Filomat 37:8 (2023), 2531–2538
https://doi.org/10.2298/FIL2308531L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The quasi-Rothberger property of Pixley–Roy hyperspaces

Zuquan Li[a]

[a] *School of Mathematics, Hangzhou Normal University, Hangzhou 311121, P.R.China*

Abstract. Let PR(X) denote the hyperspace of non-empty finite subsets of a topological space X with Pixley–Roy topology. In this paper, we investigate the quasi-Rothberger property in hyperspace PR(X). We prove that for a space X, the followings are equivalent:

(1) PR(X) is quasi-Rothberger;

(2) X satisfies $\mathsf{S}_1(\Pi_{rcf-h}, \Pi_{wrcf-h})$;

(3) X is separable and each co-finite subset of X satisfies $\mathsf{S}_1(\Pi_{pcf-h}, \Pi_{wpcf-h})$;

(4) X is separable and PR(Y) is quasi-Rothberger for each co-finite subset Y of X.

We also characterize the quasi-Menger property and the quasi-Hurewicz property of PR(X). These answer the questions posted in [8].

1. Introduction

Throughout the paper all spaces are assumed to be infinite and T_1. $\mathbb{N}$ denotes the set of natural numbers. ω is the first infinite ordinal.

Let PR(X) be the family of all non-empty finite subsets of a space X. For $A \in \mathrm{PR}(X)$ and an open set $U \subset X$, let

$$[A, U] = \{B \in \mathrm{PR}(X) : A \subset B \subset U\}.$$

The family

$$\{[A, U] : A \in \mathrm{PR}(X), U \text{ is open in } X\}$$

is a base of PR(X) for the *Pixley–Roy topology* [9] on PR(X).

We recall two very known concepts defined in a general form in 1996 by M. Scheepers [10]. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sets of an infinite set X.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{\mathsf{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ such that B_n is a finite subset of A_n for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

G. Di Maio and Lj.D.R. Kočinac [3] introduced the following quasi-version of selection principles stronger than the weakly-version of selection principles:

2020 *Mathematics Subject Classification*. Primary 54B20; Secondary 54D20

Keywords. Pixley–Roy topology, selection principle, quasi-Rothberger, quasi-Menger, quasi-Hurewicz, rcf-network on a hit-family, weakly rcf-network on a hit-family

Received: 29 March 2022; Revised: 10 August 2022; Accepted: 11 August 2022

Communicated by Ljubiša D.R. Kočinac

Email address: hzsdlzq@sina.com (Zuquan Li)

Definition 1.1. ([3, Definition 2.1]) 1. A space X is said to be quasi-Rothberger if for each closed set $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X there is a $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $F \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$.

2. A space X is said to be quasi-Menger if for each closed set $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X there is a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $F \subset \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n}$.

3. A space X is said to be quasi-Hurewicz if for each closed set $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X there is a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that for every nonempty open U of X with $U \cap F \neq \emptyset$, $U \cap (\bigcup \mathcal{V}_n) \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$.

We have the following implications:

Rothberger ⇒ quasi-Rothberger ⇒ weakly Rothberger

There are very few papers which deal with the quasi-Rothberger (resp., quasi-Menger and quasi-Hurewicz) properties. G. Di Maio and Lj.D.R. K0činac [3] pointed that a space X is quasi-Rothberger (resp., quasi-Menger and quasi-Hurewicz) if and only if every closed subspace of X is weakly Rothberger (resp., weakly Menger and weakly Hurewicz) and proved that every hereditarily separable space X is quasi-Rothberger [3, Proposition 2.2]. Z. Li studied the quasi-Rothberger property of linearly ordered spaces [6].

In [8], we investigated the weakly Rothberger property of PR(X) and posted following questions:

Question 1.2. ([8, Question 2.22]) *For a space X, find the collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of X such that:*

PR(X) is quasi-Rothberger if and only if X satisfies $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$;
PR(X) is quasi-Menger if and only if X satisfies $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$;
PR(X) is quasi-Hurewicz if and only if X satisfies $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$.

This paper is organized as follows. In the second section, we obtained some interesting properties of quasi-Rothberger property of PR(X). In the third section, we introduced a new kind of hit-and-miss networks to study the quasi-version of selection principles of PR(X) and characterize the quasi-Rothberger property, the quasi-Menger property and the quasi-Hurewicz property of PR(X).

2. Some properties of PR(X) being quasi-Rothberger

Recall that a space X is said to be *quasi-Lindelöf* [1] if for each closed subset F of X and each cover $\mathcal{U}$ of F by sets open in X, there is a countable set $\{U_n : n \in \mathbb{N}\} \subset \mathcal{U}$ such that $F \subset \overline{\bigcup\{U_n : n \in \mathbb{N}\}}$.

Theorem 2.1. *If PR(X) is quasi-Lindelöf, then X is hereditarily separable.*

Proof. Suppose that F is a subset of X, then $\mathcal{F} = \{\{x\} : x \in F\}$ is a closed subset of PR(X). Let $\mathcal{U} = \{[\{x\}, X] : x \in F\}$, then $\mathcal{U}$ is a cover of $\mathcal{F}$ open in PR(X). There exists $[\{x_n\}, X] \in \mathcal{U}$ for each $n \in \mathbb{N}$ such that

$$\mathcal{F} \subset \overline{\bigcup_{n\in\mathbb{N}} [\{x_n\}, X]}.$$

We prove that $\{x_n : n \in \mathbb{N}\}$ is a dense subset of F. In fact, for each open subset V of X with $F \cap V \neq \emptyset$, pick $y \in F \cap V$, then $[\{y\}, V]$ is a neighbourhood of $\{y\} \in \mathcal{F}$. There exists $k \in \mathbb{N}$ such that $[\{y\}, V] \cap [\{x_k\}, X] \neq \emptyset$. Thus $x_k \in V$; moreover, F is separable. So X is hereditarily separable. □

Since the quasi-Rothberger property is stronger than the quasi-Lindelöf property, we can prove the following corollary.

Corollary 2.2. *If PR(X) is quasi-Rothberger, then X is hereditarily separable.*

An open cover $\mathcal{U}$ of a space X is called an ω-cover if every finite subset of X is contained in a member of $\mathcal{U}$ and X is not a member of $\mathcal{U}$. We write Ω the collection of ω-covers of X and $\mathcal{O}$ the collection of open covers of X.

Theorem 2.3. *If* PR(X) *is quasi-Rothberger, then each subset of X satisfies* $\mathsf{S}_1(\Omega, \Omega)$.

Proof. Let F be a subset of X and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ a sequence of ω-covers of F sets open in F. Put

$$\mathcal{F} = [F]^{<\omega} \setminus \{\emptyset\}, \text{ where } [F]^{<\omega} = \{A \subset F : A \text{ is finite}\}.$$

Then $\mathcal{F}$ is a closed subset of PR(X). Indeed, if $D \notin \mathcal{F}$, then there exists $x \in D$ such that $x \notin A$ for any $A \in \mathcal{F}$. Note that $[\{x\}, X]$ is a neighbourhood of D in PR(X) and $[\{x\}, X] \cap \mathcal{F} = \emptyset$. For every $A \in \mathcal{F}$, take $U_A^{(n)} \in \mathcal{U}_n$ such that

$$A \subset U_A^{(n)}, \text{ where } U_A^{(n)} \text{ is open in } F.$$

Let $V_A^{(n)}$ be open in X such that $U_A^{(n)} = F \cap V_A^{(n)}$. Then

$$\mathcal{W}_n = \{[A, V_A^{(n)}] : A \in \mathcal{F}\}$$

is an open cover of $\mathcal{F}$ in PR(X). Since PR(X) is quasi-Rothberger, there exists $[A_n, V_{A_n}^{(n)}] \in \mathcal{W}_n$ such that

$$\mathcal{F} \subset \overline{\bigcup_{n \in \mathbb{N}} [A_n, V_{A_n}^{(n)}]}.$$

Then $U_{A_n}^{(n)} \in \mathcal{U}_n$ with $U_{A_n}^{(n)} = F \cap V_{A_n}^{(n)}$ and $\{U_{A_n}^{(n)} : n \in \mathbb{N}\}$ is an ω-cover of F. In fact, for each $A \in \mathcal{F}$, $[A, X]$ is a neighbourhood of A in PR(X). There exists $k \in \mathbb{N}$ such that $[A, X] \cap [A_k, V_{A_k}^{(n)}] \neq \emptyset$. Thus $A \subset V_{A_k}^{(k)}$. Hence $A \subset F \cap V_{A_k}^{(k)} = U_{A_k}^{(k)}$. So F satisfies $\mathsf{S}_1(\Omega, \Omega)$. □

Example 2.4. The real line $\mathbb{R}$ does not satisfy $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ [2, Proposition 2.3]. So $\mathbb{R}$ does not satisfy $\mathsf{S}_1(\Omega, \Omega)$ since $\mathsf{S}_1(\Omega, \Omega)$ is stronger than $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ [5, Fig 2]. By Theorem 2.3, PR($\mathbb{R}$) is not quasi-Rothberger. So the converse of Corollary 2.2 is not true since $\mathbb{R}$ is hereditarily separable.

Theorem 2.5. *If* PR(X) *is quasi-Rothberger, then X is quasi-Rothberger.*

Proof. From Corollary 2.2, X is hereditarily separable. By Proposition 2.2 in [3], X is quasi-Rothberger. □

Example 2.6. By the following two examples, we shall show that the converse of Theorem 2.5 is not true.

1. From Proposition 2.2 of [3], the real line $\mathbb{R}$ is quasi-Rothberger since it is hereditarily separable. But PR($\mathbb{R}$) is not quasi-Rothberger by Example 2.4.

2. Denote τ the usual topology of $\mathbb{R}$. Put

$$\mathcal{B} = \{V - A : V \in \tau, A \subset \mathbb{R}, |A| \leq \omega\}.$$

The collection $\mathcal{B}$ is a base for a new topology τ' on $\mathbb{R}$. From Example 1.5 of [6], $(\mathbb{R}, \tau')$ is quasi-Rothberger. By Example 14.7 in [4], $(\mathbb{R}, \tau')$ is not separable; moreover, $(\mathbb{R}, \tau')$ is not hereditarily separable. By Corollary 2.2, PR[$(\mathbb{R}, \tau')$] is not quasi-Rothberger.

3. Main results

Recall that a subset U of X is called a *co-finite subset of* X [7] if $0 < |X - U| < \omega$. A family $\mathcal{U}$ consisting of co-finite subsets of X is said to be a co-finite family of X. Let $Y \subsetneq X$. A subset U of Y is called a *co-finite subset of* Y [7] if $0 \leq |Y - U| < \omega$. A family $\mathcal{U}$ consisting of co-finite subsets of Y is called a co-finite family of Y.

A subset pair (C, F) of X is called a *closed-miss-finite pair of* X [7], if C is closed and F is non-empty finite with $C \cap F = \emptyset$. A *closed-miss-finite family of* X is a family of closed-miss-finite pairs of X.

First, we define hit-families of X to study closed sets in PR(X).

Definition 3.1. A co-finite family $\mathcal{U}$ of a space X is said to be a hit-family of X, for any co-finite subset W of X with $W \notin \mathcal{U}$, there exists a closed-miss-finite pair (C, F) of X with $W^c \cap C = \emptyset$ and $W \cap F = \emptyset$ such that $U^c \cap C \neq \emptyset$ or $U \cap F \neq \emptyset$ for each $U \in \mathcal{U}$.

Lemma 3.2. *Let $\mathcal{U}$ be a co-finite family of space X, then $\mathcal{U}$ is a hit-family of X if and only if $\mathcal{U}^c$ is closed in $PR(X)$.*

Proof. Let $\mathcal{U}$ be a hit-family of X and $A \in \mathrm{PR}(X) - \mathcal{U}^c$, then $A^c \notin \mathcal{U}$. There exists a closed-miss-finite pair (C, F) of X with $A \cap C = \emptyset$ and $A^c \cap F = \emptyset$ such that

$$U^c \cap C \neq \emptyset \text{ or } U \cap F \neq \emptyset \text{ for any } U \in \mathcal{U}.$$

Thus $[F, X - C]$ is a neighbourhood of A such that $[F, X - C] \cap \mathcal{U}^c = \emptyset$. So $\mathcal{U}^c$ is closed in $\mathrm{PR}(X)$.

On the other hand, let $\mathcal{U}^c$ be a closed subset of $\mathrm{PR}(X)$ and W be a co-finite subset of X with $W \notin \mathcal{U}$, then $W^c \notin \mathcal{U}^c$. There exists a neighbourhood $[A, V]$ of W^c such that $[A, V] \cap \mathcal{U}^c = \emptyset$. Then $(X - V, A)$ is a closed-miss-finite pair of X with

$$W^c \cap (X - V) = \emptyset \text{ and } W \cap A = \emptyset.$$

From $[A, V] \cap \mathcal{U}^c = \emptyset$, it is easy to see that

$$U^c \cap (X - V) \neq \emptyset \text{ or } U \cap A \neq \emptyset \text{ for any } U \in \mathcal{U}.$$

So $\mathcal{U}$ is a hit-family of X. □

Next, in order to give characterizations of the quasi-Rothberger property of $\mathrm{PR}(X)$, we introduce rcf-networks of X on a hit-family and weakly rcf-networks of X on a hit-family.

Definition 3.3. Let $\mathcal{U}$ be a hit-family of X. A closed-miss-finite family ξ of X is said to be an rcf-network of X on $\mathcal{U}$, if for each $U \in \mathcal{U}$, there exists $(C, F) \in \xi$ such that $C \subset U$ and $F \cap U = \emptyset$.

Definition 3.4. Let $\mathcal{U}$ be a hit-family of X. A closed-miss-finite family ξ of X is said to be a weakly rcf-network of X on $\mathcal{U}$, if for each $U \in \mathcal{U}$ and $C \subset U$ closed in X, there exists $(C', F') \in \xi$ such that $C' \subset U$ and $F' \cap C = \emptyset$.

For a space X, we write

- Π_{rcf-h} : the collection of rcf-networks of X on a hit-family of X;
- Π_{wrcf-h} : the collection of weakly rcf-networks of X on a hit-family of X.

Theorem 3.5. *For a space X, the following are equivalent:*
(1) $\mathrm{PR}(X)$ *is quasi-Rothberger;*
(2) X satisfies $S_1(\Pi_{rcf-h}, \Pi_{wrcf-h})$.

Proof. (1)⇒(2) Let $\mathcal{U}$ be a hit-family of X and $\{\xi_n : n \in \mathbb{N}\}$ a sequence of rcf-networks on $\mathcal{U}$. By Lemma 3.2, $\mathcal{U}^c$ is closed in $\mathrm{PR}(X)$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{[F, X - C] : (C, F) \in \xi_n\}.$$

Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of rcf-covers of $\mathcal{U}^c$ in $\mathrm{PR}(X)$. In fact, for each $U^c \in \mathcal{U}^c$, there exists $(C, F) \in \xi_n$ such that $C \subset U$ and $F \cap U = \emptyset$. Thus $U^c \in [F, X - C] \in \mathcal{U}_n$. By (1), for each $n \in \mathbb{N}$, take $[F_n, X - C_n] \in \mathcal{U}_n$ such that

$$\mathcal{U}^c \subset \overline{\bigcup_{n \in \mathbb{N}} [F_n, X - C_n]}.$$

Hence $(C_n, F_n) \in \xi_n$ and $\{(C_n, F_n) : n \in \mathbb{N}\}$ is a weakly rcf-network on $\mathcal{U}$. Indeed, let $U \in \mathcal{U}$ and $C \subset U$ closed in X, then $[U^c, X - C]$ is a neighbourhood of U^c. There is some $k \in \mathbb{N}$ such that

$$[U^c, X - C] \cap [F_k, X - C_k] \neq \emptyset.$$

So $C_k \subset U$ and $F_k \cap C = \emptyset$.

(2)⇒(1) Let $\mathcal{F}$ be a closed subset of PR(X), then $\mathcal{F}^c$ is a hit-family of X by Lemma 3.2. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of covers of $\mathcal{F}$ open sets in PR(X). Suppose now that each $\mathcal{U}_n$ is a family of basic open sets. Then

$$\xi_n = \{(X - U, A) : [A, U] \in \mathcal{U}_n\}$$

is an rcf-network on $\mathcal{F}^c$. For each $n \in \mathbb{N}$, there exists $(X - U_n, A_n) \in \xi_n$ such that $\{(X - U_n, A_n) : n \in \mathbb{N}\}$ is a weakly rcf-network on $\mathcal{F}^c$. Hence each $[A_n, U_n] \in \mathcal{U}_n$ and $\mathcal{F} \subset \overline{\bigcup_{n\in\mathbb{N}}[A_n, U_n]}$. □

Finally, in order to study the characteristic of $\mathsf{S}_1(\Pi_{rcf-h}, \Pi_{wrcf-h})$, we define a hit-family of a subset Y of X, pcf-networks of Y on a hit-family and weakly pcf-networks of Y on a hit-family.

Let Y be a subset of X. A pair (C, F) of subsets of Y is called a *proper closed-miss-finite pair of* Y, if C is closed in Y and F is non-empty finite in Y with $C \cap F = \emptyset$. A family consisting of proper closed-miss-finite pairs of Y is said to be a *proper closed-miss-finite family of* Y.

Definition 3.6. Let Y be a subset of X. A co-finite family $\mathcal{U}$ of Y is said to be a hit-family of Y, if Y is not a member of $\mathcal{U}$ and for any co-finite W of Y with $W \neq Y$ and $W \notin \mathcal{U}$, there exists a proper closed-miss-finite pair (C, F) of Y with $W^c \cap C = \emptyset$ and $W \cap F = \emptyset$ such that $U^c \cap C \neq \emptyset$ or $U \cap F \neq \emptyset$ for each $U \in \mathcal{U}$.

Definition 3.7. Let Y be a subset of X and $\mathcal{U}$ a hit-family of Y. A proper closed-miss-finite family ξ of Y is called a pcf-network of Y on $\mathcal{U}$, if for each $U \in \mathcal{U}$, there exists $(C, F) \in \xi$ such that $C \subset U$ and $F \cap U = \emptyset$.

Definition 3.8. Let Y be a subset of X and $\mathcal{U}$ a hit-family of Y. A proper closed-miss-finite family ξ of Y is called a weakly pcf-network of Y on $\mathcal{U}$, if for each $U \in \mathcal{U}$ and $C \subset U$ closed in Y, there exists $(C', F') \in \xi$ such that $C' \subset U$ and $F' \cap C = \emptyset$.

For a space X, We write

- Π_{pcf-h} : the collection of pcf-networks of $Y \subset X$ on a hit-family of Y;
- Π_{wpcf-h} : the collection of weakly pcf-networks of $Y \subset X$ on a hit-family of Y.

Theorem 3.9. *For a space X, the following are equivalent:*
(1) X satisfies $\mathsf{S}_1(\Pi_{rcf-h}, \Pi_{wrcf-h})$;
(2) X is separable and each co-finite subset of X satisfies $\mathsf{S}_1(\Pi_{pcf-h}, \Pi_{wpcf-h})$.

Proof. (1)⇒(2) By Corollary 2.2 and Theorem 3.5, X is hereditarily separable and, hence, X is separable. Let Y be a co-finite subset of X and $\mathcal{U}$ a hit-family of Y, then

$$\mathcal{V} = \{U \cup Y^c : U \in \mathcal{U}\}$$

is a hit-family of X. In fact, let W be a co-finite subset of X with $W \notin \mathcal{V}$.

Case 1. If $W = V \cup Y^c$, where V is a co-finite subset of Y with $V \neq Y$, then $V \notin \mathcal{U}$. There exists a proper closed-miss-finite pair (C_0, F_0) of Y with $V^c \cap C_0 = \emptyset$ and $V \cap F_0 = \emptyset$ such that

$$U^c \cap C_0 \neq \emptyset \text{ or } U \cap F_0 \neq \emptyset \text{ for any } U \in \mathcal{U}.$$

Let $C_1 = \overline{C_0}$ and $F_1 = F_0$, where $\overline{C_0}$ is the closure of C_0 in X. Then $C_1 - C_0 \subset Y^c$ since $C_1 \cap Y = C_0$. Thus (C_1, F_1) is a closed-miss-finite pair of X with $W^c \cap C_1 = \emptyset$ and $W \cap F_1 = \emptyset$ such that

$$(U \cup Y^c)^c \cap C_1 = (U^c \cap Y) \cap C_1 = U^c \cap C_0 \neq \emptyset$$

or $(U \cup Y^c) \cap F_1 = U \cap F_0 \neq \emptyset$ for any $U \cup Y^c \in \mathcal{V}$.

Case 2. If $W = V \cup B$, where V is a co-finite subset of Y and $B \subset Y^c$ with $Y^c - B \neq \emptyset$. Take $C_1 = B$, $F_1 = Y^c - B$. Then (C_1, F_1) is closed-miss-finite pair of X with $W^c \cap C_1 = \emptyset$ and $W \cap F_1 = \emptyset$ such that

$$(U \cup Y^c) \cap F_1 = Y^c \cap F_1 = F_1 \neq \emptyset$$

for any $U \cup Y^c \in \mathcal{V}$.

Let $\{\xi_n : n \in \mathbb{N}\}$ be a sequence of pcf-networks of Y on $\mathcal{U}$. For each $n \in \mathbb{N}$, let

$$\zeta_n = \{(\overline{C}, A) : (C, A) \in \xi_n,\ \overline{C} \text{ is the closure of } C \text{ in } X\}.$$

Then each ζ_n is a closed-miss-finite rcf-network of X on $\mathcal{V}$. Indeed, for every $U \cup Y^c \in \mathcal{V}$, there exists $(C, A) \in \xi_n$ such that $C \subset U$ and $A \cap U = \emptyset$. Thus

$$\overline{C} \subset U \cup Y^c \text{ and } A \cap (U \cup Y^c) = \emptyset$$

since $\overline{C} \cap Y = C$ and $A \subset Y$. By (1), there exists $(\overline{C_n}, A_n) \in \zeta_n$ for $n \in \mathbb{N}$ such that $\{(\overline{C_n}, A_n) : n \in \mathbb{N}\}$ is a weakly rcf-network of X on $\mathcal{V}$. We show that $\{(C_n, A_n) : n \in \mathbb{N}\}$ is a weakly pcf-network of Y on $\mathcal{U}$. Let $U \in \mathcal{U}$ and $C \subset U$ closed in Y, then $\overline{C} \subset U \cup Y^c \in \mathcal{V}$. There exists some $(\overline{C_k}, A_k) \in \{(\overline{C_n}, A_n) : n \in \mathbb{N}\}$ such that

$$\overline{C_k} \subset U \cup Y^c \text{ and } A_k \cap \overline{C} = \emptyset.$$

Thus $C_k = \overline{C_k} \cap Y \subset (U \cup Y^c) \cap Y = U$ and $A_k \cap C = \emptyset$.

(2)⇒(1) Let $\mathcal{U}$ be a hit-family of X and $\{\xi_n : n \in \mathbb{N}\}$ a sequence of rcf-networks of X on $\mathcal{U}$. Denote $\{x_m : m \in \mathbb{N}\}$ the countable dense subset of X and put $\mathbb{N}' = \{m \in \mathbb{N} : x_m \in \bigcup \mathcal{U}\}$. For each $m \in \mathbb{N}'$, let

$$\mathcal{U}_m = \{U \cap (X - \{x_m\}) :\ U \in \mathcal{U} \text{ and } x_m \in U\}.$$

Then $\mathcal{U}_m$ is a hit-family of $X - \{x_m\}$. Indeed, let W be a co-finite subset of $X - \{x_m\}$ with $W \neq X - \{x_m\}$ and $W \notin \mathcal{U}_m$. Denote

$$W = X - A - \{x_m\} \text{ with } x_m \notin A.$$

Then $X - A$ is a co-finite subset of X and $X - A \notin \mathcal{U}$. There exists a closed-miss-finite pair (C_0, F_0) of X with $(X - A)^c \cap C_0 = \emptyset$ and $(X - A) \cap F_0 = \emptyset$, i.e., $C_0 \subset X - A$ and $F_0 \subset A$ such that

$$U^c \cap C_0 \neq \emptyset \text{ or } U \cap F_0 \neq \emptyset \text{ for each } U \in \mathcal{U}.$$

Let $C_1 = C_0 \cap (X - \{x_m\})$ and $F_1 = F_0$, then (C_1, F_1) is a proper closed-miss-finite pair of $X - \{x_m\}$ with $W^c \cap C_1 = \emptyset$ and $W \cap F_1 = \emptyset$. For each $U \cap (X - \{x_m\}) \in \mathcal{U}_m$, since $x_m \in U$, we have

$$[U \cap (X - \{x_m\})]^c \cap C_1 = (U^c \cup \{x_m\}) \cap [C_0 \cap (X - \{x_m\})] = U^c \cap C_0 \neq \emptyset$$

or

$$[U \cap (X - \{x_m\})] \cap F_1 = (U \cap A) \cap F_0 = U \cap F_0 \neq \emptyset.$$

Rearrange $\{\xi_n : n \in \mathbb{N}\}$ as $\{\xi_{n,m} : n, m \in \mathbb{N}\}$. For each $m \in \mathbb{N}$, let

$$\zeta_{n,m} = \{(C \cap (X - \{x_m\}), A) : (C, A) \in \xi_{n,m} \text{ and } x_m \notin A\}.$$

Then $\{\zeta_{n,m} : n \in \mathbb{N}\}$ is a sequence of pcf-networks of $X - \{x_m\}$ on $\mathcal{U}_m$. In fact, for each $U \cap (X - \{x_m\}) \in \mathcal{U}_m$, there exists $(C, A) \in \xi_{n,m}$ such that

$$C \subset U \text{ and } A \cap U = \emptyset.$$

Since $x_m \in U$, we have $x_m \notin A$. Then $(C \cap (X - \{x_m\}), A) \in \zeta_{n,m}$ such that

$$C \cap (X - \{x_m\}) \subset U \cap (X - \{x_m\}) \text{ and } A \cap [U \cap (X - \{x_m\})] = \emptyset.$$

By (2), there exists $(C_{n,m} \cap (X - \{x_m\}), A_{n,m}) \in \zeta_{n,m}$ such that $\{(C_{n,m} \cap (X - \{x_m\}), A_{n,m}) : n \in \mathbb{N}\}$ is a weakly pcf-network of $X - \{x_m\}$ on $\mathcal{U}_m$.

We have $(C_{n,m}, A_{n,m}) \in \xi_{n,m}$ for each $n, m \in \mathbb{N}$. We want to prove that $\{(C_{n,m}, A_{n,m}) : n \in \mathbb{N}, m \in \mathbb{N}'\}$ is a weakly rcf-network of X on $\mathcal{U}$. Indeed, let $U \in \mathcal{U}$ and $C \subset U$ closed in X. Take $x_m \in U$, then $C \cap (X - \{x_m\})$ is closed in $X - \{x_m\}$ and

$$C \cap (X - \{x_m\}) \subset U \cap (X - \{x_m\}) \in \mathcal{U}_m.$$

Since $\{(C_{n,m} \cap (X - \{x_m\}), A_{n,m}) : n \in \mathbb{N}\}$ is a weakly pcf-network of $X - \{x_m\}$ on $\mathcal{U}_m$, there exists

$$(C_{k,m} \cap (X - \{x_m\}), A_{k,m}) \in \{(C_{n,m} \cap (X - \{x_m\}), A_{n,m}) : n \in \mathbb{N}\}$$

such that

$$C_{k,m} \cap (X - \{x_m\}) \subset U \cap (X - \{x_m\}) \text{ and } A_{k,m} \cap [C \cap (X - \{x_m\})] = \emptyset.$$

Thus $C_{k,m} \subset U$ and $A_{k,m} \cap C = \emptyset$ since $x_m \in U$ and $x_m \notin A_{k,m}$. So X satisfies $\mathsf{S}_1(\Pi_{rcf}, \Pi_{wrcf})$. □

Theorem 3.10. *For each co-finite subset Y of X, the following are equivalent:*
(1) Y satisfies $\mathsf{S}_1(\Pi_{pcf-h}, \Pi_{wpcf-h})$;
(2) PR(Y) is quasi-Rothberger.

Proof. Note that a co-finite family $\mathcal{U}$ of Y is a hit-family of Y if and only if $\mathcal{U}^c = \{Y - U : U \in \mathcal{U}\}$ is closed in PR(Y). It is easy to show that a proper closed-miss-finite family ξ of Y is a pcf-network of Y on a hit-family $\mathcal{U}$ of Y if and only if $\mathcal{V} = \{[F, Y - C] : (C, F) \in \xi\}$ is an rcf-cover of $\mathcal{U}^c$ in PR(Y). So the proof parallels that of Theorem 3.5. □

The following corollary is a consequence of Theorems 3.5, 3.9 and 3.10.

Corollary 3.11. *Let X be a space, the following are equivalent:*
(1) PR(X) is quasi-Rothberger;
(2) X satisfies $\mathsf{S}_1(\Pi_{rcf-h}, \Pi_{wrcf-h})$;
(3) X is separable and each co-finite subset of X satisfies $\mathsf{S}_1(\Pi_{pcf-h}, \Pi_{wpcf-h})$;
(4) X is separable and PR(Y) is quasi-Rothberger for each co-finite subset Y of X.

Similarly to the proofs of Theorems 3.5, 3.9 and 3.10, we have the following characterizations of PR(X) being quasi-Menger.

Theorem 3.12. *For a space X, the following are equivalent:*
(1) PR(X) is quasi-Menger;
(2) X satisfies $\mathsf{S}_{fin}(\Pi_{rcf-h}, \Pi_{wrcf-h})$;
(3) X is separable and each co-finite subset of X satisfies $\mathsf{S}_{fin}(\Pi_{pcf-h}, \Pi_{wpcf-h})$;
(4) X is separable and PR(Y) is quasi-Menger for each co-finite subset Y of X.

Definition 3.13. Let $\mathcal{U}$ be a hit-family of X. A partitioned closed-miss-finite family $\xi = \bigcup_{n\in\mathbb{N}} \xi_n$ of X is said to be a weakly p-rcf-network of X on $\mathcal{U}$, if for each $U \in \mathcal{U}$ and subset $C \subset U$ closed in X, there exists $(C_n, F_n) \in \xi_n$ such that $C_n \subset U$ and $F_n \cap C = \emptyset$ for all but finitely many $n \in \mathbb{N}$.

Definition 3.14. Let Y be a subset of X and $\mathcal{U}$ a hit-family of Y. A partitioned proper closed-miss-finite family $\xi = \bigcup_{n\in\mathbb{N}} \xi_n$ of Y is said to be a weakly p-cf-network on $\mathcal{U}$, if for each $U \in \mathcal{U}$ and subset $C \subset U$ closed in Y, there exists $(C_n, F_n) \in \xi_n$ such that $C_n \subset U$ and $F_n \cap C = \emptyset$ for all but finitely many $n \in \mathbb{N}$.

For a space X, we write

- Π^p_{wrcf-h} : the collection of weakly p-rcf-networks of X on a hit-family of X;
- Π^p_{wcf-h} : the collection of weakly p-cf-networks of $Y \subsetneq X$ on a hit-family of Y.

In a similar way, one can prove

Theorem 3.15. *For a space X, the following are equivalent:*
(1) PR(X) is quasi-Hurewicz;
(2) X satisfies $\mathsf{S}_{fin}(\Pi_{rcf-h}, \Pi^p_{wrcf-h})$;
(3) X is separable and each co-finite subset of X satisfies $\mathsf{S}_{fin}(\Pi_{pcf-h}, \Pi^p_{wcf-h})$;
(4) X is separable and PR(Y) is quasi-Hurewicz for each co-finite subset Y of X.

References

[1] A.V. Arhangel'skii, A generic theorem in the theory of cardinal invariants of topological spaces, Comment. Math. Univ. Carol. 36 (1995) 303–325.
[2] M. Bonanzinga, F. Cammaroto, B.A. Pansera, B. Tsaban, Diagonalizations of dense families, Topol. Appl. 165 (2014) 12–25.
[3] G. Di Maio, Lj.D.R. Kočinac, A note on quasi–Menger and similar spaces, Topol. Appl. 179 (2015) 148–155.
[4] E.K. van Douwen, The integer and topology, in: K. Kunen, J. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 395–490.
[5] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, Combinatorics of open covers (II), Topology Appl. 73 (1996) 241–266.
[6] Z. Li, The quasi-Rothberger property of linearly ordered spaces, Topol. Appl. 247 (2018) 41–49.
[7] Z. Li, Remarks on *R*-separability of Pixley–Roy hyperspaces, Filomat. 36:3 (2022) 881–886.
[8] Z. Li, The weakly Rothberger property of Pixley–Roy hyperspaces, Filomat, in press.
[9] C. Pixley, P. Roy, Uncompletable Moore spaces, in: Proc. Auburn Topology Conf. (1969) 75–85.
[10] M. Scheepers, Combinatorics of open covers I: Ramsey theory, Topology Appl. 69 (1996) 31–62.