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Further inequalities related to synchronous and asynchronous functions

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Abstract. This paper intends to show some operator and norm inequalities involving synchronous and asynchronous functions. Among other inequalities, it is shown that if $A, B \in \mathcal{B}(\mathcal{H})$ are two positive operators and $f, g : J \to \mathbb{R}$ are asynchronous functions, then

$$f(A)\,g(A) + f(B)\,g(B) \leq \frac{1}{2}\left(f^2(A) + g^2(A) + f^2(B) + g^2(B)\right).$$

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $A \geq 0$. For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ we say that $A \leq B$ if $B - A \geq 0$. The Gelfand map establishes an isometrically $*$-isomorphism ϕ between the set $C(sp(A))$ of all continuous functions on the spectrum of A, denoted $sp(A)$, and the C^*-algebra generated by A and the identity operator I on $\mathcal{H}$. For any $f, g \in C(sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

- $\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g)$;
- $\phi(fg) = \phi(f)\,\phi(g)$;
- $\left\|\phi(f)\right\| = \left\|f\right\| := \sup_{t \in sp(A)} \left|f(t)\right|$;
- $\phi(f_0) = I$ and $\phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in sp(A)$.

With this notation we define $f(A) = \phi(f)$ for all $f \in C(sp(A))$ and we call it the continuous functional calculus for a self-adjoint operator A. It is well known that, if A is a self-adjoint operator and $f \in C(sp(A))$, then $f(t) \geq 0$ for any $t \in sp(A)$ implies that $f(A) \geq 0$. It is extendable for two real valued functions on $sp(A)$.

A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It said to be normalized if $\Phi(I) = I$.

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We say that the functions $f, g : J \to \mathbb{R}$ are synchronous (asynchronous) on the interval $J \subseteq \mathbb{R}$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0 \tag{1}$$

for all $s, t \in J$.

The inequalities involving synchronous (asynchronous) functions have been of special interest; see e.g., [6, 8].

In [1, Theorem 1], it is shown that if $A \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator and $x \in \mathcal{H}$ is a unit vector, then for any synchronous functions $f, g : J \to \mathbb{R}$,

$$\langle f(A)x, x\rangle \langle g(A)x, x\rangle \leq \langle f(A)g(A)x, x\rangle \tag{2}$$

holds. More precisely, the following more general result is proved

$$\langle f(A)x, x\rangle \langle g(B)y, y\rangle + \langle f(B)y, y\rangle \langle g(A)x, x\rangle \leq \langle f(A)g(A)x, x\rangle + \langle f(B)g(B)y, y\rangle,$$

where $A, B \in \mathcal{B}(\mathcal{H})$ are two self-adjoint operators and $x, y \in \mathcal{H}$ are unit vectors.

Let $\lambda_j(A)$ to denote the j-th eigenvalue of Hermitian matrix A in the set of $n \times n$ complex matrices, when arranged in a non-increasing order, counting multiplicities. That is $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

Moradi and Sababheh [7, Theorem 4.1] proved that if $f, g : J \to \mathbb{R}$ are two asynchronous functions and A, B are $n \times n$ Hermitian matrices whose spectra are contained in J, then for every normalized positive linear map Φ

$$\lambda_j(\Phi(f(A)g(A) + f(B)g(B))) \leq \frac{1}{2}\lambda_j\left(\Phi\left(f^2(A) + g^2(A) + f^2(B) + g^2(B)\right)\right) \tag{3}$$

for $j = 1, 2, ..., n$.

In this paper, we present refinement and reverse for inequality (2). Further, we extend inequality (3) to Hilbert space operators. A related inequality for the numerical radius of Hilbert space operators is also given as well.

2. Main Results

The following lemma will be useful in the proof of our results.

Lemma 2.1. *[2, Theorem 1.4] Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and let $x \in \mathcal{H}$ be a unit vector. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle, \ (r \geq 1),$$

$$\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r, \ (0 \leq r \leq 1).$$

We start this section by refining and reversing (2).

Theorem 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator and let $x \in \mathcal{H}$ be a unit vector. If $f, g : J \to \mathbb{R}$ are synchronous functions, then*

$$\begin{aligned}
&\min\left\{\left\langle f^2(A)x, x\right\rangle - \langle f(A)x, x\rangle^2, \left\langle g^2(A)x, x\right\rangle - \langle g(A)x, x\rangle^2\right\} \\
&\leq \langle f(A)g(A)x, x\rangle - \langle f(A)x, x\rangle \langle g(A)x, x\rangle \\
&\leq \max\left\{\left\langle f^2(A)x, x\right\rangle - \langle f(A)x, x\rangle^2, \left\langle g^2(A)x, x\right\rangle - \langle g(A)x, x\rangle^2\right\}.
\end{aligned}$$

Proof. Following an idea arising in [9, Theorem 5.1], we have

$$\begin{aligned}&f(t)g(t)+f(s)g(s)-(f(t)g(s)+f(s)g(t))\\&\geq\min\left\{\left(f^2(t)+f^2(s)-2f(t)f(s)\right),g^2(t)+g^2(s)-2g(t)g(s)\right\}.\end{aligned}$$

Applying the continuous functional calculus for the self-adjoint operator A, we get

$$\begin{aligned}&f(A)g(A)+f(s)g(s)I-(g(s)f(A)+f(s)g(A))\\&\geq\min\left\{f^2(A)+f^2(s)I-2f(s)f(A),g^2(A)+g^2(s)I-2g(s)g(A)\right\}.\end{aligned}$$

Therefore, for any unit vector $x\in\mathcal{H}$, we get

$$\begin{aligned}&\langle f(A)g(A)x,x\rangle+f(s)g(s)-(g(s)\langle f(A)x,x\rangle+f(s)\langle g(A)x,x\rangle)\\&\geq\min\left\{\left\langle f^2(A)x,x\right\rangle+f^2(s)-2f(s)\langle f(A)x,x\rangle\right.\\&\left.,\left\langle g^2(A)x,x\right\rangle+g^2(s)-2g(s)\langle g(A)x,x\rangle\right\}.\end{aligned}$$

Applying again the continuous functional calculus for the self-adjoint operator A, we obtain

$$\begin{aligned}&\langle f(A)g(A)x,x\rangle I+f(A)g(A)-(\langle f(A)x,x\rangle g(A)+\langle g(A)x,x\rangle f(A))\\&\geq\min\left\{\left\langle f^2(A)x,x\right\rangle I+f^2(A)-2\langle f(A)x,x\rangle f(A)\right.\\&\left.,\left\langle g^2(A)x,x\right\rangle I+g^2(A)-2\langle g(A)x,x\rangle g(A)\right\}\end{aligned}$$

which implies,

$$\begin{aligned}&\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\langle g(A)x,x\rangle\\&\geq\min\left\{\left\langle f^2(A)x,x\right\rangle-\langle f(A)x,x\rangle^2,\left\langle g^2(A)x,x\right\rangle-\langle g(A)x,x\rangle^2\right\}\end{aligned}$$

for any unit vector $x\in\mathcal{H}$.

Since

$$\begin{aligned}&f(t)g(t)+f(s)g(s)-f(t)g(s)-f(s)g(t)\\&\leq\max\left\{f^2(t)+f^2(s)-2f(t)f(s),g^2(t)+g^2(s)-2g(t)g(s)\right\},\end{aligned}$$

we get the second inequality. □

Remark 2.3. *In the same way, we can obtain a more general result. Indeed, if $A,B\in\mathcal{B}(\mathcal{H})$ are self-adjoint operator and $x,y\in\mathcal{H}$ are two unit vectors, then*

$$\begin{aligned}&\min\left\{\left\langle f^2(A)x,x\right\rangle+\left\langle f^2(B)y,y\right\rangle-2\langle f(A)x,x\rangle\langle f(B)y,y\rangle\right.\\&\qquad\left.,\left\langle g^2(A)x,x\right\rangle+\left\langle g^2(B)y,y\right\rangle-2\langle g(A)x,x\rangle\langle g(B)y,y\rangle\right\}\\&\leq\langle f(A)g(A)x,x\rangle+\langle f(B)g(B)y,y\rangle-(\langle f(A)x,x\rangle\langle g(B)y,y\rangle+\langle f(B)y,y\rangle\langle g(A)x,x\rangle)\\&\leq\max\left\{\left\langle f^2(A)x,x\right\rangle+\left\langle f^2(B)y,y\right\rangle-2\langle f(A)x,x\rangle\langle f(B)y,y\rangle\right.\\&\qquad\left.,\left\langle g^2(A)x,x\right\rangle+\left\langle g^2(B)y,y\right\rangle-2\langle g(A)x,x\rangle\langle g(B)y,y\rangle\right\}.\end{aligned}$$

Lemma 2.4. *[5] Let $A,B\in\mathcal{B}(\mathcal{H})$ be two positive invertible operators, then for any unit vector $x\in\mathcal{H}$,*

$$\langle A\sharp Bx,x\rangle\leq\sqrt{\langle Ax,x\rangle\langle Bx,x\rangle}$$

where $A\sharp B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$ is the operator geometric mean.

Theorem 2.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive invertible operators. If $f, g : J \to \mathbb{R}$ are asynchronous functions, then*

$$f(A)\,g(A)\,\sharp f(B)\,g(B) \le \frac{1}{4}\left(f^2(A) + g^2(A) + f^2(B) + g^2(B)\right).$$

Proof. If $f, g : J \to \mathbb{R}$ are asynchronous on the interval J, then

$$f(t)\,g(t) + f(s)\,g(s) \le f(t)\,g(s) + f(s)\,g(t)$$

for any $t, s \in J$. If we fix $s \in J$ and apply the functional calculus (with $t = A$), we get

$$f(A)\,g(A) + f(s)\,g(s) \le g(s)\,f(A) + f(s)\,g(A).$$

Hence, for any unit vector $x \in \mathcal{H}$,

$$\langle f(A)\,g(A)\,x, x\rangle + f(s)\,g(s) \le g(s)\,\langle f(A)\,x, x\rangle + f(s)\,\langle g(A)\,x, x\rangle.$$

Now apply again the functional calculus (with $s = B$), we have

$$\langle f(A)\,g(A)\,x, x\rangle + f(B)\,g(B) \le g(B)\,\langle f(A)\,x, x\rangle + f(B)\,\langle g(A)\,x, x\rangle.$$

Therefore, for any unit vector $x \in \mathcal{H}$,

$$\begin{aligned}
&\langle f(A)\,g(A)\,x, x\rangle + \langle f(B)\,g(B)\,x, x\rangle \\
&\le \langle g(B)\,x, x\rangle\,\langle f(A)\,x, x\rangle + \langle f(B)\,x, x\rangle\,\langle g(A)\,x, x\rangle.
\end{aligned}$$

On the other hand, by the arithmetic-geometric mean inequality and Lemma 2.4, we have

$$\begin{aligned}
&2\,\langle f(A)\,g(A)\,\sharp f(B)\,g(B)\,x, x\rangle \\
&\le 2\sqrt{\langle f(A)\,g(A)\,x, x\rangle\,\langle f(B)\,g(B)\,x, x\rangle} \\
&\le \langle f(A)\,g(A)\,x, x\rangle + \langle f(B)\,g(B)\,x, x\rangle \\
&\le \langle g(B)\,x, x\rangle\,\langle f(A)\,x, x\rangle + \langle f(B)\,x, x\rangle\,\langle g(A)\,x, x\rangle \\
&\le \frac{1}{2}\left(\langle g(B)\,x, x\rangle^2 + \langle f(A)\,x, x\rangle^2 + \langle f(B)\,x, x\rangle^2 + \langle g(A)\,x, x\rangle^2\right) \\
&\le \frac{1}{2}\left(\left\langle g^2(B)\,x, x\right\rangle + \left\langle f^2(A)\,x, x\right\rangle + \left\langle f^2(B)\,x, x\right\rangle + \left\langle g^2(A)\,x, x\right\rangle\right) \\
&= \frac{1}{2}\left\langle\left(f^2(A) + g^2(A) + f^2(B) + g^2(B)\right)x, x\right\rangle.
\end{aligned}$$

Hence,

$$\langle f(A)\,g(A)\,\sharp f(B)\,g(B)\,x, x\rangle \le \frac{1}{4}\left\langle\left(f^2(A) + g^2(A) + f^2(B) + g^2(B)\right)x, x\right\rangle.$$

By replacing x by $\frac{y}{\|y\|}$, we get for vector $y \in \mathcal{H}$,

$$\langle f(A)\,g(A)\,\sharp f(B)\,g(B)\,y, y\rangle \le \frac{1}{4}\left\langle\left(f^2(A) + g^2(A) + f^2(B) + g^2(B)\right)y, y\right\rangle.$$

Since the above inequality holds for any vector, we get

$$f(A)\,g(A)\,\sharp f(B)\,g(B) \le \frac{1}{4}\left(f^2(A) + g^2(A) + f^2(B) + g^2(B)\right).$$

□

The following result is an immediate consequence of the proof of Theorem 2.5 and presents the operator version of the inequality shown in [7].

Corollary 2.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators. If $f, g : J \to \mathbb{R}$ are asynchronous functions, then*

$$f(A)\,g(A) + f(B)\,g(B) \le \frac{1}{2}\left(f^2(A) + f^2(B) + g^2(A) + g^2(B)\right).$$

Lemma 2.7. *[4] Let $A \in \mathcal{B}(\mathcal{H})$ and let $x, y \in \mathcal{H}$ be any vectors. If f, g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)\,g(t) = t$, $(t \ge 0)$, then*

$$\left|\langle Ax, y\rangle\right| \le \left\|f(|A|)\,x\right\| \left\|g(|A^*|)\,y\right\|.$$

The following numerical radius inequality involving synchronous functions may be stated as well. Recall that for $A \in \mathcal{B}(\mathcal{H})$, let

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\},$$
$$\omega(A) = \sup\{|\langle Ax, x\rangle| : \|x\| = 1\},$$

denote the usual operator and the numerical radius of A, respectively. It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. Namely, for $A \in \mathcal{B}(\mathcal{H})$, we have

$$\frac{1}{2}\|A\| \le \omega(A) \le \|A\|. \tag{4}$$

The inequality (4) have been improved considerably by Kittaneh in [3]. It has been shown that, if $A \in \mathcal{B}(\mathcal{H})$, then

$$\omega^2(A) \leqslant \frac{1}{2}\||A|^2 + |A^*|^2\|. \tag{5}$$

Now we establish a generalized version of (5).

Theorem 2.8. *Let $A \in \mathcal{B}(\mathcal{H})$ and let f, g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)\,g(t) = t$, $(t \ge 0)$. If $f^2, g^2 : J \to \mathbb{R}$ are synchronous functions, then*

$$\omega^2(A) \le \frac{1}{2}\left\|f^2(|A|)\,g^2(|A|) + f^2(|A^*|)\,g^2(|A^*|)\right\|.$$

Proof. From the relation (1), we obtain

$$f^2(t)\,g^2(t) + f^2(s)\,g^2(s) \ge f^2(t)\,g^2(s) + f^2(s)\,g^2(t)$$

for any $t, s \in J$. Applying the functional calculus for the positive operator $|A|$, we get

$$f^2(|A|)\,g^2(|A|) + f^2(s)\,g^2(s) \ge g^2(s)\,f^2(|A|) + f^2(s)\,g^2(|A|).$$

Hence, for any unit vector $x \in \mathcal{H}$,

$$\left\langle f^2(|A|)\,g^2(|A|)\,x, x\right\rangle + f^2(s)\,g^2(s) \ge g^2(s)\left\langle f^2(|A|)\,x, x\right\rangle + f^2(s)\left\langle g^2(|A|)\,x, x\right\rangle.$$

Applying again the functional calculus for the positive operator $|A^*|$, we have

$$\left\langle f^2(|A|)\,g^2(|A|)\,x, x\right\rangle + f^2(|A^*|)\,g^2(|A^*|) \ge g^2(|A^*|)\left\langle f^2(|A|)\,x, x\right\rangle + f^2(|A^*|)\left\langle g^2(|A|)\,x, x\right\rangle.$$

Therefore, for any unit vector $x \in \mathcal{H}$,

$$\begin{aligned} &\left\langle f^2\left(|A|\right) g^2\left(|A|\right) x, x\right\rangle + \left\langle f^2\left(|A^*|\right) g^2\left(|A^*|\right) x, x\right\rangle \\ &\geq \left\langle g^2\left(|A^*|\right) x, x\right\rangle \left\langle f^2\left(|A|\right) x, x\right\rangle + \left\langle f^2\left(|A^*|\right) x, x\right\rangle \left\langle g^2\left(|A|\right) x, x\right\rangle. \end{aligned}$$

On the other hand, by Lemma 2.7, we have

$$\left\langle g^2\left(|A^*|\right) x, x\right\rangle \left\langle f^2\left(|A|\right) x, x\right\rangle + \left\langle f^2\left(|A^*|\right) x, x\right\rangle \left\langle g^2\left(|A|\right) x, x\right\rangle \geq 2|\langle Ax, x\rangle|^2.$$

Whence,

$$|\langle Ax, x\rangle|^2 \leq \frac{1}{2}\left\langle \left(f^2\left(|A|\right) g^2\left(|A|\right) + f^2\left(|A^*|\right) g^2\left(|A^*|\right)\right) x, x\right\rangle.$$

Now, by taking supremum over all unit vector $x \in \mathcal{H}$, we get

$$\omega^2(A) \leq \frac{1}{2}\left\| f^2\left(|A|\right) g^2\left(|A|\right) + f^2\left(|A^*|\right) g^2\left(|A^*|\right)\right\|.$$

□

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