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Duality and statistical mirror symmetry in the generalized geometry setting

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Abstract. We describe statistical mirror symmetry, we introduce the notion of quasi-statistical mirror pairs and we give examples for certain quasi-statistical manifolds. As an application, we get families of almost Kähler structures on the tangent bundle manifold of almost complex 4-dimensional solvmanifolds without complex structures. Finally, we prove that statistical mirror symmetry can be understood in terms of generalized geometry.

1. Introduction

The classical mirror symmetry in complex and symplectic geometry refers to a duality between Calabi-Yau manifolds such that the complex geometry of the first manifold is related to the symplectic geometry of the second one. Strominger, Yau and Zaslow, in their pioneer paper [16], conjectured that for semi-flat Calabi-Yau manifolds mirror symmetry is T-duality.

In the present paper, following [7, 19], we describe statistical mirror symmetry, we introduce a similar notion of mirror pairs in the quasi-statistical setting, construct them in the quasi-semi-Weyl case, and give examples for certain quasi-statistical manifolds. As an application, we get families of almost Kähler structures on the tangent bundle manifold of almost complex 4-dimensional solvmanifolds without complex structures. Finally, we prove that statistical mirror symmetry can be understood in terms of generalized geometry. In our context, it is not assumed the Ricci-flatness nor compactness of the manifold. We use the splitting of the tangent bundle defined by affine connections and Sasaki metrics and we describe how statistical mirror pairs can be obtained from the same structure on the generalized tangent bundle by using duality of connections. Hence, our construction provides a bridge not only between information geometry, complex and symplectic geometry, but also with generalized geometry.

2. Dual and semi-dual connections

Dual affine connections on Riemannian manifolds play a central role in information geometry. The notion of dual connection (or conjugate connection), was firstly introduced by Amari [1] which he used in

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treating statistical inference problems. He proved its importance when dealing with certain types of family of probability densities. In the framework of Weyl geometry, the corresponding concept of duality is given by the semi-dual connection introduced by Norden [13]. We shall further recall these notions with a special view towards quasi-statistical and quasi-semi-Weyl structures.

Let (M, g) be a pseudo-Riemannian manifold. Throughout the paper, we shall denote by TM the tangent bundle of M, by T^*M its cotangent bundle and by $\Gamma^\infty(TM)$ (respectively, by $\Gamma^\infty(T^*M)$) the smooth sections of TM (respectively, of T^*M). For an arbitrary affine connection ∇ on M, we will denote by T^∇ its torsion tensor and by R^∇ its curvature tensor, given, respectively, by

$$T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y],$$

$$R^\nabla(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

for $X, Y \in \Gamma^\infty(TM)$, where $[\cdot, \cdot]$ is the Lie bracket. An affine connection is said to be *torsion-free* if its torsion tensor is zero, and *flat*, if its curvature tensor is zero.

In all the rest of the paper, we shall denote by ∇^g the Levi-Civita connection of the pseudo-Riemannian metric g.

2.1. Statistical and quasi-statistical structures

We shall recall the notions of statistical, quasi-statistical structures, and dual connections.

Definition 2.1. *[1] Let (M, g) be a pseudo-Riemannian manifold and let ∇ be a torsion-free affine connection on M. Then (M, g, ∇) is called a* statistical manifold *(and (g, ∇) a* statistical structure on *M) if*

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Remark that (M, g, ∇) is a statistical manifold if and only if ∇g is totally symmetric.

Example 2.2. *If g is a pseudo-Riemannian metric on M, then (M, g, ∇^g) is a statistical manifold.*

In 2007, Kurose introduced the notion of statistical manifold admitting torsion.

Definition 2.3. *[8] Let (M, g) be a pseudo-Riemannian manifold and let ∇ be an affine connection on M with torsion tensor T^∇. Then (M, g, ∇) is called a* quasi-statistical manifold, *or* statistical manifold admitting torsion *(and (g, ∇) a* quasi-statistical structure on *M) if*

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z) - g(T^\nabla(X, Y), Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Example 2.4. *If g is a pseudo-Riemannian metric on M and f is a positive smooth function on M, then $(M, e^f g, \nabla^g + df \otimes I)$ is a quasi-statistical manifold.*

Definition 2.5. *[1, 9] Let (M, g) be a pseudo-Riemannian manifold. Two affine connections ∇ and ∇^* on M are said to be* dual connections *with respect to g if*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$, and we call (g, ∇, ∇^*) a dualistic structure on *M.*

From the definition, we remark that $(\nabla^*)^* = \nabla$. Notice that $\nabla = \nabla^*$ if and only if ∇ is a metric connection, that is, $\nabla g = 0$. Moreover, if ∇ is torsion-free, then $\nabla = \nabla^*$ if and only if ∇ is the Levi-Civita connection of g.

Also, a dualistic structure (g, ∇, ∇^*) on M such that ∇ and ∇^* are flat and torsion-free is called a *dually flat structure* on M (and (M, g, ∇, ∇^*) a *dually flat manifold*).

From [2], we have

Proposition 2.6. *Let ∇ and ∇^* be dual connections with respect to g. Then*
(i) $R^{\nabla} = 0 \Leftrightarrow R^{\nabla^*} = 0$;
(ii) $T^{\nabla^*} = 0 \Leftrightarrow (M, g, \nabla)$ is a quasi-statistical manifold;
(iii) $T^{\nabla} = 0 \Leftrightarrow (M, g, \nabla^*)$ is a quasi-statistical manifold;
(iv) $T^{\nabla^*} = 0$, $T^{\nabla} = 0 \Leftrightarrow (M, g, \nabla)$ and (M, g, ∇^*) are both statistical manifolds.

2.2. Semi-Weyl and quasi-semi-Weyl structures

We shall recall the notions of semi-Weyl, quasi-semi-Weyl structures, and semi-dual connections.

Definition 2.7. *[10] Let (M, g) be a pseudo-Riemannian manifold, let ∇ be a torsion-free affine connection on M and let η be a 1-form. Then (M, g, η, ∇) is called a* semi-Weyl manifold *(and (g, η, ∇) a* semi-Weyl structure *on M) if*

$$(\nabla_X g)(Y, Z) + \eta(X)g(Y, Z) = (\nabla_Y g)(X, Z) + \eta(Y)g(X, Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Remark that (M, g, η, ∇) is a semi-Weyl manifold if and only if $\nabla g + \eta \otimes g$ is totally symmetric. In particular, if $\eta = 0$, then (M, g, ∇) is a statistical manifold. Moreover, if $\nabla g + \eta \otimes g = 0$, then (M, g, η, ∇) is a *Weyl manifold*.

Example 2.8. *If g is a pseudo-Riemannian metric on M and f is a positive smooth function on M, then $(M, e^f g, df, \nabla^g + 2df \otimes I + 2I \otimes df)$ is a semi-Weyl manifold.*

In [2] we introduced the notion of semi-Weyl manifold admitting torsion.

Definition 2.9. *Let (M, g) be a pseudo-Riemannian manifold, let ∇ be an affine connection on M with torsion tensor T^{∇} and let η be a 1-form. Then (M, g, η, ∇) is called a* quasi-semi-Weyl manifold, *or* semi-Weyl manifold admitting torsion *(and (g, η, ∇) a* quasi-semi-Weyl structure *on M) if*

$$(\nabla_X g)(Y, Z) + \eta(X)g(Y, Z) = (\nabla_Y g)(X, Z) + \eta(Y)g(X, Z) - g(T^{\nabla}(X, Y), Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Example 2.10. *If (M, g, ∇) is a statistical manifold and η is a nonzero 1-form on M, then $(M, g, \eta, \nabla + \eta \otimes I)$ is a quasi-semi-Weyl manifold.*

Definition 2.11. *[12, 13] Let (M, g) be a pseudo-Riemannian manifold and let η be a 1-form on M. Two affine connections ∇ and ∇^* on M are said to be* semi-dual connections *(or* generalized dual connections*) with respect to (g, η) if*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) - \eta(X)g(Y, Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$, and we call $(g, \eta, \nabla, \nabla^*)$ a semi-dualistic structure on M.

From the definition, we remark that $(\nabla^*)^* = \nabla$.

Remark 2.12. *Notice that if ∇ is torsion-free, then $\nabla = \nabla^*$ if and only if (M, g, η, ∇) is a Weyl manifold.*

Remark 2.13. *If we denote by ∇^*_g the dual connection of ∇ with respect to g and by $\nabla^*_{(g,\eta)}$ the semi-dual connection of ∇ with respect to (g, η), then $\nabla^*_g = \nabla^*_{(g,\eta)} - \eta \otimes I$ and $\nabla = (\nabla^*_{(g,\eta)})^*_g + \eta \otimes I$.*

From [2], we have

Proposition 2.14. *Let ∇ and $\nabla^*_{(g,\eta)}$ be semi-dual connections with respect to (g,η) and let ∇^*_g be the dual connection of ∇ with respect to g. Then*

(i) $R^\nabla = 0 \Leftrightarrow R^{\nabla^*_{(g,\eta)}} = 0 \Leftrightarrow R^{\nabla^*_g} = 0$;

(ii) $T^{\nabla^*_{(g,\eta)}} = 0 \Leftrightarrow (M, g, \eta, \nabla)$ *is a quasi-semi-Weyl manifold;*

(iii) $T^\nabla = 0 \Leftrightarrow (M, g, \eta, \nabla^*_{(g,\eta)})$ *is a quasi-semi-Weyl manifold;*

(iv) $T^{\nabla^*_{(g,\eta)}} = 0$, $T^\nabla = 0 \Leftrightarrow (M, g, \eta, \nabla)$ *and* $(M, g, \eta, \nabla^*_{(g,\eta)})$ *are both semi-Weyl manifolds;*

(v) $(M, g, \eta, \nabla^*_{(g,\eta)})$ *is a quasi-semi-Weyl manifold* $\Leftrightarrow (M, g, \nabla^*_g)$ *is a quasi-statistical manifold.*

3. Statistical mirror symmetry

3.1. Canonical almost Kähler structure on tangent bundles

Let (M, g) be a pseudo-Riemannian manifold and let ∇ be an affine connection on M. Then $T(TM)$ can be decomposed into the horizontal and the vertical subbundles, namely, $T(TM) = H(TM) \oplus V(TM)$, with respect to ∇.

We consider an almost complex structure $\hat{J}$ on TM as follows. Let $x \in M$, let $\{x^1, ..., x^n\}$ be local coordinates on M around x and let $\{x^1, ..., x^n, y^1, ..., y^n\}$ be the corresponding coordinates on TM. For $X \in T_xM$, we construct two tangent vectors on TM at the point (x, y), the *horizontal lift* of X, $X^H \in H_{(x,y)}(TM)$

$$X^H := X^i \frac{\partial}{\partial x^i} - X^i y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}$$

and the *vertical lift* of X, $X^V \in V_{(x,y)}(TM)$

$$X^V := X^i \frac{\partial}{\partial y^i},$$

where $X = X^i \frac{\partial}{\partial x^i}$ and Γ^k_{ij} are the connection coefficients of ∇ and we used the Einstein's summation convention. Then we define $\hat{J} =: \hat{J}^\nabla : T_{(x,y)}(TM) \to T_{(x,y)}(TM)$ by

$$\hat{J}(X^H) := X^V, \quad \hat{J}(X^V) := -X^H$$

and we have

Proposition 3.1. *[5, 19] Let $\hat{J}$ be the almost complex structure on TM defined by ∇, as above. Then $\hat{J}$ is integrable if and only if ∇ is flat and torsion-free.*

Let $\hat{G} =: \hat{G}^\nabla_g$ be the Sasaki metric on TM defined by (g, ∇) as

$$\hat{G} := g \oplus g.$$

We immediately get that $\hat{J}$ is compatible with respect to the pseudo-Riemannian metric $\hat{G}$, that is, $\hat{G}(\hat{J}X, \hat{J}Y) = \hat{G}(X, Y)$, for any $X, Y \in \Gamma^\infty(T(TM))$, and we infer

Corollary 3.2. *Let (M, g, ∇) be a pseudo-Riemannian manifold with an affine connection. Then $(TM, \hat{G}, \hat{J})$ is an almost Hermitian manifold which is Hermitian if and only if ∇ is flat and torsion-free.*

For $(TM, \hat{G}, \hat{J})$ induced by (M, g, ∇), let ω be the Kähler form, defined as

$$\omega(\xi, \zeta) := \hat{G}(\hat{J}\xi, \zeta),$$

for any $\xi, \zeta \in \Gamma^\infty(T(TM))$. If ∇^* is the dual connection of ∇ with respect to g, then we have

Proposition 3.3. *[15, 19] For $(TM, \hat{G}, \hat{J})$ induced by (M, g, ∇), the following statements are equivalent*

(i) $d\omega = 0$;

(ii) $T^{\nabla^*} = 0$;

(iii) (M, g, ∇) *is a quasi-statistical manifold.*

3.2. Statistical and quasi-statistical mirror symmetry

For a pseudo-Riemannian metric g, denote further also by g the canonical isomorphism between the tangent and the cotangent bundle induced by g, and by g^{-1} its inverse.

Next we shall enlarge the definition of statistical mirror pairs given by Zhang and Khan, to quasi-statistical mirror pairs. From Proposition 3.2 and Proposition 3.3 one can infer

Theorem 3.4. *[19] Let (M,g) be a pseudo-Riemannian manifold, let ∇ be a flat and torsion-free affine connection and let ∇^* be the dual connection of ∇ with respect to g. Then*

(i) $\mathbb{M} := \left(TM, \hat{G}_g^{\nabla}, \hat{J}^{\nabla}\right)$ is a Hermitian manifold induced by (M,g,∇);

(ii) $\mathbb{W} := \left(TM, \hat{G}_g^{\nabla^*}, \hat{J}^{\nabla^*}\right)$ is an almost Kähler manifold induced by (M,g,∇^*);

(iii) $\mathbb{M}$ and $\mathbb{W}$ are Kähler manifolds if and only if (M,g,∇) is a statistical manifold.

Therefore, the very natural way to define statistical mirror pairs is the following.

Definition 3.5. *[19] Let (g,∇,∇^*) be a dualistic structure on M such that (M,g,∇^*) is a quasi-statistical manifold with ∇^* flat. Then the Hermitian manifold $\mathbb{M} := \left(TM, \hat{G}_g^{\nabla}, \hat{J}^{\nabla}\right)$ induced by (M,g,∇) and the almost Kähler manifold $\mathbb{W} := \left(TM, \hat{G}_g^{\nabla^*}, \hat{J}^{\nabla^*}\right)$ induced by (M,g,∇^*) are called* statistical mirror pairs.

More generally, we enlarge this definition and introduce quasi-statistical mirror pairs, as follows.

Definition 3.6. *Let (g,∇,∇^*) be a dualistic structure on M such that (M,g,∇^*) is a quasi-statistical manifold. Then the almost Hermitian manifold $\mathbb{M} := \left(TM, \hat{G}_g^{\nabla}, \hat{J}^{\nabla}\right)$ induced by (M,g,∇) and the almost Kähler manifold $\mathbb{W} := \left(TM, \hat{G}_g^{\nabla^*}, \hat{J}^{\nabla^*}\right)$ induced by (M,g,∇^*) are called* quasi-statistical mirror pairs.

If M is an affine manifold with a flat and torsion-free affine connection ∇, a pseudo-Riemannian metric g on M is said to be a *pseudo-Hessian metric* if g is locally expressed by a Hessian, that is, $g = \nabla^2 f = \nabla df$, for f a locally smooth function. The pair (g,∇) is called a *pseudo-Hessian structure* and (M,g,∇) a *pseudo-Hessian manifold*. The following result was proved by Zhang and Khan.

Proposition 3.7. *[18] Let (M,g,∇) be a statistical manifold. Then it is a pseudo-Hessian manifold if and only if ∇ is flat and torsion-free, or, equivalently, if and only if (M,g,∇,∇^*) is a dually flat manifold, or, equivalently, if and only if $\mathbb{M}$ and $\mathbb{W}$ (statistical mirror pair) are Kähler manifolds.*

The special Kähler manifolds, introduced by de Witt and Van Proyen in supersymmetric fields theories [17], and mathematically by Freed [6], are in this class.

Next we construct a quasi-statistical mirror pair in the framework of Norden structures [14]. Let (M,g) be a pseudo-Riemannian manifold and let J be a g-symmetric almost complex structure on M, that is, $J: TM \to TM$, $J^2 = -I$ and $g(JX,Y) = g(X,JY)$, for any $X, Y \in \Gamma^\infty(TM)$. Then (M,g,J) is called a *Norden structure manifold* (and (g,J) a *Norden structure* on M). Moreover, the metric $\tilde{g}$ defined by $\tilde{g}(X,Y) := g(X,JY)$, is called the *twin metric* associated to (g,J). In [2] we proved the following two results.

Proposition 3.8. *Let (M,g,J) be a Norden manifold. If $d^{\nabla^g} J = 0$, then $(M,g,\bar{\nabla} := \nabla^g + \nabla^g J)$ is a statistical manifold and the dual connection of $\bar{\nabla}$ with respect to g is given by $\bar{\nabla}^*_g = \nabla^g - \nabla^g J$. Moreover, if $\tilde{g}$ is the twin metric, then $(M,\tilde{g},\nabla^g)$ is a statistical manifold and the dual connection of ∇^g with respect to $\tilde{g}$ is given by $(\nabla^g)^*_{\tilde{g}} = \nabla^g - J(\nabla^g J)$.*

Proposition 3.9. *Let (M,g,J) be a Norden manifold, let η be a 1-form and let $\bar{\nabla} := \nabla^g + J \otimes \eta$. If $\bar{\nabla}^*$ is the dual connection of $\bar{\nabla}$ with respect to g, then the curvature operator of $\bar{\nabla}^*$ is given by*

$$R^{\bar{\nabla}^*}(X,Y) = R^{\nabla^g}(X,Y) - g^{-1}(\eta) \otimes g((d^{\nabla^g} J)(X,Y))$$

$$+ \left(\eta(JX) g^{-1}(\eta) - \nabla^g_X g^{-1}(\eta)\right) \otimes g(JY) - \left(\eta(JY) g^{-1}(\eta) - \nabla^g_Y g^{-1}(\eta)\right) \otimes g(JX),$$

for any $X, Y \in \Gamma^\infty(TM)$.

For Norden manifolds, as a consequence of Proposition 3.8 and Proposition 3.9, we obtain

Proposition 3.10. *Let (M, g, J) be a Norden manifold and let $\tilde{g}$ be the twin metric defined by (g, J). If $d^{\nabla^g} J = 0$, then $(M, \tilde{g}, \nabla^g)$ and $(M, \tilde{g}, (\nabla^g)^*_{\tilde{g}})$ are statistical manifolds. In particular, $\left(TM, \hat{G}^{\nabla^g}_{\tilde{g}}, \hat{J}^{\nabla^g}\right)$ is an almost Kähler manifold and its quasi-statistical mirror pair is the almost Hermitian (actually, almost Kähler) manifold $\left(TM, \hat{G}^{(\nabla^g)^*}_{\tilde{g}}, \hat{J}^{(\nabla^g)^*}\right)$. Moreover, if ∇^g is flat, then $\left(TM, \hat{G}^{\nabla^g}_{\tilde{g}}, \hat{J}^{\nabla^g}\right)$ is a Kähler manifold.*

Proposition 3.11. *Let (M, g, J) be a Norden manifold, let η be a 1-form on M and let $\bar{\nabla} := \nabla^g + J \otimes \eta$. Then $\left(TM, \hat{G}^{\bar{\nabla}}_g, \hat{J}^{\bar{\nabla}}\right)$ is an almost Kähler manifold and its quasi-statistical mirror pair is the almost Hermitian manifold $\left(TM, \hat{G}^{\bar{\nabla}^*}_g, \hat{J}^{\bar{\nabla}^*}\right)$.*

Example 3.12. *Let M be one of the three 4-dimensional solvmanifolds without complex structures described in [4]. Then M admits natural families of Norden structures. Precisely, denoting by $\{e_1, e_2, e_3, e_4\}$ and $\{e^1, e^2, e^3, e^4\}$ the global frame for TM and T^*M respectively, by g_0, g_1, g_2 the natural neutral pseudo-Riemannian metrics on M defined as*

$$g_0 = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4$$

$$g_1 = e^1 \otimes e^1 - e^2 \otimes e^2 + e^3 \otimes e^3 - e^4 \otimes e^4$$

$$g_2 = e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 + e^4 \otimes e^4$$

and by J_0, J_1, J_2 the almost complex structures on M defined as

$$J_0(e_1) = e_2,\ J_0(e_2) = -e_1,\ J_0(e_3) = e_4,\ J_0(e_4) = -e_3$$

$$J_1(e_1) = e_3,\ J_1(e_2) = -e_4,\ J_1(e_3) = -e_1,\ J_1(e_4) = e_2$$

$$J_2(e_1) = e_4,\ J_2(e_2) = e_3,\ J_2(e_3) = -e_2,\ J_2(e_4) = -e_1,$$

then, for any $a, b \in \mathbb{R}$, $(g_0, \hat{J}_{0a,b} = aJ_1 + bJ_2)$, $(g_1, \hat{J}_{1a,b} = aJ_0 + bJ_2)$ and $(g_2, \hat{J}_{2a,b} = aJ_0 + bJ_1)$ are families of Norden structures on M [4].

Let ∇^{g_i} be the Levi-Civita connection of g_i, for $i \in \{0, 1, 2\}$, and let $\eta_j = e^j$, for $j \in \{1, 2, 3, 4\}$. Then, for any $k \in \{0, 1, 2\}$ and for any $a, b \in \mathbb{R}$,

$$\bar{\nabla}^{i,j,k} = \nabla^{g_i} + \hat{J}_{ka,b} \otimes \eta_j$$

defines the quasi-statistical structure $(g_i, \bar{\nabla}^{i,j,k})$ on M.

As a consequence, we get the following.

Proposition 3.13. *Let $(g_i, \hat{J}_{ka,b})$ be the family of Norden structures on M and let $(g_i, \bar{\nabla}^{i,j,k})$ be the corresponding family of quasi-statistical structures defined before. Then $\left(TM, \hat{G}^{\bar{\nabla}^{i,j,k}}_{g_i}, \hat{J}^{\bar{\nabla}^{i,j,k}}_{ka,b}\right)$ is an almost Kähler manifold and its quasi-statistical mirror pair is the almost Hermitian manifold $\left(TM, \hat{G}^{(\bar{\nabla}^{i,j,k})^*}_{g_i}, \hat{J}^{(\bar{\nabla}^{i,j,k})^*}_{ka,b}\right)$.*

3.3. Mirror symmetry for quasi-semi-Weyl manifolds

Now we shall construct the quasi-statistical mirror pairs in the quasi-semi-Weyl case.

Proposition 3.14. *Let (M, g, η, ∇) be a quasi-semi-Weyl manifold, let $\nabla^*_{(g,\eta)}$ be the semi-dual connection of ∇ with respect to (g, η) and let $(\nabla^*_{(g,\eta)})^*_g$ be the dual connection of $\nabla^*_{(g,\eta)}$ with respect to g. Then $\left(TM, \hat{G}^{\nabla^*_{(g,\eta)}}_g, \hat{J}^{\nabla^*_{(g,\eta)}}_g\right)$ is an almost Hermitian manifold and its quasi-statistical mirror pair is the almost Kähler manifold $\left(TM, \hat{G}^{(\nabla^*_{(g,\eta)})^*_g}_g, \hat{J}^{(\nabla^*_{(g,\eta)})^*_g}_g\right)$.*

Proof. From Proposition 2.14, (ii), we get that $T^{\nabla^*_{(g,\eta)}} = 0$, then, from Proposition 2.6, (iii), we get that $\left(M, g, (\nabla^*_{(g,\eta)})^*_g\right)$ is a quasi-statistical manifold, so we apply the definition of quasi-statistical mirror pair. □

In particular, we have

Corollary 3.15. *Let (M, g, η, ∇) be a Weyl manifold and let ∇^*_g be the dual connection of ∇ with respect to g. Then $\left(TM, \hat{G}^{\nabla}_g, \hat{J}^{\nabla}_g\right)$ is an almost Hermitian manifold and its quasi-statistical mirror pair is the almost Kähler manifold $\left(TM, \hat{G}^{\nabla^*_g}_g, \hat{J}^{\nabla^*_g}_g\right)$.*

Proof. From Remark 2.12 we get that $\nabla^*_{(g,\eta)} = \nabla$, then the statement. □

4. Statistical mirror symmetry from generalized geometry point of view

Finally, we show how the statistical mirror symmetry fits into the generalized geometry framework and we prove that two manifolds which form a statistical mirror pair provides the same structure on the generalized tangent bundle.

4.1. Geometrical structures on $TM \oplus T^*M$

Let $TM \oplus T^*M$ be the generalized tangent bundle of M. On $TM \oplus T^*M$, we consider the natural indefinite metric

$$< X + \alpha, Y + \beta >:= -\frac{1}{2}(\alpha(Y) + \beta(X))$$

and the natural symplectic structure

$$(X + \alpha, Y + \beta) := -\frac{1}{2}(\alpha(Y) - \beta(X)),$$

for all $X, Y \in \Gamma^\infty(TM)$ and $\alpha, \beta \in \Gamma^\infty(T^*M)$.

If g is a pseudo-Riemannian metric on M, we define the symmetric bilinear form, $\check{g}$, on $TM \oplus T^*M$ by

$$\check{g}(X + \alpha, Y + \beta) := g(X, Y) + g(g^{-1}(\alpha), g^{-1}(\beta)),$$

for all $X, Y \in \Gamma^\infty(TM)$ and $\alpha, \beta \in \Gamma^\infty(T^*M)$, and the generalized complex structure, $\check{J}_g$, by

$$\check{J}_g := \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix},$$

which satisfy

$$\check{g}(\check{J}_g(X + \alpha), Y + \beta) = 2(X + \alpha, Y + \beta),$$

for all $X, Y \in \Gamma^\infty(TM)$ and $\alpha, \beta \in \Gamma^\infty(T^*M)$.

Moreover, given an affine connection ∇ on M, we define the bracket $[\cdot, \cdot]_\nabla$ by

$$[X + \alpha, Y + \beta]_\nabla := [X, Y] + \nabla_X\beta - \nabla_Y\alpha,$$

for all $X, Y \in \Gamma^\infty(TM)$ and $\alpha, \beta \in \Gamma^\infty(T^*M)$. Thus, a generalized complex structure, $\check{J}$, is called ∇-*integrable* if its Nijenhuis tensor field $N^\nabla_{\check{J}}$ with respect to ∇

$$N^\nabla_{\check{J}}(\sigma, \tau) := [\check{J}\sigma, \check{J}\tau]_\nabla - \check{J}[\check{J}\sigma, \tau]_\nabla - \check{J}[\sigma, \check{J}\tau]_\nabla + \check{J}^2[\sigma, \tau]_\nabla$$

vanishes for all $\sigma = X + \alpha, \tau = Y + \beta \in \Gamma^\infty(TM \oplus T^*M)$. And we obtain the following characterization of ∇-integrability in terms of quasi-statistical structures [3], precisely

Proposition 4.1. *The generalized complex structure $\check{J}_g$ is ∇-integrable if and only if (M, g, ∇) is a quasi-statistical manifold.*

Furthermore, if we denote by p and π the canonical projections $p : TM \to M$, $\pi : T^*M \to M$, by π_* the tangent map $\pi_* : T(T^*M) \to TM$, $(\pi_*(A))(f) := A(f \circ \pi)$, for all $A \in T(T^*M)$ and for all $f \in C^\infty(M)$, and by $\Omega := d\theta$, where θ is the *Liouville's 1-form* defined by $\theta(A) := p(A)(\pi_*(A))$, for all $A \in T(T^*M)$, then the following holds [11]

Proposition 4.2. *If ∇ is an affine connection on M, then there exists a bundle morphism*

$$\Phi^\nabla : TM \oplus T^*M \to T(T^*M),$$

which is an isomorphism on the fibres, and such that

(i) Φ^∇ identifies T^*M with vertical vectors, that is, $\left(\Phi^\nabla\right)^{-1}(\ker \pi_*) = T^*M$;

(ii) $\pi_* \circ \Phi^\nabla_{|TM} = I_{|TM}$;

(iii) $\left(\Phi^\nabla\right)^*(\Omega) = -2(\cdot, \cdot)$ if and only if $T^\nabla = 0$;

(iv) $\left(\Phi^\nabla\right)([\cdot, \cdot]_\nabla) = \left[\Phi^\nabla \cdot, \Phi^\nabla \cdot\right]$ if and only if $R^\nabla = 0$, where $[\cdot, \cdot]$ denotes the Lie bracket on $T(T^*M)$.

4.2. Statistical mirror symmetry via generalized geometry

By using the pseudo-Riemannian metric g and the splitting of $T(TM)$ into the horizontal and vertical subbundles defined by the connection ∇, we define a bundle morphism [3]

$$\Psi^\nabla : TM \oplus T^*M \to T(TM).$$

Precisely, let $p : TM \to M$ be the canonical projection and $p_* : T(TM) \to TM$ its tangent map. If $a \in TM$ and $A \in T_a(TM)$, then $p_*(A) \in T_{p(a)}M$ and we denote by χ_a the standard identification between $T_{p(a)}M$ and its tangent space $T_a(T_{p(a)}M)$. Then $\Psi^\nabla : TM \oplus T^*M \to T(TM)$ is the bundle morphism defined by

$$\Psi^\nabla(X + \alpha) := X^H_a + \chi_a(g^{-1}(\alpha)),$$

where $a \in TM$ and X^H_a is the horizontal lift of $X \in T_{p(a)}M$.

In [3] (Proposition 4.4), we proved the following.

Proposition 4.3. *If $\hat{G}^\nabla_g$ is the Sasaki metric on TM defined by (g, ∇), then*

$$(\Psi^\nabla)^*(\hat{G}^\nabla_g) = \check{g}.$$

Now, by using the previous considerations, we get a description of statistical mirror symmetry in terms of generalized geometry. Precisely, the following theorem explains how two manifolds which form a statistical mirror pair give rise to the same structure on the generalized tangent bundle.

Theorem 4.4. *Let (g, ∇, ∇^*) be a dualistic structure on M such that (M, g, ∇^*) is a quasi-statistical manifold. Let $\mathbb{M} = \left(TM, \hat{G}^\nabla_g, \hat{J}^\nabla\right)$ be the almost Hermitian manifold induced by (M, g, ∇) and let $\mathbb{W} = \left(TM, \hat{G}^{\nabla^*}_g, \hat{J}^{\nabla^*}\right)$ be the almost Kähler manifold induced by (M, g, ∇^*). If $\Psi^\nabla : TM \oplus T^*M \to T(TM)$ and $\Psi^{\nabla^*} : TM \oplus T^*M \to T(TM)$ are the bundle morphisms defined by ∇ and ∇^* respectively, as before, then*

$$(\Psi^\nabla)^*(\hat{G}^\nabla_g) = \check{g} = (\Psi^{\nabla^*})^*(\hat{G}^{\nabla^*}_g),$$

$$(\Psi^\nabla)^*(\hat{J}^\nabla) = \check{J}_g = (\Psi^{\nabla^*})^*(\hat{J}^{\nabla^*}).$$

Moreover, if we denote by ω^∇ and ω^{∇^*} the Kähler forms of $\mathbb{M}$ and $\mathbb{W}$ respectively, then

$$\omega^\nabla = (\Phi^\nabla)^*(-\Omega) \text{ and } \omega^{\nabla^*} = (\Phi^{\nabla^*})^*(-\Omega)$$

if and only if (M, g, ∇) is a quasi-statistical manifold.

Proof. The statement about the metrics follows from Proposition 4.3, the statement about the almost complex structures follows from a direct computation (see [3]). Regarding Kähler forms, we apply statement (iii) from Proposition 4.2 and the fact that $T^{\nabla^*} = 0$ if and only if (M, g, ∇) is a quasi-statistical manifold. □

In particular, we have

Corollary 4.5. *If (M, g, ∇) is a statistical manifold, then $\mathbb{M}$ and $\mathbb{W}$ are both almost Kähler manifolds and*

$$\mathbb{M} = \Psi^{\nabla}(TM \oplus T^*M, \check{g}, -2(\cdot,\cdot), \check{J}_g)$$

$$\mathbb{W} = \Psi^{\nabla^*}(TM \oplus T^*M, \check{g}, -2(\cdot,\cdot), \check{J}_g).$$

Remark 4.6. *Notice that if, moreover, (g, ∇, ∇^*) is dually flat, then, by using direct computations (see [11]), we conclude that the brackets defined by ∇ and ∇^* correspond to the Lie bracket in the following sense*

$$\Psi^{\nabla}([\cdot,\cdot]_{\nabla}) = \left[\Psi^{\nabla}\cdot, \Psi^{\nabla}\cdot\right]$$

and

$$\Psi^{\nabla^*}([\cdot,\cdot]_{\nabla^*}) = \left[\Psi^{\nabla^*}\cdot, \Psi^{\nabla^*}\cdot\right],$$

where $[\cdot,\cdot]$ denotes the Lie bracket on $T(TM)$.

5. Conclusion

We showed how statistical mirror symmetry can be understood in terms of generalized geometry, describing how statistical mirror pairs can be obtained from the same structure on the generalized tangent bundle by using duality of connections. Hence, our construction provides a bridge not only between information geometry, complex and symplectic geometry, but also with generalized geometry.

Declarations

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