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The asymptotic properties for the estimators in a semiparametric regression model based on m-asymptotic negatively associated errors

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Abstract. In this paper, we investigate the parametric component and nonparametric component estimators in a semiparametric regression model based on m-asymptotic negatively associated (m-ANA, for short) random variables. The r-th ($r > 1$) mean consistency, complete consistency and uniform consistency are obtained under some suitable conditions. In order to assess the finite sample performance, we also present a numerical simulation in the last section of the paper. The results obtained in the paper extend the corresponding ones for independent random errors, φ-mixing and other dependent random errors.

1. Introduction

1.1. Concept of m-ANA random variables

In this subsection, we are going to introduce the concepts of two dependent random variables which are significant in this paper. First of all, let us recall the concept of asymptotic negatively associated (ANA, for short) random variables which was first proposed by Zhang and Wang (1999).

Definition 1.1. *A sequence $\{X_n, n \geq 1\}$ of random variables is called ANA (or ρ^--mixing) if*

$$\rho^-(s) = \sup\{\rho^-(S,T) : S, T \subset \mathbf{N}, dist(S,T) \geq s\} \to 0,$$

as $s \to \infty$, where

$$\rho^-(S,T) = 0 \vee \left\{ \frac{cov(f_1(X_i, i \in S), f_2(X_j, j \in T))}{\sqrt{Var(f_1(X_i, i \in S))Var(f_2(X_j, j \in T))}} : f_1, f_2 \in O \right\},$$

and O is the set of nondecreasing functions.

An array $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ of random variables is called rowwise ANA if for every $n \geq 1$, $\{X_{ni}, 1 \leq i \leq n\}$ are ANA.

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Since ANA random variables are widely used in real life, many scholars at home and abroad are interested in the investigation of the properties and have achieved many valuable results. For instance, the moment inequalities and the complete convergence for partial sums have been obtained in Zhang and Wang (1999); The moments of the maximum of normed partial sums have been exhibited in Liu and Liu (2009); The precise asymptotics of complete moment convergence for ρ^--mixing sequences has been gained in Fu and Wu (2017); The central limit theorems have been attained in Zhang (2000a, b); The weak convergence and some inequalities for the maximum of partial sums have been acquired in Wang and Lu (2006); The complete convergence for weighted sums without the assumption of identical distribution has been considered in Tan and Wang (2014); The law of the iterated logarithm has been established in Wang and Zhang (2007).

Wu et al. (2021) extended the concept of ANA random variables to m-ANA random variables as follows.
Definition 1.2. *Let $m \geq 1$ be a fixed integer. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be m-ANA if for any $n \geq 2$ and any $i_1, i_2, \cdots, i_n$ such that $|i_k - i_j| \geq m$ for any $1 \leq k \neq j \leq n$, we have that $X_{i_1}, X_{i_2}, \cdots, X_{i_n}$ are ANA.*

It is easy to see that the concept of m-ANA random variables is equivalent to that of ANA random variables under the condition of $m = 1$. Thus, we can consider m-ANA random variables as a more extensive dependent sequence than ANA random variables. What's more, since negatively associated (NA, for short) implies ANA, then m-negatively associated (m-NA, for short) implies m-ANA. Therefore, it's very meaningful to research m-ANA random variables. Wu et al. (2021) obtained the complete consistency for the weighted estimator in a nonparametric regression model and the strong consistency for conditional value-at-risk estimator based on m-ANA random errors.

In this article, we are going to investigate the r-th mean consistency, complete consistency and uniform consistency of the estimators under the semiparametric regression model (1.1) below based on m-ANA random errors. Besides, we provide a numerical simulation to research the numerical performance of the consistency for the least squares estimator and the nearest neighbor weight function estimator.

1.2. Semiparametric regression model

Consider the following semiparametric regression model:

$$y_i^{(n)} = x_i^{(n)}\beta + g\left(t_i^{(n)}\right) + \varepsilon_i^{(n)},\ i = 1, 2, \cdots, n,\ n \geq 1, \tag{1.1}$$

where g is an unknown function defined on a compact set A in $\mathbb{R}^p$, and β is an unknown parameter in $\mathbb{R}$, $x_i^{(n)}$ and $t_i^{(n)}$ are known to be nonrandom, $y_i^{(n)}$ represents the i-th response which is observable at points $x_i^{(n)}$ and $t_i^{(n)}$, $\varepsilon_i^{(n)}$ are random errors. Assume that for each n, $(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, \cdots, \varepsilon_n^{(n)})$ has the same distribution as $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$.

Seeing that the semiparametric regression model (1.1) contains both parametric and nonparametric components, so it is more flexible and applicable than the classical linear or nonparametric regression model. Recently, many statisticians have paid attention to the research under the semiparametric regression model (1.1). For instance, the r-th mean and uniform r-th mean consistencies, strong and uniform strong consistencies for the estimators of β and $g(t)$ based on independent errors have been established in Hu (1999); The r-th mean consistency and strong consistency for the nearest neighbor estimators based on martingale difference errors have been derived in Yan et al. (2001); The r-th mean consistency and complete consistency for the estimators based on L^q-mixingale errors have been acquired in Pan et al. (2003); The r-th mean consistency and complete consistency for the estimators based on linear time series have been obtained in Hu (2006). Inspired by the above articles, Wang et al. (2019) gained the r-th and uniform r-th ($r \geq 2$) mean consistencies, complete and uniform complete consistencies for the estimators with φ-mixing errors under the semiparametric regression model (1.1). Noting that sometimes the condition of $r \geq 2$ can not be satisfied, Wang et al. (2022) discussed the r-th ($1 < r < 2$) mean consistency and uniform consistency for the estimators with φ-mixing errors under the semiparametric regression model (1.1). Moreover, Wang et al. (2017) studied the following semiparametric regression model:

$$y^{(j)}(x_{in}, t_{in}) = t_{in}\beta + g(x_{in}) + e^{(j)}(x_{in}),\ 1 \leq j \leq m,\ 1 \leq i \leq n, \tag{1.2}$$

with $\tilde{\rho}$-mixing errors and presented the strong consistency, r-th ($r > 2$) mean consistency and complete consistency for estimators $\hat{\beta}_n$ and $\hat{g}_n(t)$ of β and $g(t)$. Wu and Wang (2018) discussed the strong consistency and r-th ($1 < r \le 2$) mean consistency for the estimators under the semiparametric regression model (1.2) with ρ^*-mixing random errors.

Considering that the convergence properties of m-ANA random variables will have a very wide range of applications in probability statistics, finance and insurance, reliability theory, complex systems and econometrics, we work on the r-th ($r > 1$) mean consistency, uniform consistency and complete consistency of the estimators under the semiparametric regression model (1.1) based on m-ANA random errors. Our results obtained in the paper extend the corresponding ones for independent random errors, φ-mixing and other dependent random errors.

The main contents of this article are organized as follows: In Section 2, we state some assumptions about fixed design points and weight functions. Meanwhile, we also provide some necessary theorems required for proving the conclusions of this article. In Section 3, we present the main results and their proofs, including the r-th mean consistency of $\hat{\beta}_n$ and $\hat{g}_n(t)$, complete consistency and uniform consistency of $\hat{g}_n(t)$. In Section 4, we carry out a simulation to investigate the numerical performance of the consistency for the least squares estimator and the nearest neighbor weight function estimator based on m-ANA samples.

Throughout the article, we assume that $C, C_1, C_2, \cdots$ are positive constants independent of n, x and t, and C can take different values in various positions, $I(E)$ denotes the indicator function of the set E, and $\|x\|$ denotes the Euclidean norm of x.

2. Assumptions and lemmas

2.1. Estimators and assumptions

For the model (1.1), based on the least squares method and the weight function method, Pan et al. (2003) obtained the estimators for β and $g(t)$ as follows:

$$\hat{\beta}_n = \frac{\sum\limits_{i=1}^{n} \widetilde{x}_i^{(n)} \widetilde{y}_i^{(n)}}{\sum\limits_{i=1}^{n} \left(\widetilde{x}_i^{(n)}\right)^2}, \tag{2.1}$$

$$\hat{g}_n(t) = \sum_{i=1}^{n} W_{ni}(t) \left(y_i^{(n)} - x_i^{(n)} \hat{\beta}_n\right), \tag{2.2}$$

where

$$\widetilde{x}_i^{(n)} = x_i^{(n)} - \sum_{k=1}^{n} W_{nk}\left(t_i^{(n)}\right) x_k^{(n)}, \quad \widetilde{y}_i^{(n)} = y_i^{(n)} - \sum_{k=1}^{n} W_{nk}\left(t_i^{(n)}\right) y_k^{(n)}, \tag{2.3}$$

$W_{ni}(t) = W_{ni}\left(t, t_1^{(n)}, \cdots, t_n^{(n)}\right)$, $i = 1, 2, \cdots, n$ are measurable weight functions satisfying the following assumptions:

A_1 $W_{ni}(t) \ge 0, 1 \le i \le n, \sum\limits_{i=1}^{n} W_{ni}(t) = 1,\ n \in \boldsymbol{N}$, for any $t \in A$;

A_2 $\max\limits_{1\le k \le n} \sum\limits_{i=1}^{n} W_{nk}\left(t_i^{(n)}\right) \le C_1$;

A_3 $\left|g(t_1) - g(t_2)\right| \le C_2 \|t_1 - t_2\|$, for any $t_1, t_2 \in A$;

A_4 $\liminf_{n\to\infty} \frac{1}{n} S_n^2 \ge C_3$, where $S_n^2 = \sum\limits_{i=1}^{n} \left(\widetilde{x}_i^{(n)}\right)^2$;

A_5 There exist some $\alpha \in \left(0, \frac{1}{2}\right)$ and h_n, $1 \le h_n \le n$, such that

$$\lim_{n\to\infty} h_n = \infty,\ \lim_{n\to\infty} \frac{h_n}{n^{1-2\alpha}} = 0,$$
$$\sum_{i=1}^{n} W_{ni}(t) I\left(\left\|t_i^{(n)} - t\right\| > \frac{h_n}{n}\right) \le C_4 \cdot \frac{h_n}{n},\ \text{for any } t \in A;$$

A_6 $\max\limits_{1\le i\le n} \left|x_i^{(n)}\right| \le C_5 n^{\alpha}$, where α is the same as in A_5;

A_7 $\max\limits_{1\le k\le n} \sup\limits_{t\in A} W_{nk}(t) = O\left(n^{-\delta}\right)$, for some $0 < \delta < 1$;

A_8 $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} W_{nk}^2(t) = 0$, for any $t \in A$;

A_9 $\lim\limits_{n\to\infty} \sup\limits_{t\in A} \sum\limits_{k=1}^{n} W_{nk}^2(t) = 0$.

Remark 2.1. *So far, many scholars have employed the above assumptions about design variables and weights. For instance, Hu (2006) has used the assumptions A_1-A_6 and A_8, Wang et al. (2019) have applied the assumptions A_1-A_6 and A_8-A_9, Wang et al. (2022) have utilized the assumptions A_1-A_7.*

Remark 2.2. *It follows by A_4 that $n^{1-\frac{2}{r}} \sum\limits_{i=1}^{n} \left(\widetilde{x}_i^{(n)}\right)^2 \to \infty$ as $n \to \infty$ for $r > 1$. This conclusion will be required in the proof of Theorem 3.2.*

2.2. Some lemmas

The following lemmas are significant to prove the main results of the paper. The first lemma provides the Rosenthal-type maximum inequality and Marcinkiewicz-Zygmund type maximum inequality applicable to m-ANA random variables, which can be found in Wu et al. (2021)

Lemma 2.1. *Suppose that $\{X_i, i \ge 1\}$ is a sequence of m-ANA random variables with $EX_i = 0$ and $E|X_i|^p < \infty$ for some $p > 1$. Then there exists a positive constant C depending only on p and $\rho^-(\cdot)$ such that for any $n \ge 1$,*

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C\left\{\sum_{i=1}^{n} E\left|X_i\right|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\},\ \textit{for } p \ge 2,$$

and

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C\sum_{i=1}^{n} E\left|X_i\right|^p,\ \textit{for } 1 < p < 2.$$

We acquire the following moment inequality for weighted sums of m-ANA random variables by using the Rosenthal-type inequality for m-ANA random variables. Since the proof is similar to that of Lemma 2.3 in Wang et al. (2019), we omit the details here.

Lemma 2.2. *Let $\{X_i, i \ge 1\}$ be a sequence of m-ANA random variables with $EX_i = 0$ and $\sup\limits_i E|X_i|^p < \infty$ for some $p \ge 2$. Then, for every real array $\{a_{ni}, 1 \le i \le n, n \ge 1\}$, there exists a positive constant C such that*

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni} X_i\right|^p\right) \le C\left(\sum_{i=1}^{n} a_{ni}^2\right)^{\frac{p}{2}},\ \textit{for each } n \ge 1.$$

The next one was exhibited by Hu (2006).

Lemma 2.3. *By A_1, A_3 and A_5, we can get that*

$$\sum_{k=1}^{n} W_{nk}(t)\left|g(t) - g\left(t_k^{(n)}\right)\right| \le \frac{Ch_n}{n},\ \textit{for any } t \in A.$$

To reduce the conditions for proving Theorem 3.2, we propose the following lemma.

Lemma 2.4. *By A_4 and the Hölder's inequality, we can get that for all n large enough,*

$$\sum_{i=1}^{n} \frac{\left|\widetilde{x}_i^{(n)}\right|}{S_n^2} \le C_3^{-\frac{1}{2}} < \infty.$$

Proof. It follows by A_4 and the Hölder's inequality that for all n large enough,

$$\sum_{i=1}^{n} \frac{\left|\widetilde{x}_i^{(n)}\right|}{S_n^2} \le \left(\sum_{i=1}^{n} \left|\widetilde{x}_i^{(n)}\right|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \left|\frac{1}{S_n^2}\right|^2\right)^{\frac{1}{2}}$$
$$= \left(\frac{n}{S_n^2}\right)^{\frac{1}{2}} \le C_3^{-\frac{1}{2}} < \infty.$$

□

3. Main results and their proofs

3.1. Consistency of $\hat{\beta}_n$ and $\hat{g}_n(t)$

In this subsection, we will present our main results and their proofs. The first one is the complete consistency for the linear combination of m-ANA random variables under some suitable conditions.

Theorem 3.1. *Let $\{\varepsilon_i, i \ge 1\}$ be a sequence of m-ANA random variables with $E\varepsilon_i = 0$, $\sup_i E\left|\varepsilon_i\right|^r < \infty$ for some $r \ge 2$. Let $\{c_{ni}, 1 \le i \le n, n \ge 1\}$ be a nonrandom array. If there exists $r \ge 2$ such that*

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} c_{ni}^2\right)^{r/2} < \infty, \tag{3.1}$$

then $\sum_{i=1}^{n} c_{ni}\varepsilon_i$ converges to zero completely.

Proof. For any $\epsilon > 0$, by Markov's inequality, Lemma 2.2 and (3.1) we get

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} c_{ni}\varepsilon_i\right| > \epsilon\right)$$
$$\le \sum_{n=1}^{\infty} E\left|\sum_{i=1}^{n} c_{ni}\varepsilon_i\right|^r / \epsilon^r$$
$$\le \frac{C}{\epsilon^r} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} c_{ni}^2\right)^{r/2} < \infty,$$

thus $\sum_{i=1}^{n} c_{ni}\varepsilon_i$ converges to zero completely. □

Next, the results mentioned in Theorem 3.2 and Theorem 3.3 are the r-th ($r > 1$) mean consistency for the estimators $\hat{\beta}_n$ and $\hat{g}_n(t)$.

Theorem 3.2. *In the model* (1.1), *assume that conditions A_1-A_7 hold. Let $\{\varepsilon_i, i \ge 1\}$ be a sequence of m-ANA random variables with $E\varepsilon_i = 0$. If there exists some $1 < r < 2$ such that*

$$\sup_i E\left|\varepsilon_i\right|^r < \infty, \tag{3.2}$$

then

$$\lim_{n\to\infty} E\left|\hat{\beta}_n - \beta\right|^r = 0. \tag{3.3}$$

Moreover, if we assume that

$$\left|\sum_{i=1}^{n} W_{ni}(t)x_i^{(n)}\right| \le C < \infty, \tag{3.4}$$

then

$$\lim_{n\to\infty} E\left|\hat{g}_n(t) - g(t)\right|^r = 0. \tag{3.5}$$

Proof. For fixed n, we denote $\widetilde{x}_i^{(n)} = \widetilde{x}_i$, $\widetilde{y}_i^{(n)} = \widetilde{y}_i$, $x_i^{(n)} = x_i$, $y_i^{(n)} = y_i$ and $t_i^{(n)} = t_i$, $i = 1, 2, \cdots, n$.
By (2.1)-(2.3), we obtain that

$$\hat{\beta}_n - \beta = \left\{\sum_{i=1}^{n} \widetilde{x}_i \left(g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k) + \varepsilon_i - \sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k\right)\right\} / S_n^2, \tag{3.6}$$

which combining with the C_r-inequality implies that

$$\begin{aligned} E\left|\hat{\beta}_n - \beta\right|^r &= E\left|\left\{\sum_{i=1}^{n} \widetilde{x}_i \left(g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k) + \varepsilon_i - \sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k\right)\right\} / S_n^2\right|^r \\ &\le 3^{r-1}\left|\sum_{i=1}^{n} \widetilde{x}_i \left(g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k)\right) / S_n^2\right|^r \\ &\quad + 3^{r-1}E\left|\sum_{i=1}^{n} \widetilde{x}_i\varepsilon_i / S_n^2\right|^r + 3^{r-1}E\left|\sum_{i=1}^{n} \widetilde{x}_i \sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k / S_n^2\right|^r. \end{aligned} \tag{3.7}$$

Firstly, we will show that

$$\lim_{n\to\infty}\left|\sum_{i=1}^{n} \widetilde{x}_i \left(g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k)\right) / S_n^2\right|^r = 0. \tag{3.8}$$

By A_1, A_5, A_6 and Lemma 2.3, we get that

$$\begin{aligned} &\left|\frac{1}{n}\sum_{i=1}^{n} \widetilde{x}_i \left(g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k)\right)\right| \\ &= \left|\frac{1}{n}\sum_{i=1}^{n} \left(x_i - \sum_{j=1}^{n} W_{nj}(t_i)x_j\right)\left(g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k)\right)\right| \\ &= \left|\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} W_{nj}(t_i)\left(x_i - x_j\right)\left(g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k)\right)\right| \\ &\le \left|\frac{2}{n}\max_{1\le i\le n}|x_i| \sum_{i=1}^{n}\left(g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k)\right)\right| \\ &\le \frac{Cn^{\alpha}}{n}\sum_{i=1}^{n}\sum_{k=1}^{n} W_{nk}(t_i)|g(t_i) - g(t_k)| \\ &\le \frac{Cn^{\alpha}}{n}\sum_{i=1}^{n}\frac{h_n}{n} = C\frac{h_n}{n^{1-\alpha}} \to 0, \ as\ n \to \infty, \end{aligned} \tag{3.9}$$

which combining with A_4 indicates that (3.8) holds. In the following, we will show that

$$\lim_{n\to\infty} E\left|\sum_{i=1}^{n}\widetilde{x}_i\varepsilon_i/S_n^2\right|^r = 0. \tag{3.10}$$

We acquire by the Hölder's inequality, Lemma 2.1, Remark 2.2 and (3.2) that

$$\begin{aligned}
E\left|\sum_{i=1}^{n}\widetilde{x}_i\varepsilon_i/S_n^2\right|^r &\leq \frac{C}{S_n^{2r}}\sum_{i=1}^{n}E\left|\widetilde{x}_i\varepsilon_i\right|^r \\
&= CS_n^{-2r}\sum_{i=1}^{n}|\widetilde{x}_i|^r E|\varepsilon_i|^r \\
&\leq CS_n^{-2r}\left(\sum_{i=1}^{n}\widetilde{x}_i^2\right)^{\frac{r}{2}}\left(\sum_{i=1}^{n}1\right)^{1-\frac{r}{2}}\sup_i E|\varepsilon_i|^r \\
&\leq CS_n^{-2r}n^{1-\frac{r}{2}}\left(\sum_{i=1}^{n}\widetilde{x}_i^2\right)^{\frac{r}{2}} \\
&\leq Cn^{1-\frac{r}{2}}S_n^{-r} \\
&= C\left(n^{1-\frac{2}{r}}S_n^2\right)^{-\frac{r}{2}} \to 0, \ as\ n\to\infty,
\end{aligned}$$

which implies that (3.10) holds. Now, we will prove that

$$\lim_{n\to\infty} E\left|\sum_{i=1}^{n}\widetilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k/S_n^2\right|^r = 0. \tag{3.11}$$

Noting that $f(x) = |x|^r$ is a convex function for $r > 1$, we can get that

$$\begin{aligned}
&E\left|\sum_{i=1}^{n}\widetilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k/S_n^2\right|^r \\
&= E\left|\sum_{i=1}^{n}\frac{\widetilde{x}_i}{\sum_{j=1}^{n}|\widetilde{x}_j|}\sum_{j=1}^{n}|\widetilde{x}_j|\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k/S_n^2\right|^r \\
&= \left(\frac{\sum_{j=1}^{n}|\widetilde{x}_j|}{S_n^2}\right)^r E\left|\sum_{i=1}^{n}\frac{\widetilde{x}_i}{\sum_{j=1}^{n}|\widetilde{x}_j|}\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|^r \\
&\leq \left(\frac{\sum_{j=1}^{n}|\widetilde{x}_j|}{S_n^2}\right)^r E\left(\sum_{i=1}^{n}\frac{|\widetilde{x}_i|}{\sum_{j=1}^{n}|\widetilde{x}_j|}\left|\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|\right)^r \\
&\leq \left(\frac{\sum_{j=1}^{n}|\widetilde{x}_j|}{S_n^2}\right)^r \sum_{i=1}^{n}\frac{|\widetilde{x}_i|}{\sum_{j=1}^{n}|\widetilde{x}_j|}E\left|\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|^r \\
&\leq \left(\frac{\sum_{j=1}^{n}|\widetilde{x}_j|}{S_n^2}\right)^r \max_{1\leq i\leq n}E\left|\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|^r.
\end{aligned}$$

By Lemma 2.1, (3.2), A_1 and A_7, we can see that

$$\begin{aligned}
&\max_{1\le i\le n} E\left|\sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k\right|^r \\
&\le \sup_{t\in A} E\left|\sum_{k=1}^{n} W_{nk}(t)\varepsilon_k\right|^r \\
&\le C\sup_{t\in A}\sum_{k=1}^{n} E\left|W_{nk}(t)\varepsilon_k\right|^r \\
&\le C\sup_{t\in A}\sum_{k=1}^{n} W_{nk}^r(t)\sup_k E\left|\varepsilon_k\right|^r \\
&\le C\left(\max_{1\le k\le n}\sup_{t\in A} W_{nk}(t)\right)^{r-1}\sum_{k=1}^{n} W_{nk}(t) \\
&\le Cn^{-\delta(r-1)} \to 0,\ as\ n\to\infty,
\end{aligned} \tag{3.12}$$

which together with Lemma 2.4 yields that (3.11) holds. Through (3.8), (3.10) and (3.11), it is clearly seen that (3.3) holds.

Next, we will prove (3.5). From (2.1)-(2.3), we can gain that

$$\begin{aligned}
&\hat{g}_n(t) - g(t) \\
&= \sum_{i=1}^{n} W_{ni}(t)\left(y_i - x_i\hat{\beta}_n\right) - g(t) \\
&= \sum_{i=1}^{n} W_{ni}(t)\left(x_i\beta + g(t_i) + \varepsilon_i - x_i\hat{\beta}_n\right) - g(t) \\
&= \sum_{i=1}^{n} W_{ni}(t)x_i\left(\beta - \hat{\beta}_n\right) + \sum_{i=1}^{n} W_{ni}(t)\left(g(t_i) - g(t)\right) + \sum_{i=1}^{n} W_{ni}(t)\varepsilon_i.
\end{aligned} \tag{3.13}$$

Hence, by the C_r-inequality and the above equality, we have

$$\begin{aligned}
&E\left|\hat{g}_n(t) - g(t)\right|^r \\
&= E\left|\sum_{i=1}^{n} W_{ni}(t)x_i\left(\beta - \hat{\beta}_n\right) + \sum_{i=1}^{n} W_{ni}(t)\left(g(t_i) - g(t)\right) + \sum_{i=1}^{n} W_{ni}(t)\varepsilon_i\right|^r \\
&\le 3^{r-1}E\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta - \hat{\beta}_n)\right|^r + 3^{r-1}\left|\sum_{i=1}^{n} W_{ni}(t)(g(t_i) - g(t))\right|^r + 3^{r-1}E\left|\sum_{i=1}^{n} W_{ni}(t)\varepsilon_i\right|^r.
\end{aligned} \tag{3.14}$$

Then, we will prove that

$$\lim_{n\to\infty} E\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta - \hat{\beta}_n)\right|^r = 0. \tag{3.15}$$

It follows by (3.4) that

$$\begin{aligned}
E\left|\sum_{i=1}^{n} W_{ni}(t)x_i\left(\beta - \hat{\beta}_n\right)\right|^r &= \left|\sum_{i=1}^{n} W_{ni}(t)x_i\right|^r E\left|\beta - \hat{\beta}_n\right|^r \\
&\le CE\left|\beta - \hat{\beta}_n\right|^r,
\end{aligned}$$

which combining with (3.3) implies that (3.15) holds. In the following, we will show that

$$\lim_{n\to\infty}\left|\sum_{i=1}^{n} W_{ni}(t)\left(g(t_i)-g(t)\right)\right|^{r}=0. \tag{3.16}$$

By A_5 and Lemma 2.3, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} W_{ni}(t)\left|g(t_i)-g(t)\right|\le\lim_{n\to\infty}C\frac{h_n}{n}=0,$$

which means that (3.16) holds. And it follows by (3.12) that

$$E\left|\sum_{i=1}^{n} W_{ni}(t)\varepsilon_i\right|^{r}\to 0,\ as\ n\to\infty. \tag{3.17}$$

Therefore, we can acquire that (3.5) follows from (3.15)-(3.17) immediately. This completes the proof of the theorem. □

Remark 3.1. *Comparing with Theorem 3.1 in Wang et al. (2022), we have the following promotions and improvements:*

(1) The results of the r-th $(1<r<2)$ *mean consistency for the estimators* $\hat{\beta}_n$ *and* $\hat{g}_n(t)$ *based on* φ*-mixing errors are generalized to that based on m-ANA random errors;*

(2) In this paper, the condition $\sum_{i=1}^{n}\left|\widetilde{x}_i^{(n)}\right|/S_n^2\le C<\infty$*, which was used in Wang et al. (2022) to prove the r-th* $(1<r<2)$ *mean consistency, is deleted. In addition, the condition* $\sum_{i=1}^{n}\left|W_{ni}(t)x_i^{(n)}\right|\le C<\infty$ *in Wang et al. (2022) is weakened to* $\left|\sum_{i=1}^{n}W_{ni}(t)x_i^{(n)}\right|\le C<\infty$*.*

Theorem 3.3. *In the model* (1.1)*, assume that conditions* A_1*-*A_6 *and* A_8 *hold. Let* $\{\varepsilon_i,\ i\ge 1\}$ *be a sequence of m-ANA random variables with* $E\varepsilon_i=0$, $\sup_i E\left|\varepsilon_i\right|^r<\infty$ *for some* $r\ge 2$.

(i) If A_5 *and* A_6 *hold for some* $\alpha\in(0,\frac{1}{2})$*, then*

$$\lim_{n\to\infty}E\left|\hat{\beta}_n-\beta\right|^{r}=0. \tag{3.18}$$

(ii) If A_5 *and* A_6 *hold for some* $\alpha\in(0,\frac{1}{4})$*, then*

$$\lim_{n\to\infty}E\left|\hat{g}_n(t)-g(t)\right|^{r}=0. \tag{3.19}$$

Proof. At first, we will prove (3.18). For fixed n, we denote $\widetilde{x}_i^{(n)}=\widetilde{x}_i$, $\widetilde{y}_i^{(n)}=\widetilde{y}_i$, $x_i^{(n)}=x_i$, $y_i^{(n)}=y_i$ and $t_i^{(n)}=t_i$, $i=1,2,\cdots,n$. By (3.7), it is evident to see that we just need to prove that (3.8), (3.10) and (3.11) hold under the condition $r\ge 2$. We gain by (3.9) and A_4 that

$$\lim_{n\to\infty}\left|\sum_{i=1}^{n}\widetilde{x}_i\left(g(t_i)-\sum_{k=1}^{n}W_{nk}(t_i)g(t_k)\right)/S_n^2\right|=0. \tag{3.20}$$

Next, we shall prove that

$$\lim_{n\to\infty}E\left|\sum_{i=1}^{n}\widetilde{x}_i\varepsilon_i/S_n^2\right|^{r}=0. \tag{3.21}$$

It follows from A_1 and A_6 that

$$|\widetilde{x}_i|\le|x_i|+\sum_{k=1}^{n}W_{nk}(t_i)\max_{1\le k\le n}|x_k|\le Cn^{\alpha},$$

and by Lemma 2.2, we can get that

$$E\left|\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\varepsilon_i\right|^r \leq C\left(\frac{1}{n^2}\sum_{i=1}^{n}\widetilde{x}_i^2\right)^{r/2} \leq C\left(\frac{1}{n^{1-2\alpha}}\right)^{r/2} \to 0,\ as\ n\to\infty, \tag{3.22}$$

which combining with A_4 yields that (3.21) holds. Then, we will exhibit that

$$\lim_{n\to\infty} E\left|\sum_{i=1}^{n}\widetilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k/S_n^2\right|^r = 0. \tag{3.23}$$

Through A_2 and A_6, we can get that

$$\left|\sum_{i=1}^{n}\widetilde{x}_iW_{nk}(t_i)\right| \leq \max_{1\leq i\leq n}\left|\widetilde{x}_i\right| \max_{1\leq k\leq n}\sum_{i=1}^{n}W_{nk}(t_i) \leq Cn^{\alpha},$$

and by Lemma 2.2, we have

$$\begin{aligned} &E\left|\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|^r \\ &\leq C\left[\frac{1}{n^2}\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\widetilde{x}_iW_{nk}(t_i)\right)^2\right]^{r/2} \\ &\leq C\left(\frac{1}{n^{1-2\alpha}}\right)^{r/2} \to 0,\ as\ n\to\infty, \end{aligned} \tag{3.24}$$

which together with A_4 yields that (3.23) holds. Therefore, it is easily checked that (3.18) holds.

We now prove (3.19). It follows by (3.14) that we just need to prove that (3.15), (3.16) and (3.17) hold under the condition $r \geq 2$. At first, we will show that

$$\lim_{n\to\infty} E\left|\sum_{i=1}^{n}W_{ni}(t)x_i(\beta-\hat{\beta}_n)\right|^r = 0. \tag{3.25}$$

It follows by A_1, A_6, (3.6) and the C_r-inequality that

$$\begin{aligned} &E\left|\sum_{i=1}^{n}W_{ni}(t)x_i\left(\beta-\hat{\beta}_n\right)\right|^r \\ &\leq E\left(\sum_{i=1}^{n}W_{ni}(t)\max_{1\leq i\leq n}|x_i|\left|\beta-\hat{\beta}_n\right|\right)^r \\ &\leq CE\left|n^{\alpha}\left(\beta-\hat{\beta}_n\right)\right|^r \\ &\leq C\left|\frac{n^{\alpha}}{S_n^2}\sum_{i=1}^{n}\widetilde{x}_i\left(g(t_i)-\sum_{k=1}^{n}W_{nk}(t_i)g(t_k)\right)\right|^r + CE\left|\frac{n^{\alpha}}{S_n^2}\sum_{i=1}^{n}\widetilde{x}_i\varepsilon_i\right|^r + CE\left|\frac{n^{\alpha}}{S_n^2}\sum_{i=1}^{n}\widetilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|^r, \end{aligned} \tag{3.26}$$

and by A_5, (3.9), (3.22) and (3.24), we have

$$n^{\alpha}\cdot\left|\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\left(g(t_i)-\sum_{k=1}^{n}W_{nk}(t_i)g(t_k)\right)\right| \leq C\frac{h_n}{n^{1-2\alpha}} \to 0,\ as\ n\to\infty, \tag{3.27}$$

$$E\left|n^{\alpha}\cdot\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\varepsilon_i\right|^r\le C\left(\frac{1}{n^{1-4\alpha}}\right)^{r/2}\to 0,\ as\ n\to\infty, \tag{3.28}$$

$$E\left|n^{\alpha}\cdot\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|^r\le C\left(\frac{1}{n^{1-4\alpha}}\right)^{r/2}\to 0,\ as\ n\to\infty, \tag{3.29}$$

which together with A_4 means that (3.25) holds. Then, we will prove that

$$\lim_{n\to\infty}E\left|\sum_{i=1}^{n}W_{ni}(t)\varepsilon_i\right|^r=0. \tag{3.30}$$

It follows by Lemma 2.2 that

$$E\left|\sum_{i=1}^{n}W_{ni}(t)\varepsilon_i\right|^r\le C\left(\sum_{i=1}^{n}W_{ni}^2(t)\right)^{r/2},$$

which combining with A_8 implies that (3.30) holds. According to (3.16) and the above proofs, we can easily get that (3.19) is established. This completes the proof of the theorem. □

Remark 3.2. *Comparing with Theorem 3.2 in Wang et al. (2019), we generalize the r-th* ($r\geqslant 2$) *mean consistency for the estimators $\hat{\beta}_n$ and $\hat{g}_n(t)$ based on φ-mixing errors to the case of m-ANA random errors.*

The following result is the complete consistency ($r>2$) for the estimators $\hat{\beta}_n$ and $\hat{g}_n(t)$.

Theorem 3.4. *In the model* (1.1), *let $\{\varepsilon_i,\ i\ge 1\}$ be a sequence of m-ANA random variables with $E\varepsilon_i=0$, $\sup_i E|\varepsilon_i|^r<\infty$ for some $r>2$. Further we assume that A_1-A_6 hold.*

(i) *If A_5 and A_6 hold for some $\alpha\in(0,\frac{1}{2}-\frac{1}{r})$, then $\hat{\beta}_n$ converges to β completely, and thus, $\hat{\beta}_n\to\beta$ a.s..*

(ii) *If A_5 and A_6 hold for some $\alpha\in(0,\frac{1}{4}-\frac{1}{2r})$, and*

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}W_{ni}^2(t)\right)^{r/2}<\infty, \tag{3.31}$$

then $\hat{g}_n(t)$ converges to $g(t)$ completely, and thus, $\hat{g}_n(t)\to g(t)$ a.s..

Proof. For fixed n, we denote $\widetilde{x}_i^{(n)}=\widetilde{x}_i$, $\widetilde{y}_i^{(n)}=\widetilde{y}_i$, $x_i^{(n)}=x_i$, $y_i^{(n)}=y_i$ and $t_i^{(n)}=t_i$, $i=1,2,\cdots,n$.

(i) For any $\epsilon>0$, by Markov's inequality, (3.22) and (3.24), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\varepsilon_i\right|>\epsilon\right)\\&\le\sum_{n=1}^{\infty}E\left|\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\varepsilon_i\right|^r/\epsilon^r\\&\le\frac{C}{\epsilon^r}\sum_{n=1}^{\infty}\left(\frac{1}{n^{1-2\alpha}}\right)^{r/2}<\infty,\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\sum_{n=1}^{\infty}P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|>\epsilon\right)\\&\le\sum_{n=1}^{\infty}E\left|\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|^r/\epsilon^r\\&\le\frac{C}{\epsilon^r}\sum_{n=1}^{\infty}\left(\frac{1}{n^{1-2\alpha}}\right)^{r/2}<\infty.\end{aligned} \tag{3.33}$$

By (3.32), (3.33) and A_4, for any $\epsilon > 0$, we can attain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} \tilde{x}_i \varepsilon_i\right| / S_n^2 > \epsilon\right) \\
&= \sum_{n=1}^{\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i \varepsilon_i\right| > \frac{S_n^2}{n} \cdot \epsilon\right) \\
&\leq C \sum_{n=1}^{\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i \varepsilon_i\right| > C_3 \cdot \epsilon\right) < \infty,
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} \tilde{x}_i \sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k\right| / S_n^2 > \epsilon\right) \\
&= \sum_{n=1}^{\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i \sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k\right| > \frac{S_n^2}{n} \cdot \epsilon\right) \\
&\leq C \sum_{n=1}^{\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i \sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k\right| > C_3 \cdot \epsilon\right) < \infty.
\end{aligned} \tag{3.35}$$

It follows by (3.6), (3.20), (3.34) and (3.35) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\beta - \hat{\beta}_n\right| > \epsilon\right) \\
&\leq \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} \tilde{x}_i (g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i) g(t_k))\right| / S_n^2 > \epsilon/3\right) + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} \tilde{x}_i \varepsilon_i\right| / S_n^2 > \epsilon/3\right) \\
&\quad + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} \tilde{x}_i \sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k\right| / S_n^2 > \epsilon/3\right) < \infty,
\end{aligned}$$

which implies that $\hat{\beta}_n$ converges to β completely.

(*ii*) We will prove that $\hat{g}_n(t)$ converges to $g(t)$ completely. Similar to the proof of (3.32) and (3.33), we have by Markov's inequality, (3.28) and (3.29) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|n^{\alpha} \cdot \frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i \varepsilon_i\right| > \epsilon\right) \\
&\leq \sum_{n=1}^{\infty} E\left|n^{\alpha} \cdot \frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i \varepsilon_i\right|^r / \epsilon^r \\
&\leq \frac{C}{\epsilon^r} \sum_{n=1}^{\infty} \left(\frac{1}{n^{1-4\alpha}}\right)^{r/2} < \infty,
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|n^{\alpha}\cdot\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|>\epsilon\right)\\
&\leq\sum_{n=1}^{\infty} E\left|n^{\alpha}\cdot\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|^{r}/\epsilon^{r}\\
&\leq\frac{C}{\epsilon^{r}}\sum_{n=1}^{\infty}\left(\frac{1}{n^{1-4\alpha}}\right)^{r/2}<\infty.
\end{aligned}\tag{3.37}$$

By (3.36), (3.37) and A_4, for any $\epsilon>0$, it is easily seen that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|n^{\alpha}\cdot\sum_{i=1}^{n}\tilde{x}_i\varepsilon_i\right|/S_n^2>\epsilon\right)\\
&=\sum_{n=1}^{\infty} P\left(\left|n^{\alpha}\cdot\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\varepsilon_i\right|>\frac{S_n^2}{n}\cdot\epsilon\right)\\
&\leq C\sum_{n=1}^{\infty} P\left(\left|n^{a}\cdot\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\varepsilon_i\right|>C_3\cdot\epsilon\right)<\infty,
\end{aligned}\tag{3.38}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|n^{\alpha}\cdot\sum_{i=1}^{n}\tilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|/S_n^2>\epsilon\right)\\
&=\sum_{n=1}^{\infty} P\left(\left|n^{\alpha}\cdot\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|>\frac{S_n^2}{n}\cdot\epsilon\right)\\
&\leq C\sum_{n=1}^{\infty} P\left(\left|n^{\alpha}\cdot\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|>C_3\cdot\epsilon\right)<\infty.
\end{aligned}\tag{3.39}$$

Hence, we have by (3.6), (3.27), (3.38), (3.39), A_1 and A_6 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n}W_{ni}(t)x_i(\beta-\hat{\beta}_n)\right|>\epsilon\right)\\
&\leq\sum_{n=1}^{\infty} P\left(\left|C_5 n^{\alpha}\cdot(\beta-\hat{\beta}_n)\right|>\epsilon\right)\\
&\leq\sum_{n=1}^{\infty} P\left(\left|C_5 n^{\alpha}\cdot\sum_{i=1}^{n}\tilde{x}_i\left(g(t_i)-\sum_{k=1}^{n}W_{nk}(t_i)g(t_k)\right)\right|/S_n^2>\epsilon/3\right)\\
&\quad+\sum_{n=1}^{\infty} P\left(\left|C_5 n^{\alpha}\cdot\sum_{i=1}^{n}\tilde{x}_i\varepsilon_i\right|/S_n^2>\epsilon/3\right)\\
&\quad+\sum_{n=1}^{\infty} P\left(\left|C_5 n^{\alpha}\cdot\sum_{i=1}^{n}\tilde{x}_i\sum_{k=1}^{n}W_{nk}(t_i)\varepsilon_k\right|/S_n^2>\epsilon/3\right)<\infty.
\end{aligned}\tag{3.40}$$

Through Markov's inequality, Lemma 2.2 and (3.31), it is apparent to check that

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n}W_{ni}(t)\varepsilon_i\right|>\epsilon\right)\leq\frac{C}{\epsilon^{r}}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}W_{ni}^2(t)\right)^{r/2}<\infty,\ for\ any\ \epsilon>0.\tag{3.41}$$

Meanwhile, by (3.13), (3.16), (3.40) and (3.41), we can acquire that for all n large enough, $\left|\sum_{i=1}^{n} W_{ni}(t)(g(t_i)-g(t))\right| \le \epsilon/3$ and thus

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\hat{g}_n(t)-g(t)\right| > \epsilon\right) \\
\le &\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta-\hat{\beta}_n)\right| > \epsilon/3\right) \\
&+\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}(t)\varepsilon_i\right| > \epsilon/3\right) < \infty,
\end{aligned}$$

which implies that $\hat{g}_n(t)$ converges to $g(t)$ completely. This completes the proof of the theorem. □

Remark 3.3. *Comparing with Theorem 3.3 in Wang et al. (2019), we generalize the complete consistency* ($r > 2$) *for the estimators $\hat{\beta}_n$ and $\hat{g}_n(t)$ based on φ-mixing errors to the case of m-ANA random errors.*

3.2. Uniform consistency of $\hat{g}_n(t)$

In this subsection, we will present the uniform r-th ($r > 1$) mean consistency for the estimator $\hat{g}_n(t)$.

Theorem 3.5. *In the model (1.1), assume that the conditions of Theorem 3.2 hold. If*

$$\sup_{t\in A}\left|\sum_{i=1}^{n} W_{ni}(t)x_i^{(n)}\right| \le C < \infty, \tag{3.42}$$

then

$$\lim_{n\to\infty}\sup_{t\in A} E\left|\hat{g}_n(t)-g(t)\right|^r = 0. \tag{3.43}$$

Proof. For fixed n, we denote $\widetilde{x}_i^{(n)} = \widetilde{x}_i$, $\widetilde{y}_i^{(n)} = \widetilde{y}_i$, $x_i^{(n)} = x_i$, $y_i^{(n)} = y_i$, $t_i^{(n)} = t_i$, $i = 1, 2, \cdots, n$. By (3.13) and the C_r-inequality, we have

$$\begin{aligned}
&\sup_{t\in A} E\left|\hat{g}_n(t)-g(t)\right|^r \\
=&\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta-\hat{\beta}_n)+\sum_{i=1}^{n} W_{ni}(t)(g(t_i)-g(t))+\sum_{i=1}^{n} W_{ni}(t)\varepsilon_i\right|^r \\
\le& 3^{r-1}\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta-\hat{\beta}_n)\right|^r + 3^{r-1}\sup_{t\in A}\left|\sum_{i=1}^{n} W_{ni}(t)(g(t_i)-g(t))\right|^r \\
&+3^{r-1}\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)\varepsilon_i\right|^r .
\end{aligned}\tag{3.44}$$

Firstly, we shall prove that

$$\lim_{n\to\infty}\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta-\hat{\beta}_n)\right|^r = 0. \tag{3.45}$$

It follows by (3.42) and (3.3) that

$$\begin{aligned}
&\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta-\hat{\beta}_n)\right|^r \\
\le& E\left(\sup_{t\in A}\left|\sum_{i=1}^{n} W_{ni}(t)x_i\right|^r \left|\beta-\hat{\beta}_n\right|^r\right) \\
\le& CE|\beta-\hat{\beta}_n|^r \to 0, \ as\ n\to\infty,
\end{aligned}$$

which indicates that (3.45) holds. In the following, we will prove

$$\lim_{n\to\infty}\sup_{t\in A}\left|\sum_{i=1}^{n} W_{ni}(t)(g(t_i)-g(t))\right|^r = 0. \tag{3.46}$$

We have by Lemma 2.3 and A_5 that

$$\sup_{t\in A}\sum_{i=1}^{n} W_{ni}(t)|g(t_i)-g(t)| \le C\frac{h_n}{n} \to 0,\ as\ n\to\infty,$$

which yields that (3.46) holds. It follows by (3.12) that

$$\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)\varepsilon_i\right|^r \le Cn^{-\delta(r-1)} \to 0,\ as\ n\to\infty. \tag{3.47}$$

Thus, from all the statements above, (3.43) has been exhibited from (3.45)-(3.47). This completes the proof of the theorem. □

Remark 3.4. *Comparing with Theorem 3.2 in Wang et al. (2022), we generalize the uniform r-th* $(1 < r < 2)$ *mean consistency for the estimator* $\hat{g}_n(t)$ *based on* φ*-mixing errors to the case of m-ANA random errors and we also delete the condition* $\sum_{i=1}^{n}\left|\widetilde{x}_i^{(n)}\right|/S_n^2 \le C < \infty$.

Theorem 3.6. *In the model (1.1), assume that the conditions of Theorem 3.3 hold and condition* A_8 *is replaced by* A_9*. Then*

$$\lim_{n\to\infty}\sup_{t\in A} E\left|\hat{g}_n(t)-g(t)\right|^r = 0. \tag{3.48}$$

Proof. For fixed n, we denote $\widetilde{x}_i^{(n)} = \widetilde{x}_i$, $\widetilde{y}_i^{(n)} = \widetilde{y}_i$, $x_i^{(n)} = x_i$, $y_i^{(n)} = y_i$ and $t_i^{(n)} = t_i$, $i = 1, 2, \cdots, n$. By (3.44), it is easily seen that we just need to prove that (3.45), (3.46) and (3.47) hold under the condition $r \ge 2$. It follows by (3.26) that

$$\begin{aligned}
&\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta-\hat{\beta}_n)\right|^r \\
&\le C\left|\frac{n^\alpha}{S_n^2}\sum_{i=1}^{n}\widetilde{x}_i\left(g(t_i)-\sum_{k=1}^{n} W_{nk}(t_i)g(t_k)\right)\right|^r \\
&\quad + CE\left|\frac{n^\alpha}{S_n^2}\sum_{i=1}^{n}\widetilde{x}_i\varepsilon_i\right|^r + CE\left|\frac{n^\alpha}{S_n^2}\sum_{i=1}^{n}\widetilde{x}_i\sum_{k=1}^{n} W_{nk}(t_i)\varepsilon_k\right|^r,
\end{aligned}$$

which together with (3.27)-(3.29) obtains

$$\lim_{n\to\infty}\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)x_i(\beta-\hat{\beta}_n)\right|^r = 0. \tag{3.49}$$

It follows by Lemma 2.3 that

$$\lim_{n\to\infty}\sup_{t\in A}\left|\sum_{i=1}^{n} W_{ni}(t)(g(t_i)-g(t))\right|^r \le \lim_{n\to\infty}\left(\frac{Ch_n}{n}\right)^r = 0. \tag{3.50}$$

Through A_9 and Lemma 2.2, we can learn that

$$\lim_{n\to\infty}\sup_{t\in A} E\left|\sum_{i=1}^{n} W_{ni}(t)\varepsilon_i\right|^r \le C\lim_{n\to\infty}\sup_{t\in A}\left(\sum_{i=1}^{n} W_{ni}^2(t)\right)^{r/2} = 0. \tag{3.51}$$

Thus, (3.48) follows by (3.49)-(3.51) immediately. This completes the proof of the theorem. □

Remark 3.5. *Comparing with Theorem 3.4 in Wang et al. (2019), we generalize the uniform r-th* $(r \geqslant 2)$ *mean consistency for the estimator* $\hat{g}_n(t)$ *based on* φ*-mixing errors to the case of m-ANA random errors.*

4. Numerical simulation

In this section, we are going to do a numerical simulation to investigate the numerical performance of the consistency for the least squares estimator $\hat{\beta}_n$ and the nearest neighbor weight function estimator $\hat{g}_n(t)$ based on m-ANA samples. To start with, it is necessary for us to generate the data that we need to use. For any $n \geq 3$ and $m \geq 1$, assume $(e_1, e_2, \cdots, e_{n+m}) \sim N_{n+m}(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{0}$ is a zero vector and

$$\mathbf{\Sigma} = \begin{pmatrix} 1+\frac{1}{n+m} & -\theta & 0 & \cdots & 0 & 0 & 0 \\ -\theta & 1+\frac{2}{n+m} & -\theta & \cdots & 0 & 0 & 0 \\ 0 & -\theta & 1+\frac{3}{n+m} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\frac{n+m-2}{n+m} & -\theta & 0 \\ 0 & 0 & 0 & \cdots & -\theta & 1+\frac{n+m-1}{n+m} & -\theta \\ 0 & 0 & 0 & \cdots & 0 & -\theta & 1+1 \end{pmatrix}_{(n+m)\times(n+m)},$$

where $0 < \theta < 1$. It is evident that $(e_1, e_2, \cdots, e_{n+m})$ is a NA vector for each $n \geq 3$ and $m \geq 1$ with finite moment of any order in accordance with Joag-Dev and Proschan (1983). For fixed positive integer m and $1 \leq i \leq n$, let $\varepsilon_i = \sum_{k=1}^{m} e_{i+k-1}$. And we can prove that $\{\varepsilon_i, 1 \leq i \leq n\}$ is a sequence of m-ANA random variables.

Furthermore, let us recall the concept of the nearest neighbor weight function estimator as follows. Put $A = [0, 1]$ and $t_i^{(n)} = \frac{i}{n}$, $x_i^{(n)} = (-1)^i \frac{i}{n}$, $1 \leq i \leq n$. For any $t \in A$, we rewrite

$$|t_1^{(n)} - t|, \ |t_2^{(n)} - t|, \cdots, |t_n^{(n)} - t|$$

as follows:

$$|t_{R_1(t)}^{(n)} - t| \leq |t_{R_2(t)}^{(n)} - t| \leq \cdots \leq |t_{R_n(t)}^{(n)} - t|,$$

if $|t_i^{(n)} - t| = |t_j^{(n)} - t|$, then $|t_i^{(n)} - t|$ is permuted before $|t_j^{(n)} - t|$ when $i < j$.

Let $1 \leq k_n \leq n$, the nearest neighbor weight functions are defined as follows:

$$\tilde{W}_{ni}(t) = \begin{cases} \frac{1}{k_n}, & \text{if } |t_i^{(n)} - t| \leq |t_{R_{k_n}(t)}^{(n)} - t|, \\ 0, & \text{otherwise.} \end{cases}$$

Let $k_n = \lfloor n^{0.8} \rfloor$, $h_n = \lfloor n^{0.4} \rfloor$, $m = 10$ and $\theta = 0.5$. It can be simply proved that all the conditions of the theorems in our paper will be satisfied as long as we choose $\alpha > 0$ sufficiently small.

Next, we will consider the simulation for the difference between the estimator $\hat{\beta}_n$ and the parameter β. Taking the sample sizes n as $n = 200, 400, 900, 1600$ respectively, we compute $\hat{\beta}_n - \beta$ with $\beta = 1.5$, 2.5 and 3.5 under the condition of $g(t) = t^6 \sin(3t) + \ln(t - \cos t + 6)$ with 1000 replications and obtain the boxplots of $\hat{\beta}_n - \beta$ in Figures 1-3 by using R software.

Figures 1, 2 and 3 are the boxplots of $\hat{\beta}_n - \beta$ with $\beta = 1.5$, 2.5 and 3.5 respectively. It is evident that no matter $\beta = 1.5$, $\beta = 2.5$ or $\beta = 3.5$, the values of $\hat{\beta}_n - \beta$ fluctuate around zero and the fluctuation ranges decrease as the sample size n increases.

Then, we will consider the simulation for the difference between the estimator $\hat{g}_n(t)$ and the function $g(t)$. To reflect the generality of $g(t)$ selection, we choose $g(t) = t^6 \sin(3t) + \ln(t - \cos t + 6)$ which contains trigonometric, power and logarithmic functions, and $g(t) = (t^5 + 2t + 7)e^t + \sin t$ which contains trigonometric, power and exponential functions, respectively. Taking the sample sizes n as $n = 200, 400, 900, 1600$ and the points $t = 0.3, 0.6, 0.9$ respectively, we compute $\hat{g}_n(t) - g(t)$ with $g(t) = t^6 \sin(3t) + \ln(t - \cos t + 6)$ and $g(t) = (t^5 + 2t + 7)e^t + \sin t$ under the condition of β =2.5 with 1000 replications and obtain the boxplots of $\hat{g}_n(t) - g(t)$ in Figures 4-9 by using R software.

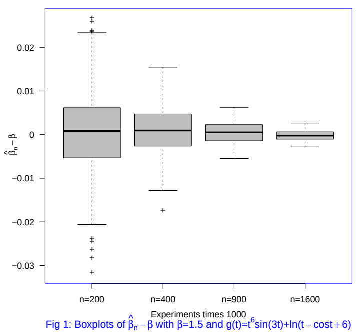

Fig 1: Boxplots of $\hat{\beta}_n-\beta$ with β=1.5 and g(t)=t^6sin(3t)+ln(t−cost+6)

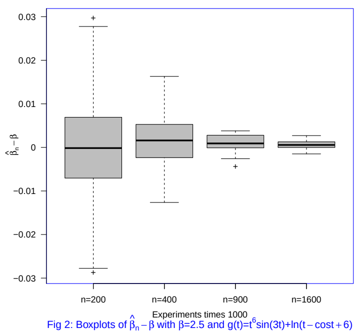

Fig 2: Boxplots of $\hat{\beta}_n-\beta$ with β=2.5 and g(t)=t^6sin(3t)+ln(t−cost+6)

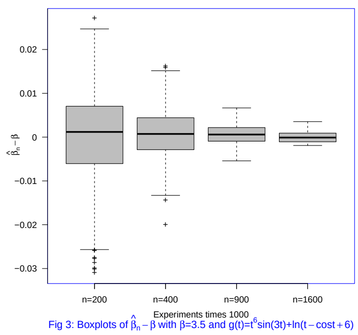

Fig 3: Boxplots of $\hat{\beta}_n-\beta$ with β=3.5 and g(t)=t^6sin(3t)+ln(t−cost+6)

Figures 4, 5 and 6 are the boxplots of $\hat{g}_n(t) - g(t)$ for $g(t) = t^6 \sin(3t) + \ln(t - \cos t + 6)$ and Figures 7, 8 and 9 are the boxplots of $\hat{g}_n(t) - g(t)$ for $g(t) = (t^5 + 2t + 7)e^t + \sin t$ with the points $t = 0.3, 0.6, 0.9$ respectively. It is easy to find that no matter $g(t) = t^6 \sin(3t) + \ln(t - \cos t + 6)$ or $g(t) = (t^5 + 2t + 7)e^t + \sin t$, for the fixed points $t = 0.3, 0.6, 0.9$, the values of $\hat{g}_n(t) - g(t)$ fluctuate around zero and the fluctuation ranges decrease as the sample size n increases. These simulations verify the validity of our theoretical results.

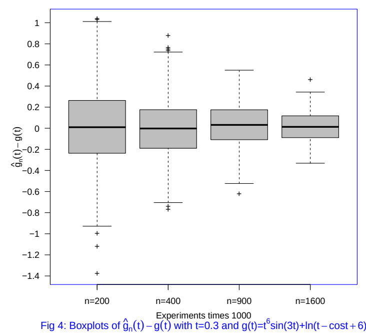

Fig 4: Boxplots of $\hat{g}_n(t)-g(t)$ with t=0.3 and g(t)=t^6sin(3t)+ln(t−cost+6)

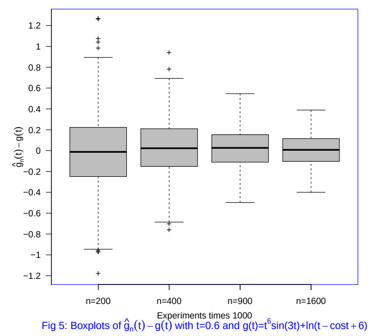

Fig 5: Boxplots of $\hat{g}_n(t)-g(t)$ with t=0.6 and g(t)=t^6sin(3t)+ln(t−cost+6)

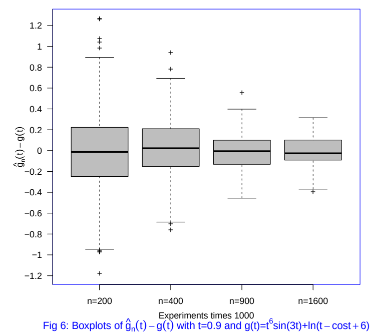

Fig 6: Boxplots of $\hat{g}_n(t)-g(t)$ with t=0.9 and g(t)=t^6sin(3t)+ln(t−cost+6)

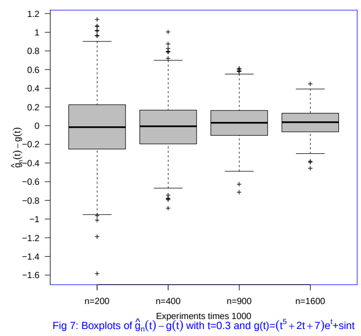

Fig 7: Boxplots of $\hat{g}_n(t)-g(t)$ with t=0.3 and g(t)=$(t^5+2t+7)e^t$+sint

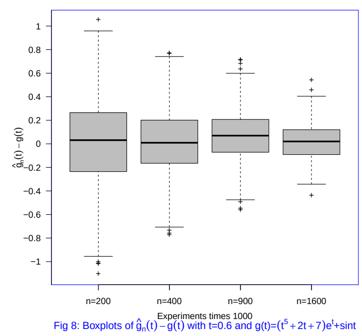

Fig 8: Boxplots of $\hat{g}_n(t)-g(t)$ with t=0.6 and g(t)=$(t^5+2t+7)e^t$+sint

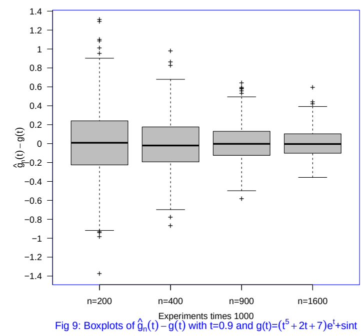

Fig 9: Boxplots of $\hat{g}_n(t)-g(t)$ with t=0.9 and g(t)=$(t^5+2t+7)e^t$+sint

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