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Almost Kenmotsu manifolds with constant Reeb or ϕ-sectional curvatures

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Abstract. In this paper, we prove that an almost Kenmotsu manifold M has constant Reeb sectional curvatures if and only if M has conformal Reeb foliation. On an almost Kenmotsu h-a-manifold of dimension three having constant ϕ-sectional curvature, the Reeb vector field is an eigenvector field of the Ricci operator if and only if the manifold is locally isometric to a non-unimodular Lie group.

1. Introduction

In geometry of almost contact metric manifolds, the following three assertions are well known:

- If a contact metric manifold is of constant curvature c, then $c = 1$ and the manifold is Sasakian when the dimension of the manifold is greater than three and $c = 0$ or 1 when the dimension of the manifold is three (see [3, 24]);
- If an almost Kenmotsu manifold is of constant curvature c, then $c = -1$ and the manifold is Kenmotsu (see [9]);
- If an almost cosymplectic manifold is of constant curvature c, then $c = 0$ and the manifold is cosymplectic (see [25, 26]).

In a sense, the above three kinds of almost contact metric manifolds correspond to the sphere of constant positive curvatures, hyperbolic space of constant negative curvatures and Euclidean space, respectively; and they are ones of the most important research objects in almost contact Riemannian geometry.

According to the above statements, one observes that the constancy of sectional curvatures in geometry of almost contact metric manifolds is too strong. Therefore, one always considers some other kinds of sectional curvatures. For an almost contact metric manifold (M, g) together with the almost contact structure (ϕ, ξ, η), the manifold M is said to have constant Reeb sectional curvature if the sectional curvature of the plane section containing the Reeb vector field ξ and a vector field $X \in \ker \eta$ is a constant which is independent of the choice of $X \in \ker \eta$ and the point in M. Similarly, M is said to have constant ϕ-sectional curvature if the sectional curvature of the plane section containing a vector field $X \in \ker \eta$ and ϕX is a constant which is independent of the choice of $X \in \ker \eta$ and the point in M. Weakening constancy of sectional curvature

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to constancy of Reeb or ϕ-sectional curvature in geometry of almost contact metric manifolds has been investigated by many authors.

Moskal in [19] proved that for a Sasakian manifold the ϕ-sectional curvatures determine the curvature completely. A Sasakian manifold of constant ϕ-sectional curvatures is called a Sasakian space form and its curvature tensor was obtained by Ogiue in [23]. Koufogiorgos [18] found a class of non-Sasakian contact metric manifolds which are of constant ϕ-sectional curvatures. With regard to the Reeb sectional curvatures, Gouli-Andreou and Xenos in [11] obtained a classification result of contact metric manifolds whose Reeb sectional and ϕ-sectional curvatures are both constant. A classification of almost cosymplectic 3-manifolds with constant Reeb sectional curvatures was considered by D. Perrone in [28]. Kenmotsu manifolds with constant ϕ-sectional curvatures and cosymplectic manifolds with constant ϕ-sectional curvatures were presented in [16] and [14], respectively. For some other results on almost contact metric manifolds with constant Reeb sectional or ϕ-sectional curvatures, or some other types of sectional curvatures, we refer the reader to [1–3, 6, 33].

The studies of constancy of certain sectional curvatures in both contact metric and almost cosymplectic geometry are rich, but the corresponding theorem in almost Kenmotsu geometry is seldom. In view of this, in the present paper, we aim to start the study of almost Kenmotsu manifolds having constant Reeb sectional or ϕ-sectional curvatures. First, we prove that an almost Kenmotsu manifold is of constant Reeb sectional curvatures if and only if the Lie derivative of the structure tensor field ϕ along the Reeb vector field ξ vanishes identically. It is interesting that such a property is much different from the contact metric and almost cosymplectic cases. We show that if an almost Kenmotsu manifold has constant Reeb sectional curvature and all ϕ-sectional curvatures are constant -1, then the manifold is a Kenmotsu manifold of constant sectional curvature -1. We classify almost Kenmotsu *h-a*-manifolds of dimension three having constant ϕ-sectional curvatures under an additional assumption.

2. Almost Kenmotsu Manifolds

If on a smooth differentiable manifold M of dimension $2n+1$, $n \geq 1$, there are three tensor fields ϕ, ξ, η of type $(1,1)$, $(1,0)$ and $(0,1)$, respectively, satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi \text{ and } \eta(\xi) = 1, \tag{1}$$

then M is said to admit an almost contact structure and it is called an almost contact manifold. If in addition there exists a Riemannian metric g on M such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any vector fields X, Y, M is said to be an almost contact metric manifold and g is said to be a compatible metric with respect to the almost contact structure. An almost Kenmotsu manifold is defined as an almost contact metric manifold satisfying $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta \wedge \Phi$, where the fundamental two-form Φ of the almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y ([15]). A contact metric (resp. almost cosymplectic) manifold is defined as an almost contact metric manifold satisfying $\mathrm{d}\eta = \Phi$ (resp. $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 0$).

On the product $M \times \mathbb{R}$ of an almost contact metric manifold M and $\mathbb{R}$, there exists an almost complex structure J defined by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to M, t is the coordinate of $\mathbb{R}$ and f is a C^∞-function on $M \times \mathbb{R}$. If J is integrable, the almost contact metric structure is said to be normal. A normal almost Kenmotsu manifold is called a Kenmotsu manifold; and a normal contact metric (resp. almost cosymplectic) manifold is called a Sasakian (resp. cosymplectic) manifold.

An almost Kenmotsu manifold is a Kenmotsu manifold if and only if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for any vector fields X, Y. If the almost Kenmotsu manifold is of dimension three, $h = 0$ is a necessary and sufficient condition for the manifold to be Kenmotsu, where $h := \frac{1}{2}\mathcal{L}_\xi \phi$. On an almost Kenmotsu manifold, one can check that h and $h' := h \circ \phi$ are both symmetric operators, and they satisfy $h\xi = 0$, $\mathrm{tr}h = 0$, $\mathrm{tr}(h') = 0$ and $h\phi + \phi h = 0$ and

$$\nabla \xi = \mathrm{id} - \eta \otimes \xi + h'. \tag{3}$$

All the above preliminaries can be seen in [3, 9, 10, 15].

3. Constant Reeb sectional and ϕ-sectional curvatures

Recall from [27, Section 3] that the Reeb foliation (generated by the Reeb vector field ξ) of an almost Kenmotsu manifold is conformal if and only if $h = 0$. From this, our first result is given as follows:

Theorem 3.1. *An almost Kenmotsu manifold has constant Reeb sectional curvatures if and only if the Reeb foliation is conformal.*

Proof. On an almost Kenmotsu manifold M, from (3) we obtain

$$R(X, Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X \tag{4}$$

for any vector fields X, Y, where the curvature operator R is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

If the Reeb foliation is conformal, or equivalently, $h = 0$, from (4) we get

$$R(X, \xi)\xi = \phi^2 X$$

for any vector field X. By this, we see that all Reeb sectional curvatures of the manifold are -1.

Conversely, suppose that an almost Kenmotsu manifold is of constant Reeb sectional curvatures, say $\kappa \in \mathbb{R}$. That is,

$$g(R(X, \xi)\xi, X) = \kappa(g(X, X) - \eta^2(X))$$

for any vector field X. By polarization, the above equality is equivalent to

$$R(X, \xi)\xi = -\kappa\phi^2 X \tag{5}$$

for any vector field X. Setting $Y = \xi$ in (4), with the aid of (3), we get

$$R(X, \xi)\xi = \phi^2 X - 2h'X - (\nabla_\xi h')X - h^2 X,$$

which is compared with (5) implying

$$\nabla_\xi h' = (\kappa + 1)\phi^2 - 2h' - h^2.$$

As $\nabla_\xi \phi = 0$ on an almost Kenmotsu manifold (see [17]), we have $\nabla_\xi h' = \nabla_\xi h \circ \phi$, which is used in the above equality giving $\nabla_\xi h \circ \phi = (\kappa + 1)\phi^2 - 2h' - h^2$. Applying this to ϕ and using (1), we have

$$\nabla_\xi h = (\kappa + 1)\phi - 2h + h^2 \circ \phi.$$

Recall that h is a self-adjoint operator on an almost Kenmotsu manifold, and so is $\nabla_\xi h$. Applying such a property on the above equality, with the aid of (1), we get

$$h^2 = (\kappa + 1)\phi^2. \tag{6}$$

Substituting (6) back into the previous equality we have

$$\nabla_\xi h = -2h.$$

Taking the covariant derivative of (6) along the Reeb vector field ξ, in view of $\nabla_\xi \phi = 0$ and $\kappa \in \mathbb{R}$, we obtain $\nabla_\xi h \circ h + h \circ \nabla_\xi h = 0$. Now, putting $\nabla_\xi h = -2h$ into the previous equality gives $h^2 = 0$. Since h is self-adjoint, we have $h = 0$. □

Remark 3.2. *There are some contact metric and almost cosymplectic manifolds with constant Reeb sectional curvatures on which $h \neq 0$ (see [11, 28]). But, by Theorem 3.1, if the Reeb sectional curvatures of an almost Kenmotsu manifold are the same constant at each point and for any vector field, then $h = 0$ everywhere. This is much different from the contact metric and almost cosymplectic cases.*

From the proof of Theorem 3.1, we also have

Corollary 3.3. *All Reeb sectional curvatures of an almost Kenmotsu manifold with conformal Reeb foliation are* -1.

On an almost Kenmotsu manifold of dimension $2n + 1$ having constant Reeb sectional curvatures, by Theorem 3.1, using $h = 0$ in (4) gives

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X$$

for any vector fields X and Y. It follows immediately that $Q\xi = -2n\xi$, where Q denotes the Ricci operator which is defined as the trace of the curvature tensor $Q := \mathrm{trace}\{X \to R(\cdot, X)X\}$.

Corollary 3.4. *On an almost Kenmotsu manifold with constant Reeb sectional curvatures, the Reeb vector field is an eigenvector field of the Ricci operator.*

Naik, Venkatesha and Kumara in [22] obtained the same conclusion of Corollary 3.4 under the constancy of the Reeb sectional curvatures and local symmetry. Obviously, from our Corollary 3.4, the local symmetry condition in [22] is redundant.

An almost Kenmotsu manifold M is Kenmotsu if and only if $h = 0$ and M is CR-integrable, i.e., the associated almost Kähler structure on $\ker \eta$ is integrable. From this we have

Corollary 3.5. *A CR-integrable almost Kenmotsu manifold with constant Reeb sectional curvatures is Kenmotsu.*

A Kenmotsu manifold of constant ϕ-sectional curvatures is said to be a Kenmotsu space form and its curvature tensor is given in [16]. From the following theorem, we see that the conclusion in Corollary 3.5 is still true if the CR-integrability is replaced by constancy of ϕ-sectional curvatures.

Theorem 3.6. *On an almost Kenmotsu manifold M having constant Reeb sectional curvatures, M has constant ϕ-sectional curvatures* -1 *if and only if M is a Kenmotsu manifold of constant sectional curvature* -1.

Proof. First, if an almost Kenmotsu manifold M is of constant Reeb sectional curvatures, from Theorem 3.1 we have $h = 0$. Let M' be the maximal integral submanifold of the contact distribution $\ker \eta$ of an almost Kenmotsu manifold M satisfying $h = 0$ whose associated metric is g. According to [17], one sees that M' is an almost Kähler manifold whose almost Kähler metric is the restriction of g on $\ker \eta$. Let ∇ and ∇' be the Levi-Civita connections of the metric g on M and the induced metric on M', respectively. Then, from (3) and $h = 0$ we have

$$\nabla_{X'} Y' = \nabla'_{X'} Y' - g(X', Y')\xi \tag{7}$$

for any vector fields X', Y' on M'. Let τ and τ' be the sectional curvatures of the manifold M and M', respectively. Then, from (7) we obtain (see also the proof of [9, Theorem 3]):

$$\tau'(X', Y') = \tau(X', Y') + 1 \tag{8}$$

for any vector fields X', Y' on M'. If the manifold M is of constant ϕ-sectional curvatures -1, then from (8) we obtain that M' is of constant holomorphic sectional curvatures zero. From [21, Theorem 2.1], one sees that M' is of constant sectional curvature zero. Consequently, by [13, Corollary 1.2], one sees that the almost Kähler structure on M' is integrable. This, together with $h = 0$, implies that the manifold M is Kenmotsu. For a Kenmotsu manifold of constant ϕ-sectional curvature τ, the curvature tensor of the manifold is obtained by K. Kenmotsu (see [16]):

$$\begin{aligned} R(X, Y)Z =& \frac{\tau - 3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \frac{\tau + 1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \\ &\quad + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \end{aligned}$$

for any vector fields X, Y, Z. Thus, if $\tau = -1$, using it in the above equality we see that the manifold is of constant sectional curvature -1.

The converse is easy to check. This completes the proof. □

Remark 3.7. *It was pointed out in [16, Theorem 13] that if a Kenmotsu manifold is of constant ϕ-sectional curvature, then it is of constant sectional -1. This is much different from the Sasakian and cosymplectic cases.*

In view of the above results, the constancy of ϕ-sectional curvatures is a strong condition on an almost Kenmotsu manifold with $h = 0$. Next, we study strictly almost Kenmotsu manifolds (which means $h \neq 0$ everywhere) having constant ϕ-sectional curvatures. On such manifolds we shall find in the following statement that the constancy of ϕ-sectional curvatures is much weaker. Therefore, we need some other assumptions.

An almost Kenmotsu manifold is said to be a *h-a*-manifold if

$$\nabla_\xi h \text{ is a constant multiple of } h'. \tag{$*$}$$

Notice that the above condition was widely used in contact metric geometry (see [3–5, 7, 12]) and almost Kenmotsu geometry (see [30–32]) and it is natural for almost Kenmotsu 3-manifolds (see the first equality in the proof of Theorem 3.9). The condition $(*)$ is meaningless for the case $h = 0$, so in the next theorem we consider the case $h \neq 0$ everywhere.

In general case, let $\mathcal{U}_1$ be the maximal open subset of an almost Kenmotsu 3-manifold M on which $h \neq 0$; and let $\mathcal{U}_2$ be the maximal open subset on which $h = 0$. Then $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we set $he = \lambda e$ and hence $h\phi e = -\lambda\phi e$, where λ is assumed to be a positive eigenfunction of h on $\mathcal{U}_1$.

Lemma 3.8 ([8, Lemma 6]). *On $\mathcal{U}_1$ we have*

$$\begin{aligned} &\nabla_\xi \xi = 0,\ \nabla_\xi e = a\phi e,\ \nabla_\xi \phi e = -ae, \\ &\nabla_e \xi = e - \lambda\phi e,\ \nabla_e e = -\xi - b\phi e,\ \nabla_e \phi e = \lambda\xi + be, \\ &\nabla_{\phi e}\xi = -\lambda e + \phi e,\ \nabla_{\phi e} e = \lambda\xi + c\phi e,\ \nabla_{\phi e}\phi e = -\xi - ce, \end{aligned} \tag{9}$$

where a, b, c are smooth functions.

Theorem 3.9. *On a strictly almost Kenmotsu 3-manifold with $(*)$ and constant ϕ-sectional curvature, the Reeb vector field is an eigenvector field of the Ricci operator if and only if the manifold is locally isometric to a non-unimodular Lie group.*

Proof. From Lemma 3.8, on an almost Kenmotsu 3-manifold M with $h \neq 0$ we have

$$\nabla_\xi h = \frac{1}{\lambda}\xi(\lambda)h + 2a\phi h.$$

So by the assumption of the theorem we have $\xi(\lambda) = 0$ and $a \in \mathbb{R}$. From a direct calculation we have

$$R(e, \phi e)\phi e = -(2b\lambda + \phi e(\lambda))\xi - (1 - \lambda^2 + e(c) + \phi e(b) + b^2 + c^2)e.$$

Thus, if the ϕ-sectional curvature of the manifold is a constant, say $\tau \in \mathbb{R}$, it follows directly that

$$1 - \lambda^2 + e(c) + \phi e(b) + b^2 + c^2 + \tau = 0.$$

From Lemma 3.8, in view of $\xi(\lambda) = 0$ and a direct calculation, we have

$$Q\xi = -2(\lambda^2 + 1)\xi,$$

$$Qe = (\tau - \lambda^2 - 1 - 2a\lambda)e + 2\lambda\phi e$$

and

$$Q\phi e = 2\lambda e + (\tau - \lambda^2 - 1 + 2a\lambda)\phi e,$$

where the scalar curvature

$$r = 2(\tau - 2\lambda^2 - 2)$$

and we used the assumption that the Reeb vector field is an eigenvector field of the Ricci operator. Applying again Lemma 3.8 and the above three equalities, with the aid of $a \in \mathbb{R}$, we have

$$(\nabla_\xi Q)\xi = 0,$$

$$(\nabla_e Q)e = (\lambda^2 - 1 - \tau + 2a\lambda)\xi + 2(2b\lambda - (\lambda + a)e(\lambda))e + 2(e(\lambda) + 2ab\lambda)\phi e$$

and

$$(\nabla_{\phi e} Q)\phi e = (\lambda^2 - 1 - \tau - 2a\lambda)\xi + 2(\phi e(\lambda) - 2ac\lambda)e + 2(2c\lambda - (\lambda - a)\phi e(\lambda))\phi e.$$

Recall that the following equality is always valid on a Riemannian manifold:

$$\frac{1}{2}\operatorname{grad} r = \operatorname{div} Q. \tag{10}$$

Taking the inner product of (10) with ξ and using the above three equalities, with the aid of $r = 2(\tau - 2\lambda^2 - 2)$, we get

$$\tau = \lambda^2 - 1,$$

and hence λ is a nonzero constant. Taking the inner product of (10) with e and using the previous three equalities we get

$$b = ac.$$

Similarly, taking the inner product of (10) with ϕe and using the previous three equalities we get

$$c = -ab.$$

Comparing this with the above equality we obtain $b = c = 0$. Now, from Lemma 3.8 we have

$$[\xi, e] = -e + (\lambda + a)\phi e, \ [e, \phi e] = 0, \ [\phi e, \xi] = (a - \lambda)e + \phi e. \tag{11}$$

From this one sees that

$$\mathrm{trace}(\mathrm{ad}_\xi) = -2,\ \mathrm{trace}(\mathrm{ad}_e) = 0 \text{ and } \mathrm{trace}(\mathrm{ad}_{\phi e}) = 0.$$

This means that the unimodular kernel of the Lie algebra generated by $\{\xi, e, \phi e\}$ is of dimension two. Thus, from [20], the manifold is locally isometric to a non-unimodular Lie group whose Lie algebra is given by (11).

Conversely, from Milnor [20], if G is a three-dimensional non-unimodular Lie group, then there exists a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying

$$[e_1, e_2] = \alpha e_2 + \beta e_3,\ [e_2, e_3] = 0,\ [e_1, e_3] = \gamma e_2 + \delta e_3 \tag{12}$$

and $\alpha + \delta = 2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Let g be a Riemannian metric defined on G by $g(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$. Let $\xi := -e_1$ and by η we mean the dual one-form of ξ. Let ϕ be a $(1,1)$-type tensor field defined by $\phi(\xi) = 0$, $\phi(e_2) = e_3$ and $\phi(e_3) = -e_2$. Now, one sees that (G, ϕ, ξ, η, g) admits a left invariant almost Kenmotsu structure. For more details regarding the above statements we refer the reader to [10, Theorem 5.2]. The Levi-Civita connection of the metric g is

$$\nabla_\xi \xi = 0,\ \nabla_{e_2}\xi = \alpha e_2 + \frac{1}{2}(\beta+\gamma)e_3,\ \nabla_{e_3}\xi = \frac{1}{2}(\beta+\gamma)e_2 + (2-\alpha)e_3,$$
$$\nabla_\xi e_2 = \frac{1}{2}(\gamma-\beta)e_3,\ \nabla_{e_2}e_2 = -\alpha\xi,\ \nabla_{e_3}e_2 = -\frac{1}{2}(\beta+\gamma)\xi,$$
$$\nabla_\xi e_3 = \frac{1}{2}(\beta-\gamma)e_2,\ \nabla_{e_2}e_3 = -\frac{1}{2}(\beta+\gamma)\xi,\ \nabla_{e_3}e_3 = (\alpha-2)\xi.$$

From this we have

$$he_2 = \frac{1}{2}(\beta+\gamma)e_2 + (1-\alpha)e_3 \text{ and } he_3 = (1-\alpha)e_2 - \frac{1}{2}(\beta+\gamma)e_3.$$

So one can check that the following equality is true (see also [32]):

$$\nabla_\xi h = (\beta-\gamma)h'.$$

This means that the the condition $(*)$ is valid. By a direct calculation we have

$$Q\xi = -2\left(\alpha^2 - 2\alpha + \frac{1}{4}(\beta+\gamma)^2 + 2\right)\xi.$$

This means that the the Reeb vector field is an eigenvector field of the Ricci operator. Moreover, the ϕ-sectional curvature τ of this almost Kenmotsu structure is

$$\tau = \alpha(\alpha-2) + \frac{1}{4}(\beta+\gamma)^2 \in \mathbb{R}.$$

This completes the proof. □

As followed from Theorems 3.1 and 3.6, a Kenmotsu 3-manifold has constant ϕ-sectional curvature if and only if it is of constant sectional curvature -1.

Except for the above left-invariant almost Kenmotsu 3-manifolds on Lie groups, next we show that there is a type of warped product almost Kenmotsu manifolds having constant ϕ-sectional curvatures.

Example 3.10. *Let (N^{2n}, g') be a non-Kähler almost Kähler manifold of constant holomorphic sectional curvatures $\tau' \neq 0 \in \mathbb{R}$ (for existence of these manifolds we refer to [29]). The warped product manifold $\mathbb{R} \times_f N^{2n}$ admits an almost Kenmotsu structure, where $f = ce^t$ for a nonzero constant c and t is the coordinate of $\mathbb{R}$ (see [9, Example 1]). This almost Kenmotsu structure is non-Kenmotsu (since the CR-integrability is invalid) and the associated tensor field $h = 0$. From the proof of Theorem 3.6, such an almost Kenmotsu manifold has constant ϕ-sectional curvature $\tau' - 1$.*

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