Filomat 37:8 (2023), 2539–2559
https://doi.org/10.2298/FIL2308539H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The semi-T_3-separation axiom of Khalimsky topological spaces

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Abstract. The paper initially studies both the s-T_3-separation and the semi-T_3-separation axiom of Khalimsky (K- for brevity) topological spaces. To do this work, first we investigate some properties of semi-open and semi-closed sets with respect to the operations of union or intersection and further, a homeomorphism, and a semi-homeomorphism. Next, we study various properties of semi-topological properties of K-topological spaces such as simple K-paths. Finally, after introducing the notion of a semi-T_3-separation axiom which is broader than the s-T_3-separation axiom, we find a sufficient and necessary condition for a Khalimsky topological space to satisfy the semi-T_3-separation axiom.

1. Introduction

Many concepts were involved in the study of semi-separation axioms such as a regular open set [45], a semi-open set [32], a preopen set [37], an s-regular set [41, 42], a semi-regular set [10, 36], an irreducible set [12], and so on. Furthermore, based on these notions, various types of mappings were also considered such as an irresolute map [8], a semi-continuous mapping [43], a semi-homeomorphism (a bijection such that the images of semi-open sets are semi-open and inverses of semi-open sets are semi-open) [8], and so forth. Since both separation axioms and semi-separation axioms play important roles in modern mathematics including pure and applied topology such as digital topology, computational topology, the paper studies some properties of Khalimsky topological spaces with respect to the semi-separation axioms.

The notion of a $T_{\frac{1}{2}}$-space with the property that every g-closed set is closed was developed in [32]. Then, it turns out that a topological space X is $T_{\frac{1}{2}}$ if and only if each singleton of X is open or closed. Thus it is obvious that a $T_{\frac{1}{2}}$-space places between a T_0- and a T_1-space [11]. Meanwhile, the semi-T_i-separation axioms, where $i = 0, \frac{1}{2}, 1$, etc (see [4, 6, 32, 34]), are obtained from the definitions of the typical T_i-separation axioms after replacing open sets by semi-open ones. Hence the axiom T_i obviously implies the axiom semi-T_i [9] but the converse does not hold. Moreover, in the case of $i \leq j$, the semi-T_j-separation axiom implies the semi-T_i-separation axiom, and the converse does not hold [5]. Besides, it turns out that a space X

2020 *Mathematics Subject Classification*. Primary 54A05; Secondary 54D10, 54F05, 54C08, 54C10, 54F65
Keywords. Semi-open, semi-closed, semi-T_3-space, s-T_3-space, Alexandroff space, semi-regular, semi-T_1-space, semi-T_2-space, Khalimsky topology, semi-homeomorphism, s-regular, semi-regular
Received: 06 April 2022; Revised: 01 July 2022; Accepted: 13 July 2022
Communicated by Ljubiša D.R. Kočinac
The first author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2019R1I1A3A03059103). Besides, the author was also supported by Korea-China (NRF-NSFC) Joint Research Program (2021K2A9A2A06039864)
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satisfies the semi-$T_{\frac{1}{2}}$ separation axiom if for each point p of X at least one of the sets $\{p\}$, $X \setminus \{p\}$ is semi-open, i.e., for each point p of X the set $\{p\}$ is either semi-open or semi-closed [9–11, 33]. As usual, a property is called a semi-topological property if the property is preserved by a semi-homeomorphism. Then the semi-T_i-separation axioms, $i \in \{0, \frac{1}{2}, 1, 2\}$, are proved to have the semi-topological property [34].

Now we may pose the following query. In the case of $i \lneq j$, what relationship exists between T_i-space and a semi-T_j-space? In relation to this query, the recent paper [5] proved that a semi-$T_{\frac{1}{2}}$-space does not imply a T_0-space (see Example 2.7 of [5]) and not vice versa (see also Example 2.8 of [5]). Besides, a locally finite $T_{\frac{1}{2}}$-space does not imply a semi-T_1-space either (see Example 2.6 of [5]). In addition, whereas a $T_{\frac{1}{2}}$-space (resp. a T_1-space) obviously implies a semi-T_0-space (resp. a semi-$T_{\frac{1}{2}}$-space), the converse does not hold.

The aim of this paper is initially to propose that the n-dimensional Khalimsky (or K-, for brevity) topological space, $(\mathbb{Z}^n, \kappa^n)$, satisfies the semi-T_3-separation axiom. In order to carry out this work, we need to investigate the following topics strongly.

- Study of some properties of the union and intersection of semi-closed and semi-open sets. In detail, under what condition is the union of two semi-closed sets semi-closed?
- Assume two homeomorphic subspaces (Y, T_Y) and (Z, T_Z) induced by a topological space (X, T), where $Y, Z \subset X$. Then we examine the semi-openness and semi-closedness of Y and Z in (X, T) and further, a comparison between two subspaces (Y, T_Y) and (Z, T_Z) with respect to a semi-homeomorphism.
- What condition makes a simple K-path semi-closed?
- Study of various properties of the union and intersection of semi-closed and semi-open sets in the n-dimensional Khalimsky topological space.
- Establishment of a semi-T_3-separation axiom and a comparison between a semi-T_3-separation axiom and an s-T_3-separation axiom.
- Examination of the semi-T_3-separation axiom of the n-dimensional Khalimsky topological space.
- Comparison between the semi-T_3-separation axiom and the typical T_1-separation axiom.

Hereinafter, we will often use the notations $\mathbb{N}$, $\mathbb{Z}_o$ (resp. $\mathbb{Z}_e$) to indicate the sets of natural numbers, i.e., positive integers, odd (resp. even) integers.

Besides, we use the notation " $\subset$" (resp. $|X|$) to indicate a 'proper subset or equal' (resp. the cardinality of the given set X). The symbol $\aleph_0$ means the cardinality of an (infinite) denumerable set. In addition, for distinct integers $a, b \in \mathbb{Z}$, we use the notations $[a, b]_{\mathbb{Z}} = \{x \in \mathbb{Z} \,|\, a \leq x \leq b\}$, $(-\infty, b]_{\mathbb{Z}} = \{x \in \mathbb{Z} \,|\, x \leq b\}$, and $[a, \infty)_{\mathbb{Z}} = \{x \in \mathbb{Z} \,|\, a \leq x\}$. Finally, the notation " $:=$" will be used to introduce new notions without proving the fact.

This paper is organized as follows. Section 2 provides some basic notions related to the K-topology and Alexandroff topological structure. Section 3 studies various properties of semi-closed and semi-open sets in the K-topological space. Section 4 examines semi-open and semi-closed properties of Y and Z in (X, T), where (Y, T_Y) and (Z, T_Z) are homeomorphic subspaces induced by a topological space (X, T), where $Y, Z \subset X$. Section 5 first proves the semi-closedness of a simple K-path with some hypothesis. Section 6 finds a sufficient and necessary condition for Khalimsky topological spaces to satisfy the semi-T_3-separation. Section 7 concludes the paper with summary and a further work.

2. Some Khalimsky topological properties with respect to the Alexandroff structure

Let us recall some properties of K-topological spaces with respect to the Alexandroff (A-, for brevity) structure which will be used in studying semi-topological properties of K-topological spaces. Besides, we will refer to some advantages of the K-topological spaces in the fields of digital topology, digital geometry, rough set theory, and so on (see Remark 2.4).

A topological space (X, T) is called an Alexandroff space [1, 2] if for each $x \in X$, the intersection of all open sets of X containing x (denoted by $SN_T(x)$) is T-open in X. As an example of the A-space, the n-dimensional K-topological space, denoted by $(\mathbb{Z}^n, \kappa^n)$ as a product topological space of the K-topological line space, can be considered because $(\mathbb{Z}^n, \kappa^n)$ is locally finite. There is now a considerable literature on the space $(\mathbb{Z}^n, \kappa^n)$ including its various properties obtained from [15–17, 20–25, 28, 29, 31, 36, 38, 40]. Besides, for $X \subset \mathbb{Z}^n, n \in \mathbb{N} \setminus \{1\}$, we will denote by $(X, (\kappa^n)_X)$ a subspace induced by $(\mathbb{Z}^n, \kappa^n)$. Hence let us now recall basic notions of the n-dimensional Khalimsky space, $n \geq 1$. Khalimsky line topology κ on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbb{Z}} \,|\, n \in \mathbb{Z}\}$ as a subbase [28]. In the present paper we call $([a,b]_{\mathbb{Z}}, \kappa_{[a,b]_{\mathbb{Z}}})$ (or for short $[a,b]_{\mathbb{Z}}$ if there is no danger of ambiguity) a Khalimsky (or K-) interval. Moreover, for a subset $X \subset \mathbb{Z}^n$, the subspace induced by $(\mathbb{Z}^n, \kappa^n)$ is obtained, denoted by $(X, (\kappa^n)_X)$ and called a K-topological space.

Let us now investigate some structures of $(\mathbb{Z}^n, \kappa^n)$. A point $x = (x_i)_{i \in [1,n]_{\mathbb{Z}}} \in \mathbb{Z}^n$ is *pure open* if all coordinates are odd, and *pure closed* if each of the coordinates is even and the other points in $\mathbb{Z}^n$ are called *mixed* [29]. These points are shown like the following symbols: The symbols ■ (resp. •) means a pure closed point (resp. a mixed point) (see Figures 1, 2, 3, 4, 5, and 6) and further, a black jumbo dot represents a pure open point. In addition, in the present paper we denote by $(\mathbb{Z}^n)_o$ (resp. $(\mathbb{Z}^n)_e$) the set of all pure open (resp. pure closed) points of $(\mathbb{Z}^n, \kappa^n)$. Besides, $(\mathbb{Z}^n)_m = \mathbb{Z}^n \setminus ((\mathbb{Z}^n)_e \cup (\mathbb{Z}^n)_o)$ stands for the set of all mixed points of $(\mathbb{Z}^n, \kappa^n), n \in \mathbb{N} \setminus \{1\}$.

In relation to the study of digital objects in $\mathbb{Z}^n$, let us recall some basic notations named by digital k-neighborhood of the given point $p \in \mathbb{Z}^2$, as follows:
For a point $p = (x, y) \in \mathbb{Z}^2$ we follow the notations [44].

$$\left.\begin{cases} N_4(p) = \{(x \pm 1, y), p, (x, y \pm 1)\} \\ N_8(p) = \{(x \pm 1, y), p, (x, y \pm 1), (x \pm 1, y \pm 1)\} \end{cases}\right\} \tag{2.1}$$

which is respectively called the 4-neighborhood and the 8-neighborhood of a given point $p = (x, y) \in \mathbb{Z}^2$.

Motivated by the digital k-connectivity for low dimensional digital images $(X, k), X \subset \mathbb{Z}^n, n \in [1,3]_{\mathbb{Z}}$ [30, 44], as a generalization of this approach, the papers [14, 19] initially developed some k-adjacency relations for high dimensional digital images $(X, k), X \subset \mathbb{Z}^n$ (see also (2.2) below), as follows: For a natural number $t \in [1, n]_{\mathbb{Z}}$, the distinct points $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n$ are $k(t, n)$-adjacent if at most t of their coordinates differ by ± 1 and the others coincide.

According to this statement, the $k(t, n)$-adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$, are formulated [14] (see also [19]) as follows:

$$k = k(t, n) = \sum_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.2}$$

For instance, the following are obtained in [14, 19]:

$$(n, t, k) \in \left.\begin{cases} (4,1,8), (4,2,32), (4,3,64), (4,4,80); \text{and} \\ (5,1,10), (5,2,50), (5,3,130), (5,4,210), (5,5,242). \end{cases}\right\}$$

Using these k-adjacency relations of $\mathbb{Z}^n$ in (2.2), $n \in \mathbb{N}$, we will call the pair (X, k) a digital image on $\mathbb{Z}^n$, $X \subset \mathbb{Z}^n$, as usual. Besides, these k-adjacency relations can be essential to studying digital products with normal adjacencies [14] and pseudo-normal adjacencies [27] and calculating digital k-fundamental groups of digital products [14]. Hereafter, (X, k) is assumed in $\mathbb{Z}^n, n \in \mathbb{N}$, with one of the k-adjacency of (2.2).

In relation to the further statement of a mixed point in $(\mathbb{Z}^2, \kappa^2)$, for the point $p = (2m, 2n+1)$(resp. $p = (2m+1, 2n)$), we call the point p *closed-open* (resp. *open-closed*). In terms of this perspective, we clearly observe that the smallest (open) neighborhood of the point $p = (p_1, p_2)$ of $\mathbb{Z}^2$, denoted by $SN_K(p) \subset \mathbb{Z}^2$, is the following [15, 29]:

$$SN_K(p) = \left.\begin{cases} \{p\} \text{ if } p \text{ is pure open}, \\ \{(p_1, p_2 \pm 1), p\} \text{ if } p \text{ is open-closed}, \\ \{(p_1 \pm 1, p_2), p\} \text{ if } p \text{ is closed-open}, \\ N_8(p) \text{ if } p \text{ is pure closed} \end{cases}\right\} \tag{2.3}$$

Hereafter, in $(X, (\kappa^2)_X)$, for a point $p \in X$ we use the notation $SN_X(p) = SN_K(p) \cap X$ [15, 25] for short. Based on (2.3), for a point p in $(\mathbb{Z}^n, \kappa^n)$, we can establish $SN_K(p)$ in $(\mathbb{Z}^n, \kappa^n)$. Using the smallest open set of (2.3), the notion of a K-adjacency in $(\mathbb{Z}^n, \kappa^n)$ is defined, as follows: For distinct points $p, q \in (\mathbb{Z}^n, \kappa^n)$, we say that p is K-adjacent to q [25, 29] if

$$p \in SN_K(p) \text{ or } q \in SN_K(q).$$

Based on the structure of the smallest open set of a point p in $(\mathbb{Z}^2, \kappa^2)$, we obtain the following:
Given the point $p = (p_1, p_2)$ of $\mathbb{Z}^2$, denoted by $Cl_K(\{p\}) \subset \mathbb{Z}^2$, is the following [17, 20, 28]:

$$Cl_K(\{p\}) = \begin{Bmatrix} \{p\} \text{ if } p \text{ is pure closed,} \\ \{(p_1, p_2 \pm 1), p\} \text{ if } p \text{ is closed-open,} \\ \{(p_1 \pm 1, p_2), p\} \text{ if } p \text{ is open-closed,} \\ N_8(p) \text{ if } p \text{ is pure open} \end{Bmatrix} \tag{2.4}$$

Based on (2.4), for a point p in $(\mathbb{Z}^n, \kappa^n)$, we can establish $Cl_K(\{p\})$ in $(\mathbb{Z}^n, \kappa^n)$. Hereinafter, in relation to the study of K-topological space, we will use the term 'Cl' for brevity instead of "Cl_K" if there is no danger of confusion.

Definition 2.1. ([25]) For $X := (X, (\kappa^n)_X)$ we define the following.

(1) For distinct points x, y in X, if there is the sequence (or a path) $(x_0, x_1, \cdots, x_l)$ on X such that $x = x_0$ and $y = x_l$ and further, x_i and x_{i+1} are K-adjacent, $i \in [0, l-1]_{\mathbb{Z}}, l \in \mathbb{N}$, then we say that the sequence is the K-path connecting the given points x and y. Furthermore, the number l is the length of this K-path. In addition, a singleton is assumed to be a K-path.

(2) For any two points $x, y \in X$, there is a K-path connecting the two points, then X is said to be K-path connected (or connected).

(3) A *simple K-path* in X is the K-path $(x_i)_{i\in[0,l]_{\mathbb{Z}}}$ in X such that x_i and x_j are K-adjacent if and only if $|i - j| = 1$.

According to the properties (2.1), using some properties of the closure and the interior operator, we obviously obtain the following:

Lemma 2.2. ([17]) *A subset B of $(\mathbb{Z}^2, \kappa^2)$ is open if and only if*

$$\begin{Bmatrix} N_8(p) \subset B \text{ whenever } p = (2m, 2n) \in B, \text{or} \\ \{2m+1\} \times [2n-1, 2n+1]_{\mathbb{Z}} \subset B \text{ whenever } (2m+1, 2n) \in B, \text{or} \\ [2m-1, 2m+1]_{\mathbb{Z}} \times \{2n+1\} \subset B \text{ whenever } (2m, 2n+1) \in B. \end{Bmatrix} \tag{2.5}$$

Based on the property (2.4) and the notion of closure of a given set [39], we obtain the following:

Corollary 2.3. ([17]) *A subset C of $(\mathbb{Z}^2, \kappa^2)$ is closed if and only if*

$$\begin{Bmatrix} N_8(q) \subset C \text{ whenever } q = (2m+1, 2n+1) \in C, \text{or} \\ [2m, 2m+2]_{\mathbb{Z}} \times \{2n\} \subset C \text{ whenever } (2m+1, 2n) \in C, \text{or} \\ \{2m\} \times [2n, 2n+2]_{\mathbb{Z}} \subset C \text{ whenever } (2m, 2n+1) \in C. \end{Bmatrix} \tag{2.6}$$

In view of the property (2.6), under $(\mathbb{Z}^2, \kappa^2)$, for the point $p = (2m+1, 2n+1)$, $m, n \in \mathbb{Z}$, the closure of the singleton $\{p\}$ is the set

$$Cl(SN_K(p)) = Cl(\{p\}) = N_8(p) = [2m, 2m+2]_{\mathbb{Z}} \times [2n, 2n+2]_{\mathbb{Z}}. \tag{2.7}$$

In view of (2.3)–(2.7), we obviously obtain that $(\mathbb{Z}^n, \kappa^n)$ is a semi-$T_{\frac{1}{2}}$ space $n \in \mathbb{N}$ [5].

When studying digital objects in $\mathbb{Z}^n$, the properties of (2.3) and (2.4) enable us to get the following advantages of the K-topological structure of X induced by $(\mathbb{Z}^n, \kappa^n)$.

Remark 2.4. (Utility of the K-topological structure)

(1) When studying a self-homeomorphism of $(\mathbb{Z}^n, \kappa^n)$, we should consider the following map

$$\left\{\begin{array}{l} h : (\mathbb{Z}^n, \kappa^n) \to (\mathbb{Z}^n, \kappa^n) \text{ defined by:} \\ \text{for each point } x := (x_1, x_2, \cdots, x_n) \in \mathbb{Z}^n, \\ h(x) = (x_1 + 2m_1, x_2 + 2m_2, \cdots, x_n + 2m_n), \\ \text{with some } m_i \in \mathbb{Z}, i \in M \subset [1, n]_{\mathbb{Z}} \end{array}\right\} \tag{2.8}$$

For instance, it is clear that the map $h : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ defined by $h(x) = x+2m, m \in \mathbb{Z}$, is a homeomorphism. Meanwhile, note that the following map g cannot be a homeomorphism, where

$$\left\{\begin{array}{l} g : (\mathbb{Z}^n, \kappa^n) \to (\mathbb{Z}^n, \kappa^n) \text{ defined by:} \\ \text{for each point } x := (x_1, x_2, \cdots, x_n) \in \mathbb{Z}^n, \\ g(x) = (x_1 + t_1, x_2 + t_2, \cdots, x_n + t_n) \\ \text{such that there is at least } t_i \in \mathbb{Z}_o, i \in [1, n]_{\mathbb{Z}}. \end{array}\right\} \tag{2.9}$$

For instance, $g : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ defined by $g(x) = x + 2m + 1, m \in \mathbb{Z}$ cannot be a homeomorphism.

(2) Since the K-topological structure is one of the fundamental frames for studying digital images on $\mathbb{Z}^n$, motivated by this structure, some more generalized topological structures on $\mathbb{Z}^n$ can be established [23, 24].

(3) Based on the K-topological structure of $\mathbb{Z}^n$, we can obtain a digital adjacency induced by the given topological structure [25, 29].

(4) When digitizing a set X in the n-dimensional real space with respect to the K-topological structure, we can use some local rule in [18] to obtain its digitized set $D_K(X) \subset \mathbb{Z}^n$ and finally use it in the fields of rough set theory, mathematical morphology, digital geometry, and so on [18, 21].

(5) Since the modern electronic devices are usually operated on the finite digital planes with more than twenty million pixels, some restriction of the given map h of (2.8) on a finite digital image can be used in the fields of pattern recognition and image processing in a K-topological setting. In particular, as introduced in [18], when studying digital rough set theory, the K-topological structure can be essential to digitizing some real objects.

3. Some properties of semi-open and semi-closed sets

This section first recalls the notions of a semi-open and semi-closed set. Namely, a subset A of a topological space (X, T) is said to be *semi-open* if there is an open set O in (X, T) such that $O \subset A \subset Cl(O)$. Besides, we say that a subset B of a topological space (X, T) is *semi-closed* if the complement of B in X, i.e., B^c, is semi-open in (X, T). Then we see that a subset A of (X, T) is semi-open if and only if $A \subset Cl(Int(A))$ [32] and a subset B of (X, T) is semi-closed if and only if $Int(Cl(B)) \subset B$ [7]. It is clear that "open" (resp. "closed") is stronger than "semi-open" (resp. "semi-closed"). Besides, an empty set is clearly both a semi-open and semi-closed. The notion of semi-openness and semi-closedness enable us to get the following [10, 32, 41]:

Now let us investigate some semi-topological properties of some subsets of $(\mathbb{Z}, \kappa)$ which correct some errors in some literature [13, 40].

Lemma 3.1. *In $(\mathbb{Z}, \kappa)$, for any $x \in \mathbb{Z}$, we obtain the following: (1) Each singleton $\{x\}$ is semi-closed.*

(2) Each singleton $\{x\}$ is not semi-open if $x \in \mathbb{Z}_e$ [40]. For any $a, b \in \mathbb{Z}$, we have the following:

(3) Any set $[a, b]_{\mathbb{Z}}$ is semi-closed.

(4) A set $[a, b]_{\mathbb{Z}}$ is semi-open if $a \neq b$.

(5) For any $a \in \mathbb{Z}$, each of the sets $(-\infty, a]_{\mathbb{Z}}$ and $[a, \infty)_{\mathbb{Z}}$ is both semi-open and semi-closed.

Proof. (1) (Case 1-1) Assume $x \in \mathbb{Z}_e$. Then we have $Int(Cl(\{x\})) = \emptyset \subset \{x\}$.

(Case 1-2) Assume $x \in \mathbb{Z}_o$. Then we have $Cl(\{x\}) = \{x_1, x, x + 1\}$ so that $Int(Cl(\{x\})) = \{x\}$.

(2) For $x \in \mathbb{Z}_e$, since $Cl(Int(\{x\})) = \emptyset$, the proof is completed.

(3) According to the numbers a, b of $A = [a, b]_{\mathbb{Z}}$, we have the following several cases.
(Case 3-1) Assume $A = [2m, 2n]_{\mathbb{Z}}$. Then, we have $Int(Cl(A)) = [2m+1, 2n-1]_{\mathbb{Z}}$, which implies the semi-closedness of A in $(\mathbb{Z}, \kappa)$.
(Case 3-2) Assume $A = [2m+1, 2n]_{\mathbb{Z}}$. Then, we have $Int(Cl(A)) = [2m+1, 2n-1]_{\mathbb{Z}}$, which implies the semi-closedness of A in $(\mathbb{Z}, \kappa)$.
(Case 3-3) Assume $A = [2m, 2n+1]_{\mathbb{Z}}$. Then, using a method similar to the proof of (Case 3-2), we see the semi-closedness of A in $(\mathbb{Z}, \kappa)$.
(Case 3-4) Assume $A = [2m+1, 2n+1]_{\mathbb{Z}}$. Then, we have $Int(Cl(A)) = A$, which implies the semi-closedness of A in $(\mathbb{Z}, \kappa)$.

(4)-(5) Using a method similar to the proof of (3), the assertions are proved. □

Lemma 3.2. *(1) Given two semi-open sets, the intersection of them need not be semi-open.*
(2) Given two semi-closed sets, the union of them may not be semi-closed.

To support Lemma 3.2, it suffices to suggest the following two counterexamples in $(\mathbb{Z}, \kappa)$.
(1) For an even number $a \in \mathbb{Z}_e$, assume $A_1 = (-\infty, a]_{\mathbb{Z}}$ and $A_2 = [a, \infty)_{\mathbb{Z}}$. While each of $A_i, i \in \{1, 2\}$, is semi-open, the intersection of them, i.e., $A_1 \cap A_1 = \{a\}$, is proved not to be semi-open because $\{a\} \not\subseteq Cl(Int(\{a\}) = \emptyset$.
(2) Assume $X = \{4n+1 \,|\, n \in \mathbb{Z}\}$ and $Y = \{4n+3 \,|\, n \in \mathbb{Z}\}$ which are semi-closed. Then the union $X \cup Y$ is not semi-closed because $Int(Cl(X \cup Y)) = \{4n+1, 4n+2, 4n+3\} \not\subseteq X \cup Y$.

Based on the notions of a semi-open and a semi-closed set, we obviously obtain the following.

Remark 3.3. (1) Given two semi-open sets, the union of them is semi-open.
(2) Given two semi-closed sets, the intersection of them is semi-closed.

By Lemma 3.1(2) and (4), we obtain the following:

Corollary 3.4. *A connected subset X of $(\mathbb{Z}, \kappa)$ with $|X| \geq 2$ is semi-open.*

Theorem 3.5. *In $(\mathbb{Z}, \kappa)$, a connected subset is semi-closed. However, the converse does not hold.*

Proof. After denoting a connected subset of $(\mathbb{Z}, \kappa)$ by A, we have the following three cases.
(Case 1) Assume the case of $|A| = \aleph_0$. Then, consider the following:
(1-1) In the case of $A = \mathbb{Z}$, the assertion is clearly proved.
(1-2) In the case of $A = (-\infty, b]_{\mathbb{Z}}$ for some $b \in \mathbb{Z}$, by Lemma 3.1(5), the assertion is obviously proved.
(1-3) In the case of $A = [a, \infty)_{\mathbb{Z}}$ for some $a \in \mathbb{Z}$, by Lemma 3.1(5), the assertion is also proved.
(Case 2) Assume the set A such that $2 \leq |A| \lneq \aleph_0$. Then A is a kind of a finite K-interval. By Lemma 3.1(3), the proof is completed.
(Case 3) Assume the set A such that $|A| = 1$, then by Lemma 3.1, the proof is completed.
Conversely, to prove that a semi-closed subset need not be connected in $(\mathbb{Z}, \kappa)$, we have the following counterexample. While the set $\{2n, 2n+2\}$ is semi-closed in $(\mathbb{Z}, \kappa)$, it is not connected in $(\mathbb{Z}, \kappa)$. □

In view of Lemma 3.1, Corollary 3.4, and Theorem 3.5, we have the following:

Remark 3.6. In $(\mathbb{Z}, \kappa)$, assume a connected subset A with $|A| \geq 2$. Then it is both semi-open and semi-closed.

Unlike Lemma 3.2(2), we obtain the following property of the union of two semi-closed sets in $(\mathbb{Z}^n, \kappa^n)$.

Theorem 3.7. *In a subspace (X, κ^n_X) of $(\mathbb{Z}^n, \kappa^n)$, assume two semi-closed sets $A_i, i \in \{1, 2\}$, such that $Cl(A_1) \cap Cl(A_2) = \emptyset$. Then the union of them is semi-closed in (X, κ^n_X). The converse does not hold.*

Proof. While we have the properties

$$\begin{Bmatrix} Int(Cl(A_1 \cup A_2)) = Int(Cl(A_1) \cup Cl(A_2)) \text{ and} \\ Int(Cl(A_1)) \cup Int(Cl(A_2)) \subset Int(Cl(A_1) \cup Cl(A_2)), \end{Bmatrix} \tag{3.1}$$

we see the following [39]:

$$Int(Cl(A_1) \cup Cl(A_2)) \text{ need not be a subset of } Int(Cl(A_1)) \cup Int(Cl(A_2)).$$

For instance, in $(\mathbb{Z}, \kappa)$, let $A_1 = \{1\}$ and $A_2 = \{3\}$. Then, we obviously obtain

$$\begin{Bmatrix} Cl(A_1) = \{0, 1, 2\} \text{ and } Cl(A_2) = \{2, 3, 4\}, \text{and} \\ Int(Cl(A_1)) = \{1\} \text{ and } Int(Cl(A_2)) = \{3\}. \end{Bmatrix}$$

Meanwhile, we obtain

$$\begin{Bmatrix} Int(Cl(A_1) \cup Cl(A_2)) = \{1, 2, 3\} \\ \text{so that } Int(Cl(A_1) \cup Cl(A_2)) \not\subset Int(Cl(A_1)) \cup Int(Cl(A_2)). \end{Bmatrix}$$

However, in the case of $Cl(A_1) \cap Cl(A_2) = \emptyset$, we now prove the identity

$$Int(Cl(A_1) \cup Cl(A_2)) = Int(Cl(A_1)) \cup Int(Cl(A_2)). \tag{3.2}$$

To be specific, with the given hypothesis, in view of (3.1), we only need to prove the following:

$$Int(Cl(A_1) \cup Cl(A_2)) \subset Int(Cl(A_1)) \cup Int(Cl(A_2)). \tag{3.3}$$

To do this work, we have the following two cases.
(Case 1) Assume the case $Int(Cl(A_1) \cup Cl(A_2)) = \emptyset$. Then the property of (3.3) clearly holds.
(Case 2) Assume the case $Int(Cl(A_1) \cup Cl(A_2)) \neq \emptyset$. Take an arbitrary point

$$p \in Int(Cl(A_1) \cup Cl(A_2)). \tag{3.4}$$

Then there is the smallest open set $O(p)(\ni p)$ in (X, T) such that $p \in O(p) \subset Cl(A_1) \cup Cl(A_2)$. Owing to the given hypothesis, i.e.,

$$Cl(A_1) \cap Cl(A_2) = \emptyset \tag{3.5}$$

and the connectedness of $O(p)$, we have

$$O(p) \subset Cl(A_1) \text{ or } O(p) \subset Cl(A_2). \tag{3.6}$$

The former implies $p \in Int(Cl(A_1))$ and the latter supports $p \in Int(Cl(A_2))$. Thus we have

$$p \in Int(Cl(A_1)) \cup Int(Cl(A_2)).$$

By (3.1) and (3.3), we have the identity as in (3.2).

As for the property of (3.6), by contrary, suppose that the property of (3.6) does not hold. Then we certainly come across the disconnectedness of $O(p)$.

Finally, owing to the identity of (3.2), we have

$$Int(Cl(A_1 \cup A_2)) = Int(Cl(A_1)) \cup Int(Cl(A_2)) \subset A_1 \cup A_2,$$

which leads to the semi-closedness of $A_1 \cup A_2$.

To prove the converse, consider the following counterexample. Let $A_1 = \{0, 1\}$ and $A_2 = \{2, 3\}$. By Lemma 3.1 and Corollary 3.4, while the union of them, $A := A_1 \cup A_2$, is semi-closed, we obtain $Cl(A_1) \cap Cl(A_2) \neq \emptyset$ because $Cl(A_1) = \{0, 1, 2\}$ and $Cl(A_2) = \{2, 3, 4\}$. □

Example 3.8. (1) In $(\mathbb{Z}, \kappa)$, consider the two sets $A_1 = (-\infty, 0]_{\mathbb{Z}}$ and $A_2 = [2, \infty)_{\mathbb{Z}}$. Then these are semi-closed and since

$$Int(Cl(A_i)) \subset A_i, i \in \{1, 2\} \text{ and } Cl(A_1) \cap Cl(A_2) = \emptyset,$$

we have $Int(Cl(A_1 \cup A_2)) \subset A_1 \cup A_2$.

(2) In $(\mathbb{Z}, \kappa)$, while each of the singletons $\{2n+1\}$ and $\{2n+3\}$ is semi-closed, $Cl(\{2n+1\}) = \{2n, 2n+1, 2n+2\}$ and $Cl(\{2n+3\}) = \{2n+2, 2n+3, 2n+4\}$ so that $Cl(\{2n+1\}) \cap Cl(\{2n+3\}) \neq \emptyset$. Then we observe that the union of them, $\{2n+1, 2n+3\}$, is not semi-closed because

$$Int(Cl(\{2n+1, 2n+3\})) = \{2n+1, 2n+2, 2n+3\} \not\subset \{2n+1, 2n+3\}.$$

Theorem 3.9. ([36]) *A subset B of $(\mathbb{Z}, \kappa)$ is semi-open if and only if $p-1 \in B$ or $p+1 \in B$ whenever $p \in B$, where $p \in \mathbb{Z}_e$.*

To support Theorem 3.9, for the sake of a contradiction, suppose both $p-1 \notin B$ and $p+1 \notin B$ whenever $p \in B$, where $p \in \mathbb{Z}_e$, i.e., $SN_K(p) \not\subset B$. Then, for $B = (B \setminus \{p\}) \cup \{p\}$, since $Int(\{p\}) = \emptyset$, we have

$$Cl(Int(B)) = Cl(Int((B \setminus \{p\}) \cup \{p\})) \subset B \setminus \{p\},$$

which means

$$B \not\subset Cl(Int(B))$$

so that B is not semi-open.

Conversely, assume that B is semi-open. Then, for any point $p \in B, p \in \mathbb{Z}_e$, we have the following three cases

$$\{p, p+1\} \subset B, \{p-1, p\} \subset B, \text{ or } \{p-1, p, p+1\} \subset B. \tag{3.7}$$

For each case of (3.7), we see the property $B \subset Cl(Int(B))$, which completes the proof.

We say that a topological space (X, T) is a cut-point space [29] if there is a point $p \in X$ such that $X \setminus \{p\}$ is not connected in (X, T). Then we call the point p a cut point in (X, T). For instance, the K-topological line space $(\mathbb{Z}, \kappa)$ is a good example for a cut-point space [29]. In addition, it is clear that $(\mathbb{Z}^n, \kappa^n), n \geq 2$, is not a cut-point space.

The paper [36] proved that a subset A of $(\mathbb{Z}, \kappa)$ is semi-closed if and only if $2n-1$ or $2n+1 \notin A$ whenever $2n \notin A$. Motivated by this approach, we obtain the following:

Theorem 3.10. *In $(\mathbb{Z}, \kappa)$, consider a set $A \subset \mathbb{Z}$ with $|A| \geq 3$. For $x, x+2 \in \mathbb{Z}_o$, $x, x+2 \in A$ implies $x+1 \in A$ if and only if A is semi-closed.*

Proof. With the hypothesis, assume that A is semi-closed in $(\mathbb{Z}, \kappa)$. By contrary, suppose $x+1 \notin A$ whenever $\{x, x+2\} \subset A$ and $x, x+2 \in \mathbb{Z}_o$. Since $Cl(A)$ contains the set $\{x, x+1, x+2\}$ (see the property of (2.4) for the case of $(\mathbb{Z}, \kappa)$), i.e., $\{x, x+1, x+2\} \subset Cl(A)$. Since the set $\{x, x+1, x+2\} = SN_K(x+1)$ is an open set in $(\mathbb{Z}, \kappa)$, we obtain $Int(Cl(A)) \not\subset A$, which invokes a contradiction to the semi-closedness of A.

Conversely, owing to the condition that $x+1 \in A$ is assumed whenever $x, x+2 \in A$ and $x, x+2 \in \mathbb{Z}_o$, we have $SN_K(x+1) \subset A$. Since $SN_K(x+1)$ is connected, A contains a connected subset. With this situation, let us prove the property $Int(Cl(A)) \subset A$. To do this work, we consider the following two cases according to either the connectedness or disconnectedness of A.

(Case 1) Assume that A is a connected subset of $(\mathbb{Z}, \kappa)$. Then, by Theorem 3.5, A is semi-closed.

(Case 2) Assume that A is not a connected subset of $(\mathbb{Z}, \kappa)$. Owing to the hypothesis of non-connectedness of A, there are at least some cut points separating the given set A. Indeed, the number of the cut points depends on the subspace (A, κ_A).

Let us assume the following two sets C and D and the union of them (see Figure 1)

$$\left\{ \begin{array}{l} C = \bigcup_{i \in M} C_i, \text{ where } Cl(C_{i_1}) \cap Cl(C_{i_2}) = \emptyset \text{ if } i_1 \neq i_2, \ i_1, i_2 \text{ in } M, \\ D = \bigcup_{j \in M'} D_j, \text{ where } Cl(D_{j_1}) \cap Cl(D_{j_2}) = \emptyset \text{ if } j_1 \neq j_2, \ j_1, j_2 \text{ in } M', \text{ and} \\ A = C \cup D, \end{array} \right\} \tag{3.8}$$

where each $C_i \subset A$ is a component in $(\mathbb{Z}, \kappa)$ relating to the property (see the given hypothesis)

$$\text{“whenever } x, x+2 \in \mathbb{Z}_o, x, x+2 \in A \Rightarrow x+1 \in A\text{”} \tag{3.9}$$

and each $D_j \subset A$ is also a component in $(\mathbb{Z}, \kappa)$) that is not related to the property of (3.9).
Indeed, by Theorem 3.5, we see that each C_i and D_j is semi-closed in $(\mathbb{Z}, \kappa)$, $i \in M$ and $j \in M'$. Besides, there are some points $p \in \mathbb{Z}_o$ separating the two sets C and D. Then we have

$$Cl(C) \cap Cl(D) = \emptyset. \tag{3.10}$$

By Theorems 3.5 and 3.7, and (3.8), we see that each of the sets C and D is semi-closed. Furthermore, by Theorem 3.5 and (3.10), we have

$$Int(Cl(A)) = Int(Cl(C \cup D)) = Int(Cl(C)) \cup Int(Cl(D)) \subset C \cup D = A,$$

which completes the proof. $\square$

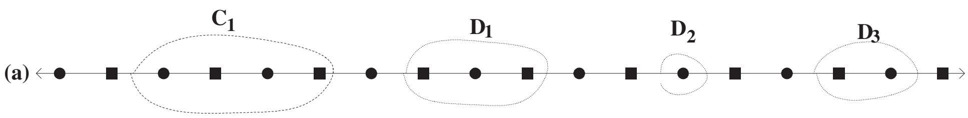

Figure 1: Configuration of the set A which is a disconnected subset $C \cup D$ referred to in the proof of Theorem 3.10 (see Case 2).

4. Some properties of semi-open and semi-closed sets with respect to a homeomorphism and a semi-homeomorphism

We first state some properties which can play important roles in studying semi-open and semi-closed sets. To support Lemma 3.2(2), we have the following:

Theorem 4.1. *In a topological space (X, T), assume two semi-closed sets $A_i, i \in \{1, 2\}$. If there is an element $x \in X \setminus (A_1 \cup A_2)$ such that $O(x) \subset Cl(A_1) \cup Cl(A_2)$, where $O(x)(\in T)$ means an open set containing the point x. Then the union $A_1 \cup A_2$ is not semi-closed in (X, T).*

Proof. Given two semi-closed sets $A_i, i \in \{1, 2\}$, in a topological space (X, T), according to the hypothesis, let us assume an element $x \in X \setminus (A_1 \cup A_2)$ and an open set $O(x)(\in T)$ such that $O(x) \subset Cl(A_1) \cup Cl(A_2)$. Then we have the property

$$Int(Cl(A_1 \cup A_2)) \not\subseteq A_1 \cup A_2.$$

For instance, assume two semi-closed sets $A_i, i \in \{1, 2\}$, in $(\mathbb{Z}^2, \kappa^2)$, where

$$\begin{Bmatrix} A_1 := \{p_1 = (0,0), p_2 = (1,1)\} \text{ and} \\ A_2 := \{p_3 = (1,-1)\} \text{ (see Figure 2(1)(a)).} \end{Bmatrix}$$

Then it is clear that both A_1 and A_2 are semi-closed sets in $(\mathbb{Z}^2, \kappa^2)$ (see Figure 2(1)(a)) and we obtain (see Figure 2(1)(b))

$$Cl(A_1) = N_8(p_2) \text{ and } Cl(A_2) = N_8(p_3),$$

i.e.,

$$Cl(A_1) \cup Cl(A_2) = N_8(p_2) \cup N_8(p_3). \tag{4.1}$$

Owing to (4.1), we see that there is a point $c := (1, 0) \notin A_1 \cup A_2$ (see Figure 2(1)(b)) such that

$$SN_K(c) \subset Cl(A_1) \cup Cl(A_2).$$

Hence we have $Int(Cl(A_1 \cup A_2)) \not\subseteq A_1 \cup A_2$, which means that $A_1 \cup A_2$ is not semi-closed in $(\mathbb{Z}^2, \kappa^2)$. $\square$

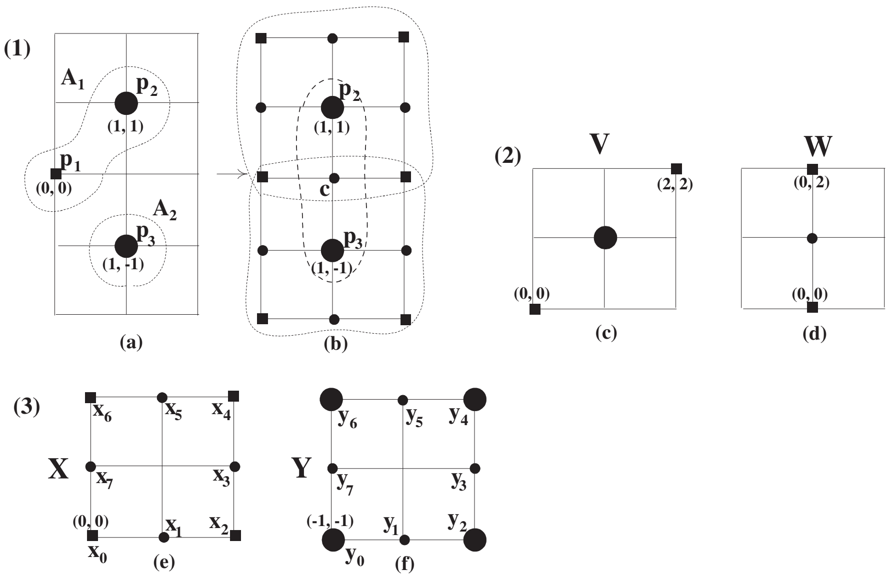

Figure 2: Configuration of a non-semi-closed set of $A_1 \cup A_2$ in $(\mathbb{Z}^2, \kappa^2)$ related to the proof of Theorem 4.1.

Let us now examine if a homeomorphism between two subspaces (Y, T_Y) and (Z, T_Z) induced by a topological space (X, T) preserves semi-openness or semi-closedness of Y and Z in (X, T).

Theorem 4.2. *Assume two homeomorphic subspaces (Y, T_Y) and (Z, T_Z) induced by a topological space (X, T), where $Y, Z \subset X$. Then, we obtain the following:*

(1) The semi-openness of Y in (X, T) need not imply that of Z in (X, T).

(2) The semi-closedness of Y in (X, T) may not imply that of Z in (X, T).

Proof. (1) To prove the assertions of (1) and (2), we suggest counterexamples (see Figure 2(2)(c) and (d), and Figure 2(3)(e) and (f)), as follows: Consider the following sets in $(\mathbb{Z}^2, \kappa^2)$ (see Figure 2(2)(c) and (d)).

$$\left\{\begin{array}{l} V = \{v_0 = (0,0), v_1 = (1,1), v_2 = (2,2)\}, \text{ and} \\ W = \{w_0 = (0,0), w_1 = (0,1), w_2 = (0,2)\}. \end{array}\right\} \tag{4.2}$$

Even though there is a homeomorphism h from $(V, (\kappa^2)_V)$ to $(W, (\kappa^2)_W)$ defined by $h(v_i) = w_i, i \in [0,2]_{\mathbb{Z}}$, the sets V and W have their own features in $(\mathbb{Z}^2, \kappa^2)$ from the viewpoint of semi-open or semi-closed properties. Namely, while V is semi-open in $(\mathbb{Z}^2, \kappa^2)$, W is not semi-open in $(\mathbb{Z}^2, \kappa^2)$. To be specific,

$$Cl(Int(V)) = Cl(\{v_1\}) = N_8(v_1) \text{ so that } V \subset Cl(Int(V)),$$

which supports the semi-openness of V in $(\mathbb{Z}^2, \kappa^2)$.
Meanwhile,

$$Cl(Int(W)) = Cl(\emptyset) = \emptyset \text{ so that } W \not\subseteq Cl(Int(W)),$$

which implies a non-semi-open set of W in $(\mathbb{Z}^2, \kappa^2)$.

(2) Consider the sets X and Y in Figure 2(3)(e) and (f), where

$$\left\{\begin{array}{l} X := \{x_0 = (0,0), x_1 = (1,0), x_2 = (2,0), x_3 = (2,1), \\ x_4 = (2,2), x_5 = (1,2), x_6 = (0,2), x_7 = (0,1)\} \text{ and} \\ Y := \{y_0 = (-1,-1), y_1 = (0,-1), y_2 = (1,-1), y_3 = (1,0), \\ y_4 = (1,1), y_5 = (0,1), y_6 = (-1,1), y_7 = (-1,0)\}. \end{array}\right\} \tag{4.3}$$

Then it is clear that each of $(X, (\kappa^2)_X)$ and $(Y, (\kappa^2)_Y)$ is a kind of a simple closed K-curve with eight elements in $(\mathbb{Z}^2, \kappa^2)$, say $SC_K^{2,8}$, so that $(X, (\kappa^2)_X)$ is K-homeomorphic to $(Y, (\kappa^2)_Y)$ by using the map $h : X \to Y$ defined by $h(x_i) = y_{i+1(mod\,8)}$. However, while X is semi-closed in $(\mathbb{Z}^2, \kappa^2)$, Y is not semi-closed in $(\mathbb{Z}^2, \kappa^2)$. To be specific, since X is a closed set in $(\mathbb{Z}^2, \kappa^2)$ and it does not contain any open subset in $(\mathbb{Z}^2, \kappa^2)$, we have

$$\begin{Bmatrix} Int(Cl(X)) = Int(X) = \emptyset \subset X \text{ and} \\ N_8(p) \subset Int(Cl(Y)), \text{ where } p = (0,0) \notin Y, \end{Bmatrix}$$

because X is a closed set in $(\mathbb{Z}^2, \kappa^2)$ and

$$Cl(Y) = \bigcup_{i \in \{0,2,4,6\}} N_8(y_i).$$

Hence we have

$$Int(Cl(Y)) = N_8(q) \nsubseteq Y, \text{ where } q = (0,0),$$

which implies the non-semi-closedness of Y in $(\mathbb{Z}^2, \kappa^2)$. $\square$

To study semi-homeomorphic properties of sets in $(\mathbb{Z}^n, \kappa^n)$, let us now recall the following: A map $f : (X, T_1) \to (Y, T_2)$ is said to be semi-continuous if and only if for each $U \in T_2$, $f^{-1}(U) \in SO(X, T_1)$ [32], where $SO(X, T_1)$ means the set of semi-open sets in the given topological space (X, T_1). Furthermore, in some literature, the notion of a semi-homeomorphism was defined by taking two approaches, which are broader than a homeomorphism. One of them is established in 1972 [7] by using the concepts of semi-open sets. However, the present paper will follow the following version which is broader than a homeomorphism.

Definition 4.3. ([8]) A bijection $h : (X, T_1) \to (Y, T_2)$ is said to be a semi-homeomorphism if $h(U) \in SO(Y, T_2)$ for each $U \in SO(X, T_1)$ (or pre-semi-open) and $h^{-1}(V) \in SO(X, T_1)$ for each $V \in SO(Y, T_2)$ (or irresolute or semi-continuous).

In addition, another approach was taken in 1971 [3] in a slightly different way using a bijection, continuity of h, and semi-openness of h. However, in the present paper, we will follow the semi-homeomorphism of Definition 4.3.

Motivated by Theorem 4.2, let us now examine if a semi-homeomorphism between two subspaces (X, T_X) and (Y, T_Y) induced by a topological space (Z, T) supports semi-openness or semi-closedness of them in (Z, T).

Corollary 4.4. *Assume two semi-homeomorphic subspaces (X, T_X) and (Y, T_Y) induced by a topological space (Z, T), where $X, Y \subset Z$. Then, we obtain the following:*
(1) The semi-openness of X in (Z, T) need not imply that of Y in (Z, T).
(2) The semi-closedness of X in (Z, T) may not imply that of Y in (Z, T).

Proof. To prove the assertion, let us first recall that a homeomorphism implies a semi-homeomorphism.
(1) While the two spaces $(V, (\kappa^2)_V)$ and $(W, (\kappa^2)_W)$ of (4.2) (see Figure 2(2)(c)-(d)) are semi-homeomorphic to each other by using the map

$$h : (V, (\kappa^2)_V) \to (W, (\kappa^2)_W) \text{ defined by } h(v_i) = w_i, i \in [0,2]_{\mathbb{Z}},$$

the two sets V and W in $(\mathbb{Z}^2, \kappa^2)$ have their own feature from the viewpoint of semi-topological structures. Namely, as mentioned in Theorem 4.2, while V is semi-open, W is not semi-open in $(\mathbb{Z}^2, \kappa^2)$ (see the sets of (4.2) and Figure 2(2)(c) and (d)).
(2) As stated in Theorem 4.2(2), the space $(X, (\kappa^2)_X)$ is semi-homeomorphic to $(Y, (\kappa^2)_Y)$ of (4.3), the two sets X and Y have their own features in $(\mathbb{Z}^2, \kappa^2)$ (see the sets of (4.3) and Figure 2(3)(e) and (f)). Namely, as referred to in Theorem 4.2, while X is semi-closed, Y is not semi-closed in $(\mathbb{Z}^2, \kappa^2)$. $\square$

5. The semi-topological properties of simple K-paths

This section studies various properties of the semi-topological property of simple K-paths, which will play an important role in studying the semi-T_3-separation axiom in Section 6. Hereinafter, a K-path is assumed to be a non-empty set.

Lemma 5.1. *Assume a K-path P in $(\mathbb{Z}^n, \kappa^n)$ such that $|P| \leq 2$. Then P is semi-closed.*

Proof. Let us consider the two cases, i.e., $|P| = 1$ or $|P| = 2$.
(Case 1) In the case of $|P| = 1$, we can assume $P = (c_0)$ is a singleton in $(\mathbb{Z}^n, \kappa^n)$ consisting of a pure closed, pure open, or mixed point. Then, according to the topological structure of $(\mathbb{Z}^n, \kappa^n)$, it satisfies the property $Int(Cl(P)) \subset P$.
To be specific, in case c_0 is a pure closed or mixed point, since $Int(Cl(P)) = \emptyset$, it is clear that P is semi-closed in $(\mathbb{Z}^n, \kappa^n)$. In case c_0 is a pure open point, since $Int(Cl(P)) = P$, it is clear that P is semi-closed in $(\mathbb{Z}^n, \kappa^n)$.
(Case 2) In the case of $|P| = 2$, according to the topological structure of $P := (c_0, c_1)$ in $(\mathbb{Z}^n, \kappa^n)$, we have the following several cases.
(2-1) c_0 is pure closed and c_1 is pure open (or c_0 is pure open and c_1 is pure closed), or
(2-2) c_0 is pure closed and c_1 is mixed (or c_0 is mixed and c_1 is pure closed), or
(2-3) c_0 is pure open and c_1 is mixed (or c_0 is mixed and c_1 is pure open).
According to these three cases, let us check the set $Int(Cl(P))$. In the case of (2-1), we obtain $Int(Cl(P)) = \{c_1\}$ (or $Int(Cl(P)) = \{c_0\}$) so that $Int(Cl(P)) \subset P$.
In the case of (2-2), we have $Int(Cl(P)) = \emptyset$ (or $Int(Cl(P)) = \emptyset$) so that $Int(Cl(P)) \subset P$.
In the case of (2-3), we have $Int(Cl(P)) = \{c_0\}$ (or $Int(Cl(P)) = \{c_1\}$) so that $Int(Cl(P)) \subset P$.
In view of Cases 1 and 2, we see the semi-closedness of P in $(\mathbb{Z}^n, \kappa^n)$. □

Unlike Lemma 5.1, we obtain the following:

Remark 5.2. Assume a K-path $P = (c_0, c_1)$ in $(\mathbb{Z}^n, \kappa^n)$ such that P contains a pure open point. Then P is semi-open in $(\mathbb{Z}^n, \kappa^n)$.

Unlike the case of a K-path P in $(\mathbb{Z}^n, \kappa^n)$ with $|P| \leq 2$ in Lemma 5.1 and Remark 5.2, let us now consider the case $|P| \geq 3$. Then we can observe that semi-topological properties of P depend on the situation, as follows:

Lemma 5.3. *Assume a simple K-path $P = (c_0, c_1, c_2)$ in $(\mathbb{Z}^n, \kappa^n)$.*
In case each of c_0 and c_2 is a pure open point, c_1 is a pure closed point or a mixed point, and there is another mixed point $c \in \mathbb{Z}^n \setminus P$ such that $c_0, c_2 \in SN_K(c) \subset Cl(\{c_0\}) \cup Cl(\{c_2\})$, P is not semi-closed.

Proof. Since $Cl(\{c_i\}) = N_{3^n-1}(c_i), i \in \{0, 2\}$, by the hypothesis, assume a mixed point $c \in \mathbb{Z}^n \setminus P$ such that $c_0, c_2 \in SN_K(c) \subset Cl(\{c_0\}) \cup Cl(\{c_2\})$ (see Figure 3(1) in the 2-dimensional case and see Figure 4 in the 3-dimensional case). Then, we have (in detail, see Example 5.4 below)

$$Cl(P) = N_{3^n-1}(c_0) \cup N_{3^n-1}(c_2) \text{ and } SN_K(c) \subset Cl(P). \tag{5.1}$$

Hence, owing to the property of $SN_K(c) \subset Int(Cl(P))$ of (5.1), we obtain $Int(Cl(P)) \not\subset P$, which implies the non-semi-closedness of P. □

Example 5.4. (1) Let us consider the simple K-path $P = (c_0, c_1, c_2)$ in $(\mathbb{Z}^2, \kappa^2)$ such that each of $c_0 = (1, -1)$, $c_1 = (0, 0)$, and $c_2 = (1, 1)$ (see Figure 3(1)(a)).

$$\left\{\begin{array}{l} Cl(P) = N_8(c_0) \cup N_8(c_2) \text{ (see Figure 3(1)(a) and (b))} \\ \text{so that there is an element } c = (1, 0) \in Cl(P) \\ \text{such that } c \notin P \text{ and } SN_K(c) \subset Cl(P). \end{array}\right\}$$

Hence, we have $SN_K(c) = \{c_0, c, c_2\} \subset Int(Cl(P))$ (see Figure 3(1)(b) and (c)), which implies that $Int(Cl(P)) \not\subseteq P$, i.e., P is not semi-closed.

(2) Assume the K-path $A = \{x_0 = (1,1), x_1 = (2,2), x_3 = (3,2)\}$(see Figure 3(2)(a)). Then we obtain $P \not\subseteq Cl(Int(P))$ (see Figure 3(2)(a)-(b)), which implies the non-semi-openness of P.

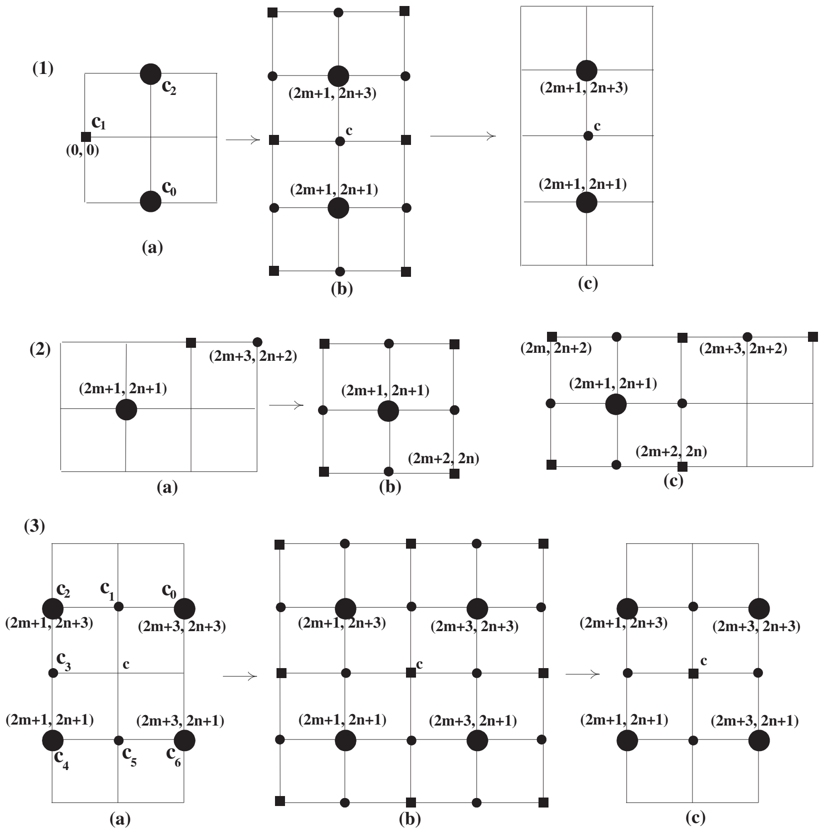

Figure 3: (1) The objects of (a)-(c) are related to the proof of the semi-closedness of a simple K-path in $(\mathbb{Z}^2, \kappa^2)$ stated in Lemmas 5.3 and 5.5, Example 5.4, and Theorem 5.7.
(2) Configuration of the non-semi-openness of the given simple K-path A in Example 5.4(2)(see also Lemma 5.3). In detail, the object of (b) is $Cl(Int(A))$ and the object of (c) is $Cl(A)$.
(3) The objects of (a)-(c) are related to the proof of the semi-closedness of a simple K-path in $(\mathbb{Z}^2, \kappa^2)$ stated in Lemma 5.6 and Theorem 5.7. Namely, (a) Assume A as a simple K-path with seven elements in $(\mathbb{Z}^2, \kappa^2)$. (b) $Cl(A)$ in $(\mathbb{Z}^2, \kappa^2)$. (c) Given A in (a), configuration of $Int(Cl(A))$ in $(\mathbb{Z}^2, \kappa^2)$ showing that $Int(Cl(A)) \not\subseteq A$ owing to the set $SN_K(p)$, where $p = (2m+2, 2n+2)$.

As a generalization of Lemma 5.3, we have the following.

Lemma 5.5. *Assume a simple K-path* $P = (c_0, c_1, \cdots, c_{l-1})$ *in* $(\mathbb{Z}^n, \kappa^n)$*. In case P has the subsequence* (c'_1, c'_2, c'_3) *of P such that each of* c'_1 *and* c'_3 *is a pure open point,* c'_2 *is a pure closed point, and there is a mixed point* $c \in \mathbb{Z}^n \setminus P$ *such that* $c'_1, c'_3 \in SN_K(c) \subset Cl(\{c'_1\}) \cup Cl(\{c'_3\})$*, P is not semi-closed in* $(\mathbb{Z}^n, \kappa^n)$*.*

Proof. With the hypothesis, let us prove the assertion.

Consider $Cl(P)$ (see Figure 3(1)(a)-(c)). Then, by hypothesis, there is a point $c \in N_{3^n-1}(c'_1) \cap N_{3^n-1}(c'_3)$ such that $SN_K(c) \subset N_{3n-1}(c'_1) \cup N_{3n-1}(c'_3)$, $c \in \mathbb{Z}^n \setminus P$, and $c \in (\mathbb{Z}^n)_m$. Thus, by Lemma 5.3, we obtain that $Int(Cl(P)) \not\subseteq P$. $\square$

Lemma 5.6. *Assume a simple K-path $P = (c_0, c_1, \cdots, c_{l-1})$ in $(\mathbb{Z}^n, \kappa^n)$. If P has the subsequence $X_1 := (c'_1, c'_2, c'_3, c'_4)$ such that*

$$\begin{Bmatrix} X_1 \subset (\mathbb{Z}^n)_o \text{ and} \\ X_1 \subset N_{3^n-1}(c), c \in \mathbb{Z}^n \setminus P. \end{Bmatrix}$$

Then P is not semi-closed in $(\mathbb{Z}^n, \kappa^n)$.

Before proving the assertion, it can be helpful to recognize the subsequence X_1 of P with an example as follows: In Figure 3(3)(a), given a K-path with seven elements $(c_i)_{i\in[0,6]_\mathbb{Z}}$, we can take the subsequence $X_1 = (c_0, c_2, c_4, c_6) \subset P$ satisfying the condition of this lemma.

Proof. With the hypothesis, consider $Cl(P)$(see Figure 3(3)(a)-(c)). Then there is a point $c \in \bigcap_{i\in M} N_{3^n-1}(c'_i)$ such that $c'_i \in X_1 \subset P, i \in M = [1,4]_\mathbb{Z}$, and

$$SN_K(p) \subset \bigcup_{i\in M} N_{3^n-1}(c'_i) \subset Cl(P) \text{ and } c \in \mathbb{Z}^n \setminus P \text{ and } c \in (\mathbb{Z}^n)_e.$$

Thus we see that $Int(Cl(P)) \not\subseteq P$ because $SN_K(p) \not\subseteq P$. □

By Lemmas 5.5 and 5.6, we have the following:

Theorem 5.7. *Assume a simple K-path $P = (c_0, c_1, \cdots, c_{l-1})$ in $(\mathbb{Z}^n, \kappa^n)$ such that*
(1) P does not have the subset $\{c'_1, c'_2, c'_3\}$ such that
(1-1) each of c'_1 and c'_3 is a pure open point and c'_2 is a pure closed point or a mixed point and
(1-2) for some mixed point $c \in \mathbb{Z}^n \setminus P$, the two points c'_1 and c'_3 satisfy the property

$$\{c'_1, c'_3\} \subset SN_K(c) \subset Cl(\{c'_1\}) \cup Cl(\{c'_3\}),$$

and
(2) P does not have the subsequence $Y_1 := (c'_1, c'_2, c'_3, c'_4)$ of P such that $Y_1 \subset (\mathbb{Z}^n)_o$ and $Y_1 \subset N_{3^n-1}(c), c \in \mathbb{Z}^n \setminus P$. Then P is semi-closed.

Before proving the assertion, we strongly need to recall the hypothesis of Lemma 5.5 and 5.6. Without the hypothesis, as shown in Figure 2(3)(f), put $Y_2 = Y \setminus \{y_7\}$. Then the set Y_2 cannot be semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Proof. Assume a simple K-path with l elements in $\mathbb{Z}^n$, say $P = (c_0, c_1, \cdots, c_{l-1})$. Then, depending on the situation, P consists of pure closed, pure open, or mixed points. Using mathematical induction, we will prove the assertion.
(Case 1) Assume $|P| = 1$. Then, it is obvious that P satisfies the conditions (1) and (2). Owing to the n-dimensional cases of the properties of (2.3) and (2.4), it is clear that P is semi-closed in $(\mathbb{Z}^n, \kappa^n)$. More precisely, assume $P = \{c_0\}$. According to $c_0 \in (\mathbb{Z}^n)_o$, $c_0 \in (\mathbb{Z}^n)_e$, or $c_0 \in (\mathbb{Z}^n)_m$, we obtain the following:
In the case of $c_0 \in (\mathbb{Z}^n)_o$, we have $Int(Cl(\{c_0\})) = \{c_0\} \subset \{c_0\}$.
In the case of $c_0 \in (\mathbb{Z}^n)_e$, we obtain $Int(Cl(\{c_0\})) = \emptyset \subset \{c_0\}$.
In the case of $c_0 \in (\mathbb{Z}^n)_m$, we have $Int(Cl(\{c_0\})) = \emptyset \subset \{c_0\}$.
In view of these three case, we now complete the proof.
(Case 2) For any l, with the hypothesis, assume that $A = (c_0, c_1, \cdots, c_{l-2}) \subset P$ is semi-closed in $(\mathbb{Z}^n, \kappa^n)$, $2 \leq l \in \mathbb{N}$. Then we now prove that $P = (c_0, c_1, \cdots, c_{l-2}, c_{l-1})$ is semi-closed. Owing to the n-dimensional cases of properties of (2.3) and (2.4), we first examine the semi-closedness of the subset of P consisting of the consecutive two elements c_{l-2} and c_{l-1} in P according to the topological properties of the points c_{l-2} and c_{l-1}. In particular, in case the subset $\{c_{l-2}, c_{l-1}\}$ consists of mixed points, it is not connected in $(\mathbb{Z}^n, \kappa^n)$. Hence it suffices to investigate the other several cases. Namely, take the set $\{c_{l-2}, c_{l-1}\} \subset P$ according to the several

cases depending on the situation of P, as follows:
(Case 2-1) Assume the case that c_{l-2} is pure open and c_{l-1} is pure closed. Then we obtain

$$Int(Cl(\{c_{l-2}, c_{l-1}\})) = \{c_{l-2}\} \subset \{c_{l-2}, c_{l-1}\},$$

because $Cl(\{c_{l-2}, c_{l-1}\}) = N_{3^n-1}(c_{l-2})$ in $(\mathbb{Z}^n, \kappa^n)$. Thus we see that the set $\{c_{l-2}, c_{l-1}\}$ is semi-closed. Based on this approach, denote the set $(c_0, c_1, \cdots, c_{l-2})$ by A. Then, for $P = A \cup \{c_{l-1}\}$, we have

$$\begin{Bmatrix} Int(Cl(P)) = Int(Cl(A \cup \{c_{l-1}\})) \\ = Int(Cl(A) \cup Cl(\{c_{l-1}\})) = Int(Cl(A)) \subset A \subset P, \end{Bmatrix}$$

which implies the property $Int(Cl(P)) \subset P$.
(Case 2-2) Assume the case that c_{l-2} is pure closed and c_{l-1} is pure open (this case is related to the hypothesis, see also Lemma 5.5). Then we obtain

$$Int(Cl(\{c_{l-2}, c_{l-1}\})) = \{c_{l-1}\} \subset \{c_{l-2}, c_{l-1}\},$$

because $Cl(\{c_{l-2}, c_{l-1}\}) = N_{3^n-1}(c_{l-1})$ in $(\mathbb{Z}^n, \kappa^n)$. Thus we see that the set $\{c_{l-2}, c_{l-1}\}$ is semi-closed. Based on this approach, denote the set $(c_0, c_1, \cdots, c_{l-2})$ by A. Then, for $P = A \cup \{c_{l-1}\}$, we have

$$\begin{Bmatrix} Int(Cl(A \cup \{c_{l-1}\})) \\ = Int(Cl(A) \cup Cl(\{c_{l-1}\})) = Int(Cl(P)) \subset P, \end{Bmatrix}$$

which implies the property $Int(Cl(P)) \subset P$.
(Case 2-3) Assume the case that c_{l-2} is pure open and c_{l-1} is mixed.
Then we obtain

$$Int(Cl(\{c_{l-2}, c_{l-1}\})) = \{c_{l-2}\} \subset \{c_{l-2}, c_{l-1}\},$$

because $Cl(\{c_{l-2}, c_{l-1}\}) = N_{3^n-1}(c_{l-2})$ in $(\mathbb{Z}^n, \kappa^n)$. Thus we see that the set $\{c_{l-2}, c_{l-1}\}$ is semi-closed. Based on this feature, put $A = (c_0, c_1, \cdots, c_{l-2})$. Then, for $P = A \cup \{c_{l-1}\}$, we have

$$\begin{Bmatrix} Int(Cl(P)) = Int(Cl(A \cup \{c_{l-1}\})) \\ = Int(Cl(A) \cup Cl(\{c_{l-1}\})) = Int(Cl(A)) \subset A \subset P, \end{Bmatrix}$$

which implies the property $Int(Cl(P)) \subset P$.
(Case 2-4) Assume the case that c_{l-2} is mixed and c_{l-1} is pure open (this case is related to the hypothesis, see also Lemma 5.6).
Then we obtain

$$Int(Cl(\{c_{l-2}, c_{l-1}\})) = \{c_{l-1}\} \subset \{c_{l-2}, c_{l-1}\},$$

because $Cl(\{c_{l-2}, c_{l-1}\}) = N_{3^n-1}(c_{l-1})$. Thus we see that the set $\{c_{l-2}, c_{l-1}\}$ is semi-closed. Based on this feature, put $A = (c_0, c_1, \cdots, c_{l-2})$. Then, for $P = A \cup \{c_{l-1}\}$, we have

$$\begin{Bmatrix} Int(Cl(A \cup \{c_{l-1}\})) \\ = Int(Cl(A) \cup Cl(\{c_{l-1}\})) = Int(Cl(P)) \subset P, \end{Bmatrix}$$

which implies the property $Int(Cl(P)) \subset P$.
(Case 2-5) Assume the case that c_{l-2} is pure closed and c_{l-1} is mixed. Then, we obtain

$$Int(Cl(\{c_{l-2}, c_{l-1}\})) = \emptyset \subset \{c_{l-2}, c_{l-1}\},$$

because $Cl(\{c_{l-2}, c_{l-1}\}) = Cl(\{c_{l-1}\})$ and $Int(Cl(\{c_{l-1}\})) = \emptyset$, which implies that the set $\{c_{l-2}, c_{l-1}\}$ is semi-closed. Based on the set $A = (c_0, c_1, \cdots, c_{l-2})$, for $P = A \cup \{c_{l-1}\}$, we have

$$\begin{Bmatrix} Int(Cl(P)) = Int(Cl(A \cup \{c_{l-1}\})) \\ = Int(Cl(A) \cup Cl(\{c_{l-1}\})) \subset Int(Cl(A)) \subset A \subset P, \end{Bmatrix}$$

which implies the property $Int(Cl(P)) \subset P$.
(Case 2-6) Assume the case that c_{l-2} is mixed and c_{l-1} is pure closed. Then we obtain

$$Int(Cl(\{c_{l-2}, c_{l-1}\})) = \emptyset \subset \{c_{l-2}, c_{l-1}\},$$

because $Cl(\{c_{l-2}, c_{l-1}\}) = Cl(\{c_{l-1}\})$ and $Int(Cl(\{c_{l-1}\})) = \emptyset$, which implies that the set $\{c_{l-2}, c_{l-1}\}$ is semi-closed. After considering the set $A = (c_0, c_1, \cdots, c_{l-2})$, we have

$$\begin{Bmatrix} Int(Cl(P)) = Int(Cl(A \cup \{c_{l-1}\})) \\ = Int(Cl(A) \cup Cl(\{c_{l-1}\})) \subset P, \end{Bmatrix}$$

which implies the property $Int(Cl(P)) \subset P$.

Based on these two cases above, the simple K-path P is proved to be semi-closed in $(\mathbb{Z}^n, \kappa^n)$. □

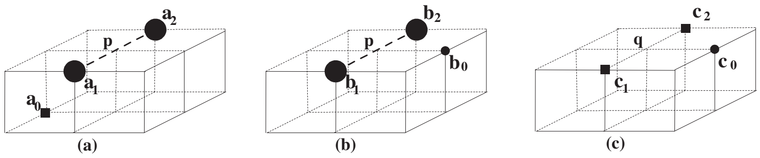

Figure 4: The simple K-paths in (a)-(c) are related to the proof of the semi-closedness of a simple K-path in $(\mathbb{Z}^3, \kappa^3)$ stated in Lemma 5.3 and Theorem 5.7.

Example 5.8. Assume a simple K-path A in $(\mathbb{Z}^2, \kappa^2)$ satisfying the hypothesis of Theorem 5.7. Then we show the semi-closedness of A in $(\mathbb{Z}^2, \kappa^2)$. To support this finding, consider the simple K-path $A = (x_0 = (0,0), x_1 = (1,1), x_2 = (1,2), x_3 = (1,3), x_4 = (2,4), x_5 = (3,4))$ as a sequence with six elements in $(\mathbb{Z}^2, \kappa^2)$. Then we observe that $Int(A) = \{x_1, x_2, x_3\}$ and $Cl(A)$ is equal to the set $N_8(x_1) \cup N_8(x_3) \cup \{x_5, (4,4)\}$. Based on this fact, we can confirm the semi-closedness of A in $(\mathbb{Z}^2, \kappa^2)$ because

$$\begin{Bmatrix} Int(Cl(A)) = \\ Int(N_8(x_1) \cup N_8(x_3) \cup \{x_5, (4,4)\}) \\ = \{x_1, x_2, x_3\} \subset A. \end{Bmatrix}$$

6. The semi-T_3-separation axiom of Khalimsky topological spaces

This section proves that $(\mathbb{Z}^n, \kappa^n)$ satisfies the semi-T_3-separation axiom. Thus we need some efficient tools to support this work. Namely, given a topological space (X, T) and a set $A(\subset X)$, to examine if A is semi-open or semi-closed in (X, T), the paper [40] proposed some results related to this need, which can be very interesting (see Theorems 2.2 and 2.3 in [40]). Although the ideas are significant, the author used some unclear notations, concepts, and redundance associated with these assertions and their proofs. For instance, see the equivalent condition of the semi-closedness in the last line of the Introduction in [40], the condition "if " which should be written by "if and only if", Lemma 2.2, and $(3^n - 1)$-adjacency instead of the K-adjacency used in the proof of Theorem 2.2 of [40]. Besides, it has a mistake in Example 2.4(1). In detail, in Example 2.4(1), even though the author claimed $\{2n + 1\}$ is not semi-closed in $(\mathbb{Z}, \kappa)$, it should be written as semi-closed (see Lemma 3.3(1) in the present paper). Thus let us first make some errors occurred in [40] fixed and improved. Then we can use the ideas in [40]. For our purpose, let us recall some notations and concepts as follows: In $(\mathbb{Z}^n, \kappa^n)$, given a set $A \subset \mathbb{Z}^n$, we will use the following notation as in [40]

$$A_{op} := \{x \,|\, x \text{ is a pure open point in } A\}. \tag{6.1}$$

Besides, owing to the topological structure of $(\mathbb{Z}^n, \kappa^n)$ as a product topology induced by $(\mathbb{Z}, \kappa)$, where $(\mathbb{Z}, \kappa)$ generated by the set $\mathcal{B} = \{\{2n+1\}, [2n-1, 2n+1]_{\mathbb{Z}} \mid n \in \mathbb{Z}\}$ as a base as mentioned in Section 2, we obviously have the following:

Remark 6.1. In $(\mathbb{Z}^n, \kappa^n)$, we have the following: (1) For $x, y \in \mathbb{Z}^n$, $x \in SN_K(y)$ if and only if $y \in Cl(\{x\})$, i.e., $y \in Cl_K(\{x\})$ [29](see the properties of (2.3) and (2.4) in the present paper).

(2) If A is an open set in $(\mathbb{Z}^n, \kappa^n)$, then there is a pure open point $x \in A$ (see the property of (2.3)). However, the converse does not hold.

(3) The set A_{op} of (6.1) is an open set in $(\mathbb{Z}^n, \kappa^n)$.

Proof. (1) The proof is straightforward.

(2) Owing to the product topology $(\mathbb{Z}^n, \kappa^n)$ induced by $(\mathbb{Z}, \kappa)$ generated by the above set $\mathcal{B}$ as a base, the proof is completed.

(3) Based on the product topological structure of $(\mathbb{Z}^n, \kappa^n)$ induced by $(\mathbb{Z}, \kappa)$ generated by the above set $\mathcal{B}$ as a base, the singleton $\{x\}$ consisting of the pure open point $x \in \mathbb{Z}^n$ is equal to $SN_K(x)$. Hence the set $A_{op} = \bigcup_{x \in A_{op}} \{x\}$ is an open set in $(\mathbb{Z}^n, \kappa^n)$. □

Let us now give the original version of Theorem 2.2 of [40] and its improved proof, as follows:

Lemma 6.2. ([40]) *In $(\mathbb{Z}^n, \kappa^n)$, a non-empty set $A (\subset \mathbb{Z}^n)$ is semi-open if and only if for each $x \in A$, $SN_K(x) \cap A_{op} \neq \emptyset$.*

Before proving the assertion, in the case of $A = \emptyset$, the proof is straightforward.

Proof. ($\Rightarrow$) According to the choice of a point $x \in A$, we can consider the following two cases.
(Case 1) Assume that $x (\in A)$ is a pure open point. From the hypothesis, we have $x \in A \subset Cl(Int(A))$ so that we obtain

$$SN_K(x) \cap Int(A) \neq \emptyset. \tag{6.2}$$

Since $SN_K(x) = \{x\}$, we obtain $x \in Int(A)$ and further, $x \in A_{op}$. Hence, owing to (6.2), we have $SN_K(x) \cap A_{op} \neq \emptyset$.

(Case 2) Assume that $x (\in A)$ is not a pure open point. Owing to the hypothesis, we have $x \in A \subset Cl(Int(A))$ so that we obtain $SN_K(x) \cap Int(A) \neq \emptyset$ as mentioned in (6.2). Since $SN_K(x) \cap Int(A)$ is an open set in $(\mathbb{Z}^n, \kappa^n)$, by Remark 6.1(2), we now take a pure open point z in $(\mathbb{Z}^n, \kappa^n)$ such that

$$z \in SN_K(x) \cap Int(A), \text{ i.e., } SN_K(z) = \{z\} \subset SN_K(x) \cap Int(A). \tag{6.3}$$

By the property of (6.3), since $z \in Int(A)$ because of $SN_K(z) = \{z\}$, we have $z \in A_{op}$ so that $z \in SN_K(x) \cap A_{op} \neq \emptyset$. In addition, we see that the point z is indeed K-adjacent to x.

($\Leftarrow$) According to the choice of a point $x \in A$, we can consider the following two cases.
(Case 1) For a point $x \in A$, assume that x is a pure open point in $(\mathbb{Z}^n, \kappa^n)$. Since $\{x\} = SN_K(x)$, owing to the hypothesis of $SN_K(x) \cap A_{op} \neq \emptyset$, we have $x \in A_{op}$, i.e., $\{x\} \cap A_{op} \neq \emptyset$. Furthermore, owing to the identity $SN_K(x) = \{x\}$, by Remark 6.1(3), it is clear that

$$x \in A_{op} \Rightarrow \{x\} \subset Int(A) \Rightarrow x \in Cl(Int(A)). \tag{6.4}$$

(Case 2) For a point $x \in A$, assume that x is not a pure open point in $(\mathbb{Z}^n, \kappa^n)$. Owing to the hypothesis, since $SN_K(x) \cap A_{op} \neq \emptyset$, by Remark 6.1(2) and (3), there is a pure open point z in $(\mathbb{Z}^n, \kappa^n)$ such that $z \in SN_K(x) \cap A_{op}$. Hence we get $z \in SN_K(x)$, by Remark 6.1(1), we have

$$x \in Cl(\{z\}) \subset Cl(Int(A)) \Rightarrow x \in Cl(Int(A)). \tag{6.5}$$

By (6.4) and (6.5), the proof is completed. □

Owing to the notion of semi-closedness, using Lemma 6.2, we obtain the following:

Lemma 6.3. ([40]) *In $(\mathbb{Z}^n, \kappa^n)$, $A(\subset \mathbb{Z}^n)$ is semi-closed if and only if each $x \in \mathbb{Z}^n \setminus A$, $SN_K(x) \cap (\mathbb{Z}^n \setminus A)_{op} \neq \emptyset$.*

To support Lemmas 6.2 and 6.3, we have the following examples as a generalization of Lemma 3.1.

Example 6.4. Under $(\mathbb{Z}^n, \kappa^n)$, we have the following:
(1) For a point $p \in (\mathbb{Z}^n)_e$, the set $\mathbb{Z}^n \setminus \{p\}$ is not semi-closed but semi-open.
(2) For a point $p \in (\mathbb{Z}^n)_o$, the set $\mathbb{Z}^n \setminus \{p\}$ is both semi-closed and semi-open.
(3) For a point $p \in (\mathbb{Z}^n)_m$, the set $\mathbb{Z}^n \setminus \{p\}$ is not semi-closed but semi-open.

Let us now support the assertion of Example 6.4 more precisely.

(1) For a point $p \in (\mathbb{Z}^n)_e$ let $\mathbb{Z}^n \setminus \{p\} = F$. Since $Cl(F) = \mathbb{Z}^n$, $Int(Cl(F)) = \mathbb{Z}^n \subsetneq F$, which implies that F is not semi-closed in $(\mathbb{Z}^n, \kappa^n)$. In addition, using Lemmas 6.2 and 6.3, we find out that $\{p\}$ is not semi-closed but semi-open.

(2) For a point $p \in (\mathbb{Z}^n)_o$ let $\mathbb{Z}^n \setminus \{p\} = A$. Then, since $Cl(A) = A$, $Int(Cl(A)) \subset A$, which implies that A is semi-closed in $(\mathbb{Z}^n, \kappa^n)$. In addition, when using Lemma 6.2, we find out that $\{p\}$ is semi-closed so that $\{p\}^c$ is semi-open.

(3) For a point $p \in (\mathbb{Z}^n)_m$ let $\mathbb{Z}^n \setminus \{p\} = B$. Then, we prove not to hold the property $Int(Cl(B)) \subset B$. For instance, under $(\mathbb{Z}^2, \kappa^2)$, for a point $p = (x, y) \in (\mathbb{Z}^2)_m$ let $\mathbb{Z}^2 \setminus \{p\} = B$. Then $Cl(B) = \mathbb{Z}^2$ so that $Int(Cl(B)) \not\subseteq B$. In addition, by Lemmas 6.2 and 6.3, we find out that $\{p\}$ is not semi-open but semi-closed.

Theorem 6.5. *In $(\mathbb{Z}^n, \kappa^n)$, for any point $p \in \mathbb{Z}^n$, $SN_K(p)$ is not closed but semi-closed.*

Proof. It is clear that $SN_K(p)$ is not a closed set in $(\mathbb{Z}^n, \kappa^n)$, let us prove the semi-closedness of $SN_K(p)$ for any point $p \in \mathbb{Z}^n$. Indeed, we will prove the identity $Int(Cl(SN_K(p))) = SN_K(p)$. To be specific, for an arbitrary element $x \in Int(Cl(SN_K(p))) \neq \emptyset$, we have $SN_K(x)$ in $(\mathbb{Z}^n, \kappa^n)$ such that $SN_K(x) \subset Cl(SN_K(p))$, which implies that $x \in SN_K(p)$.

Conversely, take an arbitrary element $x \in SN_K(p)$. Then, it is clear $x \in Cl(SN_K(p))$. Owing to the property of $SN_K(p)$, we see $x \in Int(Cl(SN_K(p)))$. $\square$

Unlike Theorem 5.7, as a generalization of Lemma 5.3 and Lemma 6.2, we have the following:

Remark 6.6. A simple K-path in $(\mathbb{Z}^n, \kappa^n)$ may not be semi-open. To be specific, given a simple K-path P, if there is a point $x \in P$ such that $SN_K(x) \cap P_{op} = \emptyset$, then by Lemma 6.2, P is not semi-open.

Based on the above properties, let us now investigate some topological properties of $(\mathbb{Z}^n, \kappa^n)$ with respect to the s-T_3-separation axiom and the semi-T_3-separation axiom, and so on. The paper [42] defined the notion of s-regular as follows: We say that a topological space (X, T) is s-regular if for each closed subset F of X and point $x \in F^c$, there are $U, V \in SO(X, T)$ such that $F \subset U$ and $x \in V$ and $U \cap V = \emptyset$. The paper [42] proved that this s-regularity has the finite product property. Hence the paper [5] proved that s-regular spaces with the T_0-separation axiom also has the finite product property. Since $(\mathbb{Z}, \kappa)$ satisfies both the T_0-separation axiom and the s-regularity to establish the semi-T_2-separation axiom of $(\mathbb{Z}, \kappa)$, we can conclude that $(\mathbb{Z}^n, \kappa^n)$ is an s-regular space [5]. Let us now generalize the notion of s-regularity instead of the Noiri's approach, as follows:

Definition 6.7. ([10]) A topological space (X, T) is said to be semi-regular if for each semi-closed set C and each $x \notin C$ there are two semi-open sets $SO(C)$ and $SO(q)$ in $(\mathbb{Z}^n, \kappa^n)$ such that $SO(C) \cap SO(q) = \emptyset$, where $SO(C)$ and $SO(q)$ mean semi-open subsets of containing C and q, respectively.

In view of Definition 6.7, after comparing between the s-regularity of Noiri and the semi-regularity of Dorsett (see Definition 6.7), it is clear that the latter is broader than the former. Thus, to evade from some confusion and make a distinction between them, for our purpose hereinafter, the latter is called semi-regularity.

Definition 6.8. ([34]) A topological space (X, T) is said to be a semi-T_1-space if any distinct points $p, q \in X$ have their own semi-open sets $SO(p)$ and $SO(q)$ such that $q \notin SO(p)$ and $p \notin SO(q)$.

Definition 6.9. ([34]) A topological space (X, T) is said to be a semi-T_2-space if any distinct points $p, q \in X$ have their own semi-open sets $SO(p)$ and $SO(q)$ such that $SO(p) \cap SO(q) = \emptyset$.

Based on the s-regularity and the semi-regularity, we now define the following:

Definition 6.10. (1) We say that a topological space (X, T) is an s-T_3-space if it is both a semi-T_1-space and an s-regular space.

(2) We say that a topological space (X, T) is a semi-T_3-space if it is both a semi-T_1-space and a semi-regular space.

Definition 6.10 enable us to get the following:

Remark 6.11. (1) An s-T_3-space is more restrictive than a semi-T_3-space.
(2) $(\mathbb{Z}^n, \kappa^n)$ is an s-T_3-space [5, 40].

Motivated by Remark 6.11, we need to prove the following:

Theorem 6.12. *The n-dimensional K-topological space $(\mathbb{Z}^n, \kappa^n)$ is a semi-T_3-space.*

Proof. In $(\mathbb{Z}^n, \kappa^n)$, let $C(\neq \emptyset)$ be semi-closed and $x \notin C$. According to the choice of the point x, we can consider the following three cases.
(Case 1) Assume that x is pure open point. Since $SN_K(x) = \{x\}$, we have the two sets $U := \mathbb{Z}^n \setminus \{x\}$ and $V := \{x\}$, then we find out that $U, V \in SO(\mathbb{Z}^n, \kappa^n)$ and by Example 6.4, $C \subset U, x \in V, U \cap V = \emptyset$. Hence $(\mathbb{Z}^n, \kappa^n)$ is semi-regular. For instance, see the cases in Figure 5(a) as a one dimensional case and Figure 6(a) as a two dimensional case.
(Case 2) Assume that x is a pure closed point. Since $x \notin C$ and C^c is semi-open, by Lemma 6.2, for each $x \in C^c$, we obtain the following property

$$SN_K(x) \cap (C^c)_{op} \neq \emptyset. \tag{6.6}$$

Namely, we may take $p \in SN_K(x) \cap (C^c)_{op}$ so that $p \in (\mathbb{Z}^n)_o$ and there is a simple K-path $\{x, p\}$ because $p \in SN_K(x)$. Besides, by Lemmas 5.1 and 5.2, the set $\{x, p\}$ is both semi-open and semi-closed (see also Lemma 5.1). Then, consider the set $\mathbb{Z}^n \setminus \{x, p\}$ so that this set is semi-open because $\{x, p\}$ is semi-closed (see Lemmas 5.1 and 6.2) and $C \subset \mathbb{Z}^n \setminus \{x, p\}$. Hence, after putting $U := \mathbb{Z}^n \setminus \{x, p\}, V := \{x, p\}$, we have $U, V \in SO(\mathbb{Z}^n, \kappa^n)$ such that $U \cap V = \emptyset$. For instance, see the cases in Figure 5(b) as a one dimensional case and Figure 6(b) as a two dimensional case.
(Case 3) Assume that x is a mixed point. By using a method similar to the proof of (Case 2), the proof is completed, To be specific, assume that x is a mixed point. Since $x \notin C$ and C^c is semi-open, by Lemma 6.2, for each $x \in C^c$, we obtain the following:

$$SN_K(x) \cap (C^c)_{op} \neq \emptyset.$$

Then there is a pure open point $q \in (C^c)_{op}$ such that the set $\{x, q\}$ is a simple K-path in $(\mathbb{Z}^n, \kappa^n)$ because $q \in SN_K(x)$ so that $\{x, q\}$ is also a semi-open set (see Lemma 6.2) containing the point x. Then consider the sets $U := \mathbb{Z}^n \setminus \{x, q\}$ and $V := \{x, q\}$. Indeed, by Lemma 6.2, the sets U and V are semi-open sets containing C and x, respectively. Finally, it is clear that $U \cap V = \emptyset$. For instance, see the case in Figure 6(c) as a two dimensional case.
In view of the above three cases, we now complete the proof that $(\mathbb{Z}^n, \kappa^n)$ is a semi-regular space. □

In the papers [5, 40], it turns out that $(\mathbb{Z}^n, \kappa^n)$ is s-regular. Owing to Theorem 6.12, we obtain the following result because $(\mathbb{Z}^n, \kappa^n)$ does not satisfy the T_1-separation axiom.

Corollary 6.13. *The semi-T_3-separation axiom is not stronger than the T_1-separation axiom.*

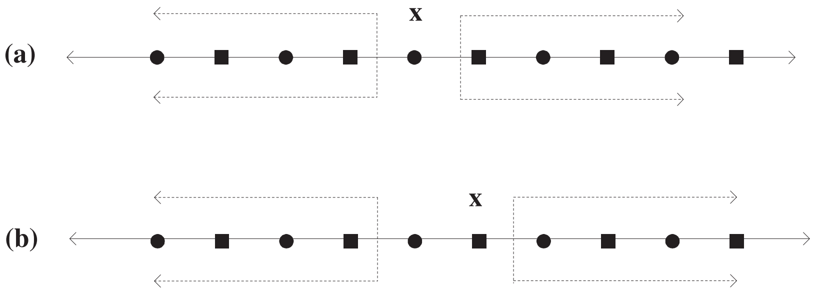

Figure 5: (a) Configuration of x related to the proof of (Case 1) of Theorem 6.12. (b) Configuration of x related to the proof of (Case 2) of Theorem 6.12.

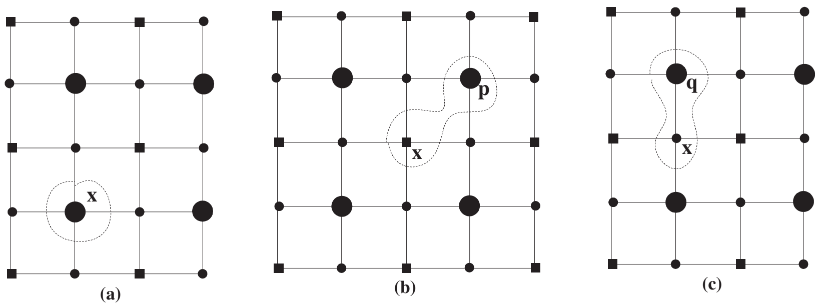

Figure 6: In $(\mathbb{Z}^2, \kappa^2)$, according to the choice of the point $x \in \mathbb{Z}^2$, configuration of the corresponding semi-open sets $SO(x)$. (a) Configuration of $SO(x)$, where x is a pure open point. (b) Configuration of $SO(x)$, where x is a pure closed point. (c) Configuration of $SO(x)$, where x is mixed.

7. Concluding remark and further work

After having studied various properties of the s-T_3-separation axiom and the semi-T_3-separation axiom, we finally proved that $(\mathbb{Z}^n, \kappa^n)$ is a semi-T_3-space. This finding facilitates to the study in the fields of pure and applied topology. As a further work, we can investigate these properties of the infinite K-circle and K-sphere, i.e., one point compactifications of the K-topological line and K-topological plane.

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