B}_4 \left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right| + \mathcal{A}_2\mathcal{B}_3 \left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|\Bigg],$$

$$\begin{aligned} &\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)} \int_0^1\int_0^1 |\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)| \\ &\times \left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1, s\kappa_2+(1-s)\tau_4)\right| dsdt \\ \le\; &\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)} \Bigg[\mathcal{A}_4\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + \mathcal{A}_4\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right| \\ &+ \mathcal{A}_3\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right| + \mathcal{A}_3\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|\Bigg], \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)} \int_0^1\int_0^1 |\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)|\,|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)| \\ &\times \left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1, s\kappa_2+(1-s)\tau_3)\right| dsdt \\ \le\; &\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)} \Bigg[\mathcal{A}_4\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + \mathcal{A}_4\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right| \\ &+ \mathcal{A}_3\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right| + \mathcal{A}_3\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|\Bigg]. \end{aligned} \tag{22}$$

If we substitute the inequalities (19)-(22) in (18), then we obtain the desired result. □

Corollary 4.2. *Under assumptions of Theorem 4.1 with $\kappa_1 = \frac{\tau_1+\tau_2}{2}$ and $\kappa_2 = \frac{\tau_3+\tau_4}{2}$, we have the following Simpson type inequality for generalized fractional integrals*

$$\begin{aligned} &|\mho(\tau_1,\tau_2;\tau_3,\tau_4)| \\ \le\; &\frac{(\tau_2-\tau_1)(\tau_4-\tau_3)}{144\Upsilon_1(1)\Upsilon_2(1)}(\mathcal{A}_5+\mathcal{A}_6)(\mathcal{B}_5+\mathcal{B}_6) \\ &\times\left[\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right| + \left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right| + \left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right| + \left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|\right] \end{aligned}$$

where $\mho(\tau_1,\tau_2;\tau_3,\tau_4)$ is defined as in Corollary 3.2 and

$$\mathcal{A}_5 = \int_0^1 t\,|\Upsilon_1(1)-3\Upsilon_1(t)|\,dt,\; \mathcal{A}_6 = \int_0^1 (1-t)\,|\Upsilon_1(1)-3\Upsilon_1(t)|\,dt, \tag{23}$$

and

$$\mathcal{B}_5 = \int_0^1 s\,|\Upsilon_2(1)-3\Upsilon_2(s)|\,ds,\; \mathcal{B}_6 = \int_0^1 (1-s)\,|\Upsilon_2(1)-3\Upsilon_2(s)|\,ds. \tag{24}$$

Corollary 4.3. *In Theorem 4.1, if we choose $\varphi(t) = t$ and $\psi(s) = s$ for all $(t,s) \in \triangle$, then we obtain the following inequality for Riemann integrals*

$$\begin{aligned}
&|\aleph(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)| \\
\leq\; & \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36}\left[\frac{29^2}{54^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + \frac{29}{54}\frac{8}{27}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|\right.\\
&\left.+\frac{8}{27}\frac{29}{54}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right| + \frac{8^2}{27^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|\right]\\
&+\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36}\left[\frac{29^2}{54^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + \frac{29}{54}\frac{8}{27}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|\right.\\
&\left.+\frac{8}{27}\frac{29}{54}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right| + \frac{8^2}{27^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|\right]\\
&+\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36}\left[\frac{29^2}{54^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + \frac{29}{54}\frac{8}{27}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|\right.\\
&\left.+\frac{8}{27}\frac{29}{54}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right| + \frac{8^2}{27^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|\right]\\
&+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36}\left[\frac{29^2}{54^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + \frac{29}{54}\frac{8}{27}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|\right.\\
&\left.+\frac{8}{27}\frac{29}{54}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right| + \frac{8^2}{27^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|\right]
\end{aligned}$$

where $\aleph(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$ is defined as in Corollary 3.3.

Remark 4.4. *If we choose $\kappa_1 = \frac{\tau_1+\tau_2}{2}$ and $\kappa_2 = \frac{\tau_3+\tau_4}{2}$ in Corollary 4.3, then Corollary 4.3 reduces to [30, Theorem 3].*

Corollary 4.5. *In Theorem 4.1, if we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ for all $(t,s) \in \triangle$, then we obtain the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&|\Phi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)| \\
\leq\; & \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36}\left[C_1(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + C_1(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|\right.\\
&\left.+C_2(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right| + C_2(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|\right]\\
&+\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36}\left[C_1(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + C_1(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|\right.\\
&\left.+C_2(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right| + C_2(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|\right]\\
&+\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36}\left[C_1(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + C_1(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|\right.\\
&\left.+C_2(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right| + C_2(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|\right]\\
&+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36}\left[C_1(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right| + C_1(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|\right.\\
&\left.+C_2(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right| + C_2(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|\right]
\end{aligned}$$

where $\Phi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$ *is defined as in Corollary 3.4 and*

$$\begin{aligned} C_1(\varsigma) &= 2\left(\frac{1}{3}\right)^{\frac{2}{\varsigma}}\left[\frac{1}{2}-\frac{1}{\varsigma+2}\right]+\frac{3}{\varsigma+2}-\frac{1}{2} \\ C_2(\varsigma) &= 2\left(\frac{1}{3}\right)^{\frac{1}{\varsigma}}\left[1-\frac{1}{\varsigma+1}\right]-2\left(\frac{1}{3}\right)^{\frac{2}{\varsigma}}\left[\frac{1}{2}-\frac{1}{\varsigma+2}\right]+\frac{3}{(\varsigma+1)(\varsigma+2)}-\frac{1}{2}. \end{aligned} \tag{25}$$

Corollary 4.6. *In Theorem 4.1, if we choose* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ *for all* $(t,s)\in\triangle$, *then we obtain the following inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned} &|\Xi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)| \\ \leq\ & \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36}\Bigg[\mathcal{D}_1(\alpha,k)\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|+\mathcal{D}_1(\alpha,k)\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right| \\ &+\mathcal{D}_2(\alpha,k)\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|+\mathcal{D}_2(\alpha,k)\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|\Bigg] \\ &+\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36}\Bigg[\mathcal{D}_1(\alpha,k)\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|+\mathcal{D}_1(\alpha,k)\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right| \\ &+\mathcal{D}_2(\alpha,k)\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|+\mathcal{D}_2(\alpha,k)\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|\Bigg] \\ &+\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36}\Bigg[\mathcal{D}_1(\alpha,k)\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|+\mathcal{D}_1(\alpha,k)\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right| \\ &+\mathcal{D}_2(\alpha,k)\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|+\mathcal{D}_2(\alpha,k)\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|\Bigg] \\ &+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36}\Bigg[\mathcal{D}_1(\alpha,k)\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|+\mathcal{D}_1(\alpha,k)\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right| \\ &+\mathcal{D}_2(\alpha,k)\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|+\mathcal{D}_2(\alpha,k)\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|\Bigg] \end{aligned}$$

where $\Xi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$ *is defined as in Corollary 3.5 and*

$$\begin{aligned} \mathcal{D}_1(\varsigma,k) &= 2\left(\frac{1}{3}\right)^{\frac{2k}{\varsigma}}\left[\frac{1}{2}-\frac{k}{\varsigma+2k}\right]+\frac{3k}{\varsigma+2k}-\frac{1}{2} \\ \mathcal{D}_2(\varsigma,k) &= 2\left(\frac{1}{3}\right)^{\frac{k}{\varsigma}}\left[1-\frac{k}{\varsigma+k}\right]-2\left(\frac{1}{3}\right)^{\frac{2k}{\varsigma}}\left[\frac{1}{2}-\frac{k}{\varsigma+2k}\right]+\frac{3k^2}{(\varsigma+k)(\varsigma+2k)}-\frac{1}{2}. \end{aligned} \tag{26}$$

Theorem 4.7. *Suppose that the assumptions of Lemma 3.1 hold. If the mapping* $\left|\frac{\partial^2\mathbb{F}}{\partial t\partial s}\right|^q$ *is co-ordinated convex on* Δ, $q>1$, *then we have the following inequality for generalized fractional integrals,*

$$\begin{aligned} &|\Omega(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)| \\ \leq\ & \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|^p\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|^p\,dsdt\right)^{\frac{1}{p}} \\ &\times\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)\right]^{\frac{1}{q}} \end{aligned}$$

$$+\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|^p\,|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)|^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)|^p\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\right)\right]^{\frac{1}{q}}$$
$$+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)|^p\,|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)|^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)\right]^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Hölder inequality and co-ordinated convexity of $\left|\frac{\partial^2\mathbb{F}}{\partial t\partial s}\right|^q$, we have

$$\begin{aligned}
&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\\
&\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_4)\right|dsdt\\
\le\;&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|^p\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|^p\,dsdt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_4)\right|^q dsdt\right)^{\frac{1}{q}}\\
\le\;&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|^p\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|^p\,dsdt\right)^{\frac{1}{p}}\\
&\times\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}\tag{27}$$

Similarly, we get

$$\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)|\tag{28}$$

$$\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\tau_2, s\kappa_2+(1-s)\tau_3\right)\right|dsdt$$

$$\leq \frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\left(\int_0^1\int_0^1\left|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)\right|^p\left|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)\right|^p dsdt\right)^{\frac{1}{p}}$$

$$\times\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\right)\right]^{\frac{1}{q}},$$

$$\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\int_0^1\int_0^1\left|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)\right|\left|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)\right| \tag{29}$$

$$\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\tau_1, s\kappa_2+(1-s)\tau_4\right)\right|dsdt$$

$$\leq \frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1\left|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)\right|^p\left|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)\right|^p dsdt\right)^{\frac{1}{p}}$$

$$\times\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\right)\right]^{\frac{1}{q}}$$

and

$$\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\int_0^1\int_0^1\left|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)\right|\left|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)\right| \tag{30}$$

$$\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(t\kappa_1+(1-t)\tau_1, s\kappa_2+(1-s)\tau_3\right)\right|dsdt$$

$$\leq \frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\left(\int_0^1\int_0^1\left|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)\right|^p\left|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)\right|^p dsdt\right)^{\frac{1}{p}}$$

$$\times\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)\right]^{\frac{1}{q}}.$$

By substituting the inequalities (27)-(30) in (18), we establish required result. □

Corollary 4.8. *Under assumptions of Theorem 4.7 with* $\kappa_1=\frac{\tau_1+\tau_2}{2}$ *and* $\kappa_2=\frac{\tau_3+\tau_4}{2}$*, we have the following Simpson type inequality for generalized fractional integrals*

$$\left|\mho(\tau_1,\tau_2;\tau_3,\tau_4)\right|$$

$$\leq \frac{(\tau_2-\tau_1)(\tau_4-\tau_3)}{144\Upsilon_1(1)\Upsilon_2(1)}\left(\int_0^1\int_0^1\left|\Upsilon_1(1)-3\Upsilon_1(t)\right|^p\left|\Upsilon_2(1)-3\Upsilon_2(t)\right|^p dsdt\right)^{\frac{1}{p}}$$

$$\times\left[\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_4\right)\right|^q\right.\right.\right.$$

$$\left.\left.+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\tau_2,\frac{\tau_3+\tau_4}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)\right]^{\frac{1}{q}}$$

$$+\frac{1}{4}\left[\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_3\right)\right|^q\right.$$
$$\left.+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\tau_2,\frac{\tau_3+\tau_4}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\right)\Bigg]^{\frac{1}{q}}$$
$$+\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_4\right)\right|^q\right.\right.$$
$$\left.\left.+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\tau_1,\frac{\tau_3+\tau_4}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\right)^{\frac{1}{q}}\right]$$
$$+\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_3\right)\right|^q\right.\right.$$
$$\left.\left.+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\tau_1,\frac{\tau_3+\tau_4}{2}\right)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)^{\frac{1}{q}}\right]\Bigg]$$

where $\mho(\tau_1,\tau_2;\tau_3,\tau_4)$ *is defined as in Corollary 3.2.*

Corollary 4.9. *In Theorem 4.7, if we choose* $\varphi(t)=t$ *and* $\psi(s)=s$ *for all* $(t,s)\in\triangle$*, then we obtain the following inequality for Riemann-Liouville fractional integrals*

$$|\aleph(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)|$$
$$\le \frac{1}{36}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{2}{p}}$$
$$\times\left[(\tau_2-\kappa_1)(\tau_4-\kappa_2)\left(\frac{\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$+(\tau_2-\kappa_1)(\kappa_2-\tau_3)\left(\frac{\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q}{4}\right)^{\frac{1}{q}}$$
$$+(\kappa_1-\tau_1)(\tau_4-\kappa_2)\left(\frac{\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q}{4}\right)^{\frac{1}{q}}$$
$$\left.+(\kappa_1-\tau_1)(\kappa_2-\tau_3)\left(\frac{\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

where $\aleph(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$ *is defined as in Corollary 3.3.*

Corollary 4.10. *In Theorem 4.7, if we choose* $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ *and* $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ *for all* $(t,s)\in\triangle$*, then we obtain the inequality*

$$|\Phi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)|$$
$$\le \frac{1}{36}\left(\int_0^1|1-3t^\alpha|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|1-3s^\beta\right|^p ds\right)^{\frac{1}{p}}$$
$$\times\left[(\tau_2-\kappa_1)(\tau_4-\kappa_2)\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)\right]^{\frac{1}{q}}\right.$$

$$+(\tau_2-\kappa_1)(\kappa_2-\tau_3)\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\right)\right]^{\frac{1}{q}}$$
$$+(\kappa_1-\tau_1)(\tau_4-\kappa_2)\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\right)\right]^{\frac{1}{q}}$$
$$+(\kappa_1-\tau_1)(\kappa_2-\tau_3)\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)\right]^{\frac{1}{q}}\Bigg]$$

where $\Phi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$ *is defined as in Corollary 3.4.*

Corollary 4.11. *In Theorem 4.7, if we choose* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ *for all* $(t,s)\in\triangle$, *then we obtain the following inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}&|\Xi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)|\\ \leq\ &\frac{1}{36}\left(\int_0^1\left|1-3t^{\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|1-3s^{\frac{\beta}{k}}\right|^p ds\right)^{\frac{1}{p}}\\ &\times\Bigg[(\tau_2-\kappa_1)(\tau_4-\kappa_2)\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)\right]^{\frac{1}{q}}\\ &+(\tau_2-\kappa_1)(\kappa_2-\tau_3)\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\right)\right]^{\frac{1}{q}}\\ &+(\kappa_1-\tau_1)(\tau_4-\kappa_2)\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\right)\right]^{\frac{1}{q}}\\ &+(\kappa_1-\tau_1)(\kappa_2-\tau_3)\left[\frac{1}{4}\left(\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)\right]^{\frac{1}{q}}\Bigg]\end{aligned}$$

where $\Xi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$ *is defined as in Corollary 3.5.*

Theorem 4.12. *Suppose that the assumptions of Lemma 3.1 hold. If the mapping* $\left|\frac{\partial^2\mathbb{F}}{\partial t\partial s}\right|^q$ *is co-ordinated convex on* Δ, $q>1$, *then we have the following inequality for generalized fractional integrals,*

$$\begin{aligned}&|\Omega(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)|\\ \leq\ &\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}\\ &\times\left(\mathcal{A}_1\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{A}_1\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q\right.\\ &\left.+\mathcal{A}_2\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\mathcal{A}_2\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)^{\frac{1}{q}}\\ &+\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)||\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}\\ &\times\left(\mathcal{A}_1\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{A}_1\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q\right.\end{aligned}$$

$$+\mathcal{A}_2\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\mathcal{A}_2\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}$$

$$\times\Bigg(\mathcal{A}_4\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{A}_4\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q$$

$$+\mathcal{A}_3\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\mathcal{A}_3\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}$$

$$\times\left(\int_0^1\int_0^1|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)|\,|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}$$

$$\times\Bigg(\mathcal{A}_4\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{A}_4\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q$$

$$+\mathcal{A}_3\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\mathcal{A}_3\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\Bigg)^{\frac{1}{q}}$$

where $\mathcal{A}_i$, $i=1,2,3,4$ are defined as in (16) and $\mathcal{B}_i$, $i=1,2,3,4$ are defined as in (17).

Proof. By using power mean inequality and co-ordinated convexity of $\left|\frac{\partial^2\mathbb{F}}{\partial t\partial s}\right|^q$, we have

$$\begin{aligned}
&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\\
&\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_4)\right|dsdt\\
\le\;&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\\
&\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_4)\right|^q dsdt\Bigg)^{\frac{1}{q}}\\
\le\;&\frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\\
&\times\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}
\end{aligned}\tag{31}$$

$$\times\left[\int_0^1\int_0^1\left(|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,ts\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q\right.\right.$$

$$+\ |\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,t(1-s)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q$$

$$+\ |\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,(1-t)s\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q$$

$$\left.\left.+\ |\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,(1-t)(1-s)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)dsdt\right]^{\frac{1}{q}}$$

$$=\ \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36\Lambda_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\mathcal{A}_1\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{A}_1\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\mathcal{A}_2\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\mathcal{A}_2\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)^{\frac{1}{q}}$$

Similarly, we obtain

$$\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|(\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s))| \tag{32}$$

$$\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_2,s\kappa_2+(1-s)\tau_3)\right|dsdt$$

$$=\ \frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36\Lambda_1(\kappa_1,1)\Delta_2(\kappa_2,1)}$$

$$\times\left(\int_0^1\int_0^1|\Lambda_1(\kappa_1,1)-3\Lambda_1(\kappa_1,t)|\,|\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\mathcal{A}_1\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{A}_1\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q\right.$$

$$\left.+\mathcal{A}_2\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\mathcal{A}_2\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\right)^{\frac{1}{q}},$$

$$\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}\int_0^1\int_0^1|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)| \tag{33}$$

$$\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1,s\kappa_2+(1-s)\tau_4)\right|dsdt$$

$$\leq\ \frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36\Delta_1(\kappa_1,1)\Lambda_2(\kappa_2,1)}$$

$$\times\left(\int_0^1\int_0^1|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)|\,|\Lambda_2(\kappa_2,1)-3\Lambda_2(\kappa_2,s)|\,dsdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\mathcal{A}_4\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{A}_4\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q\right.$$
$$\left.+\mathcal{A}_3\mathcal{B}_1\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\mathcal{A}_3\mathcal{B}_2\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\right)^{\frac{1}{q}}$$

and

$$\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}\int_0^1\int_0^1|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)||\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)| \tag{34}$$
$$\times\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(t\kappa_1+(1-t)\tau_1,s\kappa_2+(1-s)\tau_3)\right|dsdt$$
$$\leq \frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36\Delta_1(\kappa_1,1)\Delta_2(\kappa_2,1)}$$
$$\times\left(\int_0^1\int_0^1|\Delta_1(\kappa_1,1)-3\Delta_1(\kappa_1,t)||\Delta_2(\kappa_2,1)-3\Delta_2(\kappa_2,s)|dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\mathcal{A}_4\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{A}_4\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q\right.$$
$$\left.+\mathcal{A}_3\mathcal{B}_4\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\mathcal{A}_3\mathcal{B}_3\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)^{\frac{1}{q}}.$$

If we substitute the inequalities (31)-(34) in (18), then we establish desired result. □

Corollary 4.13. *Under assumptions of Theorem 4.12 with* $\kappa_1=\frac{\tau_1+\tau_2}{2}$ *and* $\kappa_2=\frac{\tau_3+\tau_4}{2}$, *we have the following Simpson type inequality for generalized fractional integrals*

$$|\mho(\tau_1,\tau_2;\tau_3,\tau_4)|$$
$$\leq \frac{(\tau_2-\tau_1)(\tau_4-\tau_3)}{144\Upsilon_1(1)\Upsilon_2(1)}\left(\int_0^1\int_0^1|\Upsilon_1(1)-3\Upsilon_1(t)||\Upsilon_2(1)-3\Upsilon_2(s)|dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\mathcal{A}_5\mathcal{B}_5\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)\right|^q+\mathcal{A}_5\mathcal{B}_6\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_4\right)\right|^q\right.\right.$$
$$\left.+\mathcal{A}_6\mathcal{B}_5\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\tau_2,\frac{\tau_3+\tau_4}{2}\right)\right|^q+\mathcal{A}_6\mathcal{B}_6\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\mathcal{A}_5\mathcal{B}_5\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)\right|^q+\mathcal{A}_5\mathcal{B}_6\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_3\right)\right|^q\right.$$
$$\left.+\mathcal{A}_6\mathcal{B}_5\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\tau_2,\frac{\tau_3+\tau_4}{2}\right)\right|^q+\mathcal{A}_6\mathcal{B}_6\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\mathcal{A}_5\mathcal{B}_5\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)\right|^q+\mathcal{A}_5\mathcal{B}_6\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_4\right)\right|^q\right.$$
$$\left.+\mathcal{A}_6\mathcal{B}_5\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\tau_1,\frac{\tau_3+\tau_4}{2}\right)\right|^q+\mathcal{A}_6\mathcal{B}_6\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\mathcal{A}_5\mathcal{B}_5\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\frac{\tau_3+\tau_4}{2}\right)\right|^q+\mathcal{A}_5\mathcal{B}_6\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\frac{\tau_1+\tau_2}{2},\tau_3\right)\right|^q\right.$$
$$\left.\left.+\mathcal{A}_6\mathcal{B}_5\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}\left(\tau_1,\frac{\tau_3+\tau_4}{2}\right)\right|^q+\mathcal{A}_6\mathcal{B}_6\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)^{\frac{1}{q}}\right]$$

where $\mho(\tau_1,\tau_2;\tau_3,\tau_4)$ is defined as in Corollary 3.2, $\mathcal{A}_5$, $\mathcal{A}_6$ are defined as in (23) and $\mathcal{B}_5$, $\mathcal{B}_6$ are defined as in (24).

Corollary 4.14. *In Theorem 4.12, if we choose $\varphi(t)=t$ and $\psi(s)=s$ for all $(t,s)\in\triangle$, then we obtain the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&|\aleph(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)|\\
\leq\;& \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36}\left(\frac{25}{36}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{29^2}{54^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\frac{29}{54}\frac{8}{27}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\frac{8}{27}\frac{29}{54}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\frac{8^2}{27^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)^{\frac{1}{q}}\\
&+\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36}\left(\frac{25}{36}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{29^2}{54^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\frac{29}{54}\frac{8}{27}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\frac{8}{27}\frac{29}{54}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+\frac{8^2}{27^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_3)\right|^q\right)^{\frac{1}{q}}\\
&+\frac{(\kappa_1-\tau_1)(\tau_4-\kappa_2)}{36}\left(\frac{25}{36}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{29^2}{54^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\frac{29}{54}\frac{8}{27}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q+\frac{8}{27}\frac{29}{54}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\frac{8^2}{27^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_4)\right|^q\right)\\
&+\frac{(\kappa_1-\tau_1)(\kappa_2-\tau_3)}{36}\left(\frac{25}{36}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{29^2}{54^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\frac{29}{54}\frac{8}{27}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q+\frac{8}{27}\frac{29}{54}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\frac{8^2}{27^2}\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

where $\aleph(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$ is defined as in Corollary 3.3.

Corollary 4.15. *In Theorem 4.12, if we choose $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ for all $(t,s)\in\triangle$, then we obtain the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&|\Phi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)|\\
\leq\;& \frac{(\tau_2-\kappa_1)(\tau_4-\kappa_2)}{36}\left(C_3(\alpha)C_3(\beta)\right)^{1-\frac{1}{q}}\\
&\times\left(C_1(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+C_1(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_4)\right|^q\right.\\
&\left.+C_2(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\kappa_2)\right|^q+C_2(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_2,\tau_4)\right|^q\right)^{\frac{1}{q}}\\
&+\frac{(\tau_2-\kappa_1)(\kappa_2-\tau_3)}{36}\left(C_3(\alpha)C_3(\beta)\right)^{1-\frac{1}{q}}\\
&\times\left(C_1(\alpha)C_1(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+C_1(\alpha)C_2(\beta)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q\right.
\end{aligned}$$

$$
\begin{aligned}
&+ C_2(\alpha) C_1(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_2, \kappa_2) \right|^q + C_2(\alpha) C_2(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_2, \tau_3) \right|^q \Bigg)^{\frac{1}{q}} \\
&+ \frac{(\kappa_1 - \tau_1)(\tau_4 - \kappa_2)}{36} \left(C_3(\alpha) C_3(\beta) \right)^{1-\frac{1}{q}} \\
&\times \Bigg(C_1(\alpha) C_1(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \kappa_2) \right|^q + C_1(\alpha) C_2(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \tau_4) \right|^q \\
&+ C_2(\alpha) C_1(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_1, \kappa_2) \right|^q + C_2(\alpha) C_2(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_1, \tau_4) \right|^q \Bigg)^{\frac{1}{q}} \\
&+ \frac{(\kappa_1 - \tau_1)(\kappa_2 - \tau_3)}{36} \left(C_3(\alpha) C_3(\beta) \right)^{1-\frac{1}{q}} \\
&\times \Bigg(C_1(\alpha) C_1(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \kappa_2) \right|^q + C_1(\alpha) C_2(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \tau_3) \right|^q \\
&+ C_2(\alpha) C_1(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_1, \kappa_2) \right|^q + C_2(\alpha) C_2(\beta) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_1, \tau_3) \right|^q \Bigg)^{\frac{1}{q}}
\end{aligned}
$$

where $\Phi(\tau_1, \tau_2, \kappa_1; \tau_3, \tau_4, \kappa_2)$ is defined as in Corollary 3.4, C_1, C_2 are defined as in (25) and C_3 is defined as

$$
C_3(\varsigma) = 2\left(\frac{1}{3}\right)^{\frac{1}{\varsigma}} \left[1 - \frac{1}{\varsigma + 1}\right] + \frac{3}{\varsigma + 1} - 1.
$$

Corollary 4.16. *In Theorem 4.12, if we choose $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s) = \frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ for all $(t, s) \in \triangle$, then we obtain the following inequality for k-Riemann-Liouville fractional integrals*

$$
\begin{aligned}
&\left| \Xi(\tau_1, \tau_2, \kappa_1; \tau_3, \tau_4, \kappa_2) \right| \\
\leq\ & \frac{(\tau_2 - \kappa_1)(\tau_4 - \kappa_2)}{36} \left(\mathcal{D}_3(\alpha, k) \mathcal{D}_3(\beta, k) \right)^{1-\frac{1}{q}} \\
&\times \Bigg(\mathcal{D}_1(\alpha, k) \mathcal{D}_1(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \kappa_2) \right|^q + \mathcal{D}_1(\alpha, k) \mathcal{D}_2(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \tau_4) \right|^q \\
&+ \mathcal{D}_2(\alpha, k) \mathcal{D}_1(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_2, \kappa_2) \right|^q + \mathcal{D}_2(\alpha, k) \mathcal{D}_2(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_2, \tau_4) \right|^q \Bigg)^{\frac{1}{q}} \\
&+ \frac{(\tau_2 - \kappa_1)(\kappa_2 - \tau_3)}{36} \left(\mathcal{D}_3(\alpha, k) \mathcal{D}_3(\beta, k) \right)^{1-\frac{1}{q}} \\
&\times \Bigg(\mathcal{D}_1(\alpha, k) \mathcal{D}_1(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \kappa_2) \right|^q + \mathcal{D}_1(\alpha, k) \mathcal{D}_2(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \tau_3) \right|^q \\
&+ \mathcal{D}_2(\alpha, k) \mathcal{D}_1(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_2, \kappa_2) \right|^q + \mathcal{D}_2(\alpha, k) \mathcal{D}_2(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_2, \tau_3) \right| \Bigg)^{\frac{1}{q}} \\
&+ \frac{(\kappa_1 - \tau_1)(\tau_4 - \kappa_2)}{36} \left(\mathcal{D}_3(\alpha, k) \mathcal{D}_3(\beta, k) \right)^{1-\frac{1}{q}} \\
&\times \Bigg(\mathcal{D}_1(\alpha, k) \mathcal{D}_1(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \kappa_2) \right|^q + \mathcal{D}_1(\alpha, k) \mathcal{D}_2(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\kappa_1, \tau_4) \right|^q \\
&+ \mathcal{D}_2(\alpha, k) \mathcal{D}_1(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_1, \kappa_2) \right|^q + \mathcal{D}_2(\alpha, k) \mathcal{D}_2(\beta, k) \left| \frac{\partial^2}{\partial t \partial s} \mathbb{F}(\tau_1, \tau_4) \right|^q \Bigg)^{\frac{1}{q}} \\
&+ \frac{(\kappa_1 - \tau_1)(\kappa_2 - \tau_3)}{36} \left(\mathcal{D}_3(\alpha, k) \mathcal{D}_3(\beta, k) \right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$\times\left(\mathcal{D}_1(\alpha,k)\,\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\kappa_2)\right|^q+\mathcal{D}_1(\alpha,k)\,\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\kappa_1,\tau_3)\right|^q\right.$$
$$\left.+\,\mathcal{D}_2(\alpha,k)\,\mathcal{D}_1(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\kappa_2)\right|^q+\mathcal{D}_2(\alpha,k)\,\mathcal{D}_2(\beta,k)\left|\frac{\partial^2}{\partial t\partial s}\mathbb{F}(\tau_1,\tau_3)\right|^q\right)^{\frac{1}{q}}$$

where $\Xi(\tau_1,\tau_2,\kappa_1;\tau_3,\tau_4,\kappa_2)$ *is defined as in Corollary 3.5,* $\mathcal{D}_1$*,* $\mathcal{D}_2$ *are defined as in (26) and* $\mathcal{D}_3$ *is defined as*

$$\mathcal{D}_3(\varsigma,k)=2\left(\frac{\varsigma}{\varsigma+k}\right)\left(\frac{1}{3}\right)^{\frac{k}{\varsigma}}+\frac{3k}{(\varsigma+k)}-1.$$

5. Concluding Remarks

In this paper, we present several generalized fractional Simpson type inequalities for functions whose partial derivatives in absolute value are co-ordinated convex functions. We also show that the results given here are a strong generalization of some already published ones. In the forthcoming papers, researchers can use the techniques of this work to obtain similar inequalities for other kinds of co-ordinated convexity.

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