Filomat 37:8 (2023), 2421–2435
https://doi.org/10.2298/FIL2308421R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Orlicz-Lacunary bicomplex sequence spaces of difference operators

**Kuldip Raj[a], Ayhan Esi[b], Charu Sharma[a]**

[a]*School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K, India*
[b]*Malatya Turgut Ozal University, Dept.of Basic Eng.Sci. (Math.Section) 44100, Malatya, Türkiye*

**Abstract.** In the present paper we introduce and study some lacunary difference bicomplex sequence spaces by means of Orlicz functions. We make an effort to study some algebraic and topological properties of these sequence spaces. We also show that these spaces are complete paranormed spaces. Further, some inclusion relations between these spaces and some interesting examples are established. Finally, we prove some results on modified complex Banach Algebra in the third section of the paper.

## 1. Introduction and Preliminaries

The algebra of bicomplex numbers is a generalization of the field of complex numbers. In [12] Luna-Elizarrarás and Shapiro have described how to define elementary functions in such an algebra as well as their inverse functions. They also emphasized the deep similarities between the properties of complex and bicomplex numbers. The bicomplex numbers were apparently first introduced in 1892 by Segre [20] that the origin of their function theory is due to the Italian school of Scorza-Dragoni and that a first theory of differentiability in bicomplex numbers was developed by Price in [13]. The set of bicomplex numbers are denoted by $\mathbb{C}_2$ and defined as follows:

$$\begin{aligned}\mathbb{C}_2 &= \{a_1 + ia_2 + ja_3 + ija_4 : a_k \in \mathbb{R},\ 1 \le k \le 4\}\\ &= \{z_1 + jz_2 : z_1, z_2 \in \mathbb{C}\},\end{aligned}$$

where $i$ and $j$ are commuting imaginary units that is, $ij = ji$, $i^2 = j^2 = -1$ and $\mathbb{C}$ is the set of complex numbers with the imaginary unit $i$. The set of bicomplex numbers $\mathbb{C}_2$ have exactly two non-trivial idempotent elements which are denoted by $e_1$ and $e_2$ defined as $e_1 = (1 + ij)/2$ and $e_2 = (1 - ij)/2$. Note that $e_1 + e_2 = 1$ and $e_1.e_2 = 0$. The number $\eta = z_1 + jz_2$ can uniquely expressed as a complex combination of $e_1$ and $e_2$ (see [18]).

$$\eta = z_1 + jz_2 = {}^1\eta e_1 + {}^2\eta e_2, \tag{1}$$

where ${}^1\eta = z_1 - iz_2$ and ${}^2\eta = z_1 + iz_2$. The complex coefficients ${}^1\eta$ and ${}^2\eta$ are called the idempotent components of $\eta$ and ${}^1\eta e_1 +^2 \eta e_2$ is known as idempotent representation of bicomplex number $\eta$. In [18], the auxiliary complex spaces $\mathbb{A}_1$ and $\mathbb{A}_2$ are defined as

$$\mathbb{A}_1 = \{{}^1\eta : \eta \in \mathbb{C}_2\} \text{ and } \mathbb{A}_2 = \{{}^2\eta : \eta \in \mathbb{C}_2\}.$$

2020 *Mathematics Subject Classification*. Primary 40A05; Secondary 40A30
*Keywords*. Orlicz function, paranorm space, lacunary sequence spaces, bicomplex sequence spaces, solidity
Received: 24 March 2022; Revised: 23 July 2022; Accepted: 24 July 2022
Communicated by Eberhard Malkowsky
*Email addresses:* kuldipraj68@gmail.com (Kuldip Raj), aesi23@hotmail.com (Ayhan Esi), charu145.cs@gmail.com (Charu Sharma)

Also, the norm in $\mathbb{C}_2$ is defined as follows:

$$\|\eta\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} \;=\; \sqrt{|z_1|^2 + |z_2|^2} \;=\; \sqrt{\frac{|^1\eta|^2 + |^2\eta|^2}{2}} \tag{2}$$

The space $(\mathbb{C}_2, +, \cdot, \|\cdot\|)$ is a Banach space by the norm defined in (2). By $\omega_4, c, c_0$ and $\ell_\infty$ we denote the classes of all bicomplex sequences, convergent sequences, null sequences and all bounded sequences, respectively. Let $p = \{p_k\}$ be a sequence of positive real numbers and $\{p_k^{-1}\} = \{t_k\}$. The set of all real numbers and the set of all natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively.

In 1971 Lindenstrauss and Tzafriri [11] first investigated Orlicz sequence spaces in detail with certain aims in Banach space theory. An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Now by using the idea of Orlicz function, we define the following sequence space on bicomplex numbers:

$$\ell^M_{\mathbb{C}_2} = \left\{\eta = \{\eta_k\} \in \omega_4 : \sum_{k=1}^{\infty} M\left(\frac{\|\eta_k\|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$$

which is known as an Orlicz $\mathbb{C}_2$-sequence space. The space $\ell^M_{\mathbb{C}_2}$ is a Banach space with the norm,

$$\|\eta\|_M = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\eta_k\|}{\rho}\right) \le 1\right\}.$$

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function. A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy $\Delta_2$-*condition* if there exist constants $a,\ K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell^1_+$ (the positive cone of $\ell^1$) such that the inequality

$$M_k(2u) \le KM_k(u) + c_k$$

hold for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \le a$.
Many authors studied the bicomplex sequence spaces and their property in details. Recently, Değirmen and Sağır [2] studied different bicomplex $\ell^p_{\mathbb{C}_2}$ spaces. They proved that spaces $\ell^p_{\mathbb{C}_2}$ are Banach $\mathbb{C}_2$-module for $1 \le p \le \infty$ and the spaces $\ell^p_{\mathbb{C}_2}$ are $p-$Banach $\mathbb{C}_2$-module for $0 < p < 1$. Now we study some more results on bicomplex sequence spaces $\ell^M_{\mathbb{C}_2}$.

**Theorem 1.1.** *The Orlicz $\mathbb{C}_2$-sequence space is convex.*

*Proof.* Suppose $\{\eta_k\}, \{\xi_k\} \in \ell^M_{\mathbb{C}_2}$, $\rho = \max\{\rho_1, \rho_2\}$ and $\lambda \in \mathbb{R}$ satisfying $\lambda \in [0, 1]$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\lambda\eta_k + (1-\lambda)\xi_k\|_M &= \sum_{k=1}^{\infty} M\left(\frac{\|\lambda\eta_k + (1-\lambda)\xi_k\|}{\rho}\right) \\
&\le K\left[\sum_{k=1}^{\infty} M\left(\frac{\|\lambda\eta_k\|}{\rho_1}\right) + \sum_{k=1}^{\infty} M\left(\frac{\|(1-\lambda)\xi_k\|}{\rho_2}\right)\right] \\
&= K\left[\lambda\sum_{k=1}^{\infty} M\left(\frac{\|\eta_k\|}{\rho_1}\right) + M(1-\lambda)\sum_{k=1}^{\infty}\left(\frac{\|\xi_k\|}{\rho_2}\right)\right]
\end{aligned}$$

which implies $\lambda\eta_k + (1-\lambda)\xi_k \in \ell^M_{\mathbb{C}_2}$. □

**Remark :** The Orlicz $\mathbb{C}_2$-sequence space is not strictly convex. Let us show this by an example.
Suppose $\{\eta_k\} = (i, 0, 0, \cdots)$ and $\{\xi_k)\} = (0, -i, 0, 0, \cdots)$. Then, we have

$$\|\eta_k\|_M = \|\xi_k\|_M = 1$$

and

$$\begin{aligned}\|\lambda\eta_k + (1-\lambda)\xi_k\|_M &= \sum_{k=1}^{\infty} M\Big(\frac{\|\lambda\eta_k + (1-\lambda)\xi_k\|}{\rho}\Big)\\ &= M\Big(\|\lambda i\| + \|(1-\lambda)(-i)\|\Big)\\ &= \lambda + (1-\lambda)\\ &= 1,\end{aligned}$$

for $\rho = 1$, $M(\eta) = \eta$ and for all $\lambda \in (0, 1)$. Here $K = \max(1, 2^{H-1})$. This implies that the Orlicz $\mathbb{C}_2$-sequence space is not strictly convex.

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0, 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Then $\theta$ is called a lacunary sequence. The intervals determined by $\theta$ are denoted by $I_r = (k_{r-1}, k_r]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by $q_r$.

The space of all lacunary strongly convergent sequences $|\omega_\theta|$ was defined by Freedman et al. in [7] as

$$|\omega_\theta| = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - \lambda| = 0, \text{ for some } \lambda\Big\}. \tag{3}$$

The notion of difference sequence spaces was introduced by Kızmaz [10], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [5] by introducing the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Later the concept have been studied by Bektaş et al. [1] and Et and Esi [6]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [22] who studied the spaces $\ell_\infty(\Delta_n)$, $c(\Delta_n)$ and $c_0(\Delta_n)$. Recently, Esi et al. [4] and Tripathy et al. [21] have introduced a new type of generalized difference operators and unified those as follows.

If $n$, $m$ are non-negative integers, then for a given sequence space $Z$ we have

$$Z(\Delta_n^m) = \{x = (x_k) : (\Delta_n^m x_k) \in Z\}$$

for $Z = c, c_0$ and $\ell_\infty$ where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta_n^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} x_{k+ni}.$$

Taking $n = 1$, we get the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [5]. Taking $m = n = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [10]. For more details about sequence spaces (see [8], [14], [15], [16], [19]) and references therein.

A sequence space $E$ is said to be solid (or normal) if $\{\alpha_k\eta_k\} \in E$ , whenever $\{\eta_k\} \in E$ and for any sequence $\{\alpha_k\}$ of complex numbers such that $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$.

A sequence space $E$ is said to be symmetric if $\{\eta_k\} \in E$ implies $\{\eta_{\pi(k)}\} \in E$, where $\pi(k)$ is a permutation of elements of $\mathbb{N}$.

A linear metric space $(X, d)$ is a linear space $X$ with a translation invariant metric $d$ on $X$ such that addition and scalar multiplication are continuous in $(X, d)$.

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \ge 0$ for all $x \in X$;
2. $p(-x) = p(x)$ for all $x \in X$;
3. $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$;
4. if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [23] Theorem 10.4.2, pp. 183).

**Remark 1.2.** *Let $M$ be an Orlicz function and $\lambda \in (0,1)$, then $M(\lambda x) \leq \lambda M(x)$, $\forall x > 0$.*

Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of positive real numbers and $\theta = (k_r)$, $r \in \mathbb{N}$ be a lacunary sequence. In this paper we define the following lacunary Orlicz $\mathbb{C}_2$-sequence spaces:

$$c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) = \left\{\{\eta_k\} \in \omega_4 : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\frac{\|u_k \Delta_n^m \eta_k - L\|}{\rho}\right)\right]^{p_k} = 0, \text{ for some } \rho > 0 \text{ and } L \in \mathbb{C}_2\right\},$$

$$c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) = \left\{\{\eta_k\} \in \omega_4 : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\frac{\|u_k \Delta_n^m \eta_k\|}{\rho}\right)\right]^{p_k} = 0, \text{ for some } \rho > 0\right\},$$

$$\ell^{\infty}(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) = \left\{\{\eta_k\} \in \omega_4 : \sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\frac{\|u_k \Delta_n^m \eta_k\|}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0\right\},$$

$$\ell(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) = \left\{\{\eta_k\} \in \omega_4 : \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\frac{\|u_k \Delta_n^m \eta_k\|}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0, \ r \in \mathbb{N}\right\}.$$

**Proposition 1.3.** *Any $\mathbb{C}_2$−sequence $\{\eta_k\}$ belongs to $Z(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ if and only if $\{^1\eta_k\} \in Z(\mathbb{A}_1, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ and $\{^2\eta_k\} \in Z(\mathbb{A}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$, where $Z = c, c_0, \ell_\infty, \ell$.*

*Proof.* It is easy to prove. For more details one can see ([13], [18]). □

The following inequality will be used throughout the paper. If $0 < p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$, then

$$\|\eta_k + \xi_k\|^{p_k} \leq K\{\|\eta_k\|^{p_k} + \|\xi_k\|^{p_k}\} \tag{4}$$

for all $k$ and $\{\eta_k\}, \{\xi_k\} \in \mathbb{C}_2$. Also, $\|\eta\|^{p_k} \leq \max\{1, \|\eta\|^H\}$, for all $\eta \in \mathbb{C}_2$.

The aim of the paper is to introduce some lacunary difference $\mathbb{C}_2$-sequence spaces by using a sequence of Orlicz functions. We investigate some topological properties such as completeness, solidness, symmetric and establish some inclusion relations concerning these spaces in second section of this paper. We make an effort to study some results on modified complex Banach Algebra in the section third of the paper.

## 2. Lacunary Orlicz $\mathbb{C}_2$-sequence spaces

**Theorem 2.1.** *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. Then the spaces $c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$, $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$, $\ell^{\infty}(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ and $\ell(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ are linear spaces over the complex field $\mathbb{C}$.*

*Proof.* Let $\eta = \{\eta_k\}$, $\xi = \{\xi_k\} \in c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\frac{\|u_k \Delta_n^m \eta_k\|}{\rho_1}\right)\right]^{p_k} = 0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\xi_k\|}{\rho_2}\right)\right]^{p_k}=0.$$

Let $\rho_3=\max\{2\|\alpha\|\rho_1, 2\|\beta\|\rho_2\}$. Since $(M_k)$ is non-decreasing and convex by using inequality (4), we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k[\alpha(\Delta_n^m\eta_k)+\beta(\Delta_n^m\xi_k)]\|}{\rho}\right)\right]^{p_k} &\leq K\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\eta_k\|}{\rho_1}\right)\right]^{p_k}+K\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\xi_k\|}{\rho_2}\right)\right]^{p_k}\\ &\to 0 \text{ as } r\to\infty.\end{aligned}$$

Thus, $\{\alpha\eta+\beta\xi\}\in c_0(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,\|.\|)$. Hence, $c_0(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,\|.\|)$ is a linear space. Similarly, we can prove $c(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,\|.\|)$, $\ell^\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,\|.\|)$ and $\ell(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,\|.\|)$ are linear spaces over the complex field $\mathbb{C}$. □

**Theorem 2.2.** *Let $\mathcal{M}=(M_k)$ be a sequence of Orlicz functions, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of positive real numbers. Then $\ell^\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,\|.\|)$ is a paranormed space with the paranorm*

$$g(\eta)=\|\eta_1\|+\inf\left\{(\rho)^{\frac{p_k}{H}}:\sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\eta_k\|}{\rho}\right)\right](t_k)^{\frac{1}{p_k}}\right)\leq 1,\ \textit{for some}\ \ \rho>0\right\},$$

*where* $H=\max(1,\sup_k p_k)<\infty$.

*Proof.* (i) Clearly, $g(\eta)\geq 0$, for $\eta=\{\eta_k\}\in\ell^\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,\|.\|)$. Since $M_k(\theta_1)=0$, we get $g(\theta_1)=0$,
(ii) $g(-\eta)=g(\eta)$,
(iii) Let $\eta=\{\eta_k\},\xi=\{\xi_k\}\in\ell^\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,\|.\|)$. Then there exist $\rho_1>0$ and $\rho_2>0$ such that

$$\sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\eta_k\|}{\rho_1}\right)\right](t_k)^{\frac{1}{p_k}}\right)\leq 1$$

and

$$\sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\xi_k\|}{\rho_2}\right)\right](t_k)^{\frac{1}{p_k}}\right)\leq 1.$$

Suppose $\rho=\rho_1+\rho_2$, then by Minkowski's inequality, we have

$$\begin{aligned}\sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m(\eta_k+\xi_k)\|}{\rho}\right)\right](t_k)^{\frac{1}{p_k}}\right) &\leq\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\eta_k\|}{\rho_1}\right)\right](t_k)^{\frac{1}{p_k}}\right)\\ &+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\xi_k\|}{\rho_2}\right)\right](t_k)^{\frac{1}{p_k}}\right)\\ &\leq 1.\end{aligned}$$

Also,

$$g(\eta+\xi) = \|\eta_1\| + \inf\left\{(\rho)^{\frac{p_k}{H}} : \sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m(\eta_k+\xi_k)\|}{\rho_1+\rho_2}\right)\right](t_k)^{\frac{1}{p_k}}\right) \le 1\right\}$$

$$\le \|\eta_1\| + \inf\left\{(\rho_1)^{\frac{p_k}{H}} : \sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\eta_k\|}{\rho_1}\right)\right](t_k)^{\frac{1}{p_k}}\right) \le 1\right\}$$

$$+ \|\eta_1\| + \inf\left\{(\rho_2)^{\frac{p_k}{H}} : \sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\xi_k\|}{\rho_2}\right)\right](t_k)^{\frac{1}{p_k}}\right) \le 1\right\}$$

$$\le g(\eta) + g(\xi).$$

Finally, we prove that scalar multiplication is continuous. Let $\lambda$ be any complex number by definition

$$g(\lambda\eta) = \|\lambda\eta_1\| + \inf\left\{(\rho)^{\frac{p_k}{H}} : \sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m(\lambda\eta_k)\|}{\rho}\right)\right](t_k)^{\frac{1}{p_k}}\right) \le 1\right\}$$

$$\le |\lambda|\|\eta_1\| + \inf\left\{(|\lambda|P)^{\frac{p_k}{H}} : \sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\eta_k\|}{P}\right)\right](t_k)^{\frac{1}{p_k}}\right) \le 1,\ P>0\right\},$$

where $P = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_k} \le \max(1, |\lambda| \sup p_k)$. This completes the proof. □

**Theorem 2.3.** *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. Then $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ is a complete paranormed space, paranormed defined by $g$.*

*Proof.* Suppose $\{\eta^n\}$ is a Cauchy sequence in $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$, where $\eta^n = \{\eta_k^n\}_{k=1}^\infty$ for all $n \in \mathbb{N}$, so that $g(\eta_k^i - \eta_k^j) \to 0$ as $i, j \to \infty$. Suppose $\epsilon > 0$ is given and let some $s > 0$ and $x_0 > 0$ be such that $\frac{\epsilon}{sx_0} > 0$ and $\sup_k (p_k)^{t_k} \le M_k(\frac{sx_0}{2})$. Since $g(\eta_k^i - \eta_k^j) \to 0$, as $i, j \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$g(\eta_k^i - \eta_k^j) < \frac{\epsilon}{sx_0}, \quad \text{for all } \ i, j \ge n_0.$$

Therefore,

$$\|\eta_1^i - \eta_1^j\| + \inf\left\{(\rho)^{\frac{p_k}{H}} : \sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\eta_k\|}{\rho}\right)\right](t_k)^{\frac{1}{p_k}}\right) \le 1, \text{ for some } \rho>0\right\} < \frac{\epsilon}{sx_0}.$$

This implies $\|\eta_1^i - \eta_1^j\| < \frac{\epsilon}{sx_0}$ and

$$\inf\left\{(\rho)^{\frac{p_k}{H}} : \sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m\eta_k\|}{\rho}\right)\right](t_k)^{\frac{1}{p_k}}\right) \le 1, \text{ for some } \rho>0\right\}.$$

It shows that $\{\eta_1^i\}$ is a Cauchy sequence in $\mathbb{C}_2$. Since $\mathbb{C}_2$ is a modified complex Banach algebra, then $\{\eta_1^i\}$ converges in $\mathbb{C}_2$. Suppose $\lim_{i\to\infty} \eta_1^i = \eta_1$. Thus then $\lim_{j\to\infty} \|\eta_1^i - \eta_1^j\| < \frac{\epsilon}{sx_0}$, we get

$$\|\eta_1^i - \eta_1\| < \frac{\epsilon}{sx_0}.$$

Thus, we have

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^m(\eta_k^i - \eta_k^j)\|}{g(\eta_k^i - \eta_k^j)}\right)\right](t_k)^{\frac{1}{p_k}}\right) \le 1.$$

This implies

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||u_k\Delta_n^m(\eta_k^i-\eta_k^j)||}{g(\eta_k^i-\eta_k^j)}\right)\right](t_k)^{\frac{1}{p_k}}\right)\le 1\le M_k\left(\frac{sx_0}{2}\right)$$

and thus,

$$||u_k\Delta_m^n\eta_k^i-u_k\Delta_m^n\eta_k^j||\le\left(\frac{sx_0}{2}\right)\left(\frac{\epsilon}{sx_0}\right)=\frac{\epsilon}{2}$$

which shows that $(u_k\Delta_m^n\eta_k^i)$ is a Cauchy sequence in $\mathbb{C}_2$ for all $k\in\mathbb{N}$. Therefore, $(u_k\Delta_m^n\eta_k^i)$ converges in $\mathbb{C}_2$. Suppose $\lim_{i\to\infty}u_k\Delta_m^n\eta_k^i=\xi_k$ for all $k\in\mathbb{N}$.
Also, we have $\lim_{i\to\infty}\Delta_m^n\eta_2^i=\xi_1-\eta_1$. On repeating the same procedure, we obtain $\lim_{i\to\infty}\Delta_m^n\eta_{k+1}^i=\xi_k-\eta_k$ for all $k\in\mathbb{N}$. Therefore, by continuity of $(M_k)$, we have

$$\lim_{j\to\infty}\sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||u_k\Delta_n^m(\eta_k^i-\eta_k^j)||}{\rho}\right)\right](t_k)^{\frac{1}{p_k}}\right)\le 1,$$

so that

$$\sup_r\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||u_k\Delta_n^m(\eta_k^i-\eta_k^j)||}{\rho}\right)\right](t_k)^{\frac{1}{p_k}}\right)\le 1.$$

Let $i\ge n_0$ and taking infimum of each $\rho>0$, we have

$$g(\eta^i-\eta)<\epsilon.$$

So $\{\eta^i-\eta\}\in\ell_\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,||.||)$. Hence, $\eta=\{\eta_k\}\in\ell_\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,||.||)$. Therefore, $\ell_\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,||.||)$ is complete paranormed space. □

**Theorem 2.4.** *Let $\mathcal{M}=(M_k)$ be a sequence of Orlicz functions, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of positive real numbers. If $\sup_k[M_k(x)]^{p_k}<\infty$ for all fixed $x>0$, then*

$$c_0(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,||.||)\subseteq\ell^\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,||.||).$$

*Proof.* Let $\eta=\{\eta_k\}\in c_0(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,||.||)$. Then there exists positive number $\rho>0$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||u_k\Delta_n^m\eta_k||}{\rho}\right)\right]^{p_k}\to 0\text{ as }r\to\infty.$$

Define $\rho=2\rho_1$. Since $(M_k)$ is non-decreasing and convex, also using inequality (4), we have

$$\begin{aligned}\sup_r\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||u_k\Delta_n^m\eta_k||}{\rho}\right)\right]^{p_k}&=\sup_r\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||u_k\Delta_n^m\eta_k+L-L||}{\rho}\right)\right]^{p_k}\\&\le K\frac{1}{2^{p_k}}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||u_k\Delta_n^m\eta_k-L||}{\rho_1}\right)\right]^{p_k}+K\frac{1}{2^{p_k}}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||L||}{\rho_1}\right)\right]^{p_k}\\&\le K\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||u_k\Delta_n^m\eta_k-L||}{\rho_1}\right)\right]^{p_k}+K\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{||L||}{\rho_1}\right)\right]^{p_k}\\&<\infty.\end{aligned}$$

Hence, $\{\eta_k\}\in\ell^\infty(\mathbb{C}_2,\theta,\mathcal{M},\Delta_n^m,p,u,||.||)$. □

**Theorem 2.5.** *Let* $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ *and* $\mathcal{M} = (M_k), \mathcal{M}' = (M_k')$ *be two sequences of Orlicz functions satisfying* $\Delta_2$*-condition. Then we have*

*(i)* $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) \subset c_0(\mathbb{C}_2, \theta, \mathcal{M} \circ \mathcal{M}', \Delta_n^m, p, u, \|.\|)$;

*(ii)* $c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) \subset c(\mathbb{C}_2, \theta, \mathcal{M} \circ \mathcal{M}', \Delta_n^m, p, u, \|.\|)$;

*(iii)* $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) = \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) \subset \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M} \circ \mathcal{M}', \Delta_n^m, p, u, \|.\|)$.

*Proof.* If $\eta = \{\eta_k\} \in c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$, then we have

$$\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{\|u_k \Delta_n^m \eta_k\|}{\rho} \right) \right]^{p_k} \to 0 \text{ as } r \to \infty.$$

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Let

$\xi_k = M_k'\left(\frac{\|u_k \Delta_n^m \eta_k\|}{\rho}\right)$ for all $k \in \mathbb{N}$. We can write

$$\frac{1}{h_r} \sum_{k \in I_r} M_k[\xi_k]^{p_k} = \frac{1}{h_r} \sum_{k \in I_r, \xi_k \leq \delta} M_k[\xi_k]^{p_k} + \frac{1}{h_r} \sum_{k \in I_r, \xi_k \geq \delta} M_k[\xi_k]^{p_k}.$$

So we have

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r, \xi_k \leq \delta} M_k[\xi_k]^{p_k} &\leq [M_k(1)]^H \frac{1}{h_r} \sum_{k \in I_r, \xi_k \leq \delta} M_k[\xi_k]^{p_k} \\ &\leq [M_k(2)]^H \frac{1}{h_r} \sum_{k \in I_r, \xi_k \leq \delta} M_k[\xi_k]^{p_k}. \end{aligned} \tag{5}$$

For $\xi_k > \delta, \xi_k < \frac{\xi_k}{\delta} < 1 + \frac{\xi_k}{\delta}$. Since $M_k's$ are non-deceasing and convex, it follows that

$$M_k(\xi_k) < M_k\left(1 + \frac{\xi_k}{\delta}\right) < \frac{1}{2} M_k(2) + \frac{1}{2} M_k\left(\frac{2\xi_k}{\delta}\right).$$

Since $\mathcal{M} = (M_k)$ satisfies $\Delta_2$-condition, we can write

$$M_k(\xi_k) < \frac{1}{2} T \frac{\xi_k}{\delta} M_k(2) + \frac{1}{2} T \frac{\xi_k}{\delta} M_k(2) = T \frac{\xi_k}{\delta} M_k(2).$$

Hence,

$$\frac{1}{h_r} \sum_{k \in I_r, \xi_k \geq \delta} M_k[\xi_k]^{p_k} \leq \max\left(1, \left(T \frac{M_k(2)}{\delta}\right)^H\right) \frac{1}{h_r} \sum_{k \in I_r, \xi_k \geq \delta} [\xi_k]^{p_k}. \tag{6}$$

From equation (5) and (6), we have $\eta = \{\eta_k\} \in c_0(\mathbb{C}_2, \theta, \mathcal{M} \circ \mathcal{M}', \Delta_n^m, p, u, \|.\|)$. This completes the proof of (*i*). Similarly, we can prove the others. □

**Theorem 2.6.** *Let* $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. *Then for a sequence of Orlicz functions* $\mathcal{M} = (M_k)$ *which satisfies* $\Delta_2$*-condition, we have*

*(i)* $c_0(\mathbb{C}_2, \theta, \Delta_n^m, p, u, \|.\|) \subset c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$;

*(ii)* $c(\mathbb{C}_2, \theta, \Delta_n^m, p, u, \|.\|) \subset c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$;

*(iii)* $\ell^\infty(\mathbb{C}_2, \theta, \Delta_n^m, p, u, \|.\|) \subset \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$.

*Proof.* It is easy to prove so we omit the details. □

**Theorem 2.7.** *Let $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. Then for a sequence of Orlicz functions $\mathcal{M} = (M_k)$ which satisfies $\Delta_2$-condition, we have*

*(i)* $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m-1}, p, u, \|.\|) \subset c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m}, p, u, \|.\|)$;

*(ii)* $c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m-1}, p, u, \|.\|) \subset c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m}, p, u, \|.\|)$;

*(iii)* $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m-1}, p, u, \|.\|) \subset \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m}, p, u, \|.\|)$.

*Proof.* Here we prove the result for $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m}, p, u, \|.\|)$ and for other cases it will follow on applying similar arguments. Let $\eta = \{\eta_k\} \in c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m-1}, p, u, \|.\|)$. Then there exist $\rho > 0$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^{m-1}\eta_k\|}{\rho}\right)\right]^{p_k} \to 0 \text{ as } r \to \infty. \tag{7}$$

On considering $2\rho$, by the convexity of Orlicz function, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^{m-1}\eta_k\|}{2\rho}\right)\right] \le \frac{1}{2}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^{m-1}\eta_k\|}{\rho}\right)\right] + \frac{1}{2}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^{m-1}\eta_{k+n}\|}{\rho}\right)\right].$$

Hence, we have

$$\frac{1}{\lambda_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^{m}\eta_k\|}{2\rho}\right)\right]^{p_k} \le K\left\{\frac{1}{2}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^{m-1}\eta_k\|}{\rho}\right)\right]^{p_k} + \frac{1}{2}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^{m-1}\eta_{k+n}\|}{\rho}\right)\right]^{p_k}\right\}.$$

Then using (7), we get

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\|u_k\Delta_n^{m}\eta_k\|}{2\rho}\right)\right]^{p_k} = 0.$$

Thus, $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m-1}, p, u, \|.\|) \subset c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m}, p, u, \|.\|)$. □

**Theorem 2.8.** *Let $0 \le p_k \le s_k$ for all $k$ and let $(\frac{s_k}{p_k})$ be bounded. Then*

$$c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m}, s, u, \|.\|) \subset c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m}, p, u, \|.\|).$$

*Proof.* Let $\eta = \{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^{m}, s, u, \|.\|)$, write

$$r_k = \left[M_k\left(\frac{\|u_k\Delta_n^{m}\eta_k - L\|}{\rho}\right)\right]^{s_k}$$

and $\mu_k = \frac{p_k}{s_k}$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \le 1$ for all $k \in \mathbb{N}$. Take $0 < \mu \le \mu_k$ for $k \in \mathbb{N}$. Define sequences $\{v_k\}$ and $\{w_k\}$ as follows :
For $r_k \ge 1$, let $v_k = r_k$ and $w_k = 0$ and for $r_k < 1$, let $v_k = 0$ and $w_k = r_k$. Then, clearly for all $k \in \mathbb{N}$, we have

$$r_k = v_k + w_k, \quad r_k^{\mu_k} = v_k^{\mu_k} + w_k^{\mu_k}.$$

Now it follows that $v_k^{\mu_k} \le v_k \le r_k$ and $w_k^{\mu_k} \le w_k^{\mu}$. Therefore,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} r_k^{\mu_k} &= \frac{1}{h_r}\sum_{k\in I_r}(v_k^{\mu_k} + w_k^{\mu_k})\\ &\le \frac{1}{h_r}\sum_{k\in I_r} r_k + \frac{1}{h_r}\sum_{k\in I_r} w_k^{\mu}.\end{aligned}$$

Now for each $k$,

$$\frac{1}{h_r}\sum_{k\in I_r} w_k^{\mu} = \sum_{k\in I_r}\Big(\frac{1}{h_r}w_k\Big)^{\mu}\Big(\frac{1}{h_r}\Big)^{1-\mu}$$
$$\leq \Big(\sum_{k\in h_r}\Big[\Big(\frac{1}{h_r}w_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu}$$
$$= \Big(\frac{1}{h_r}\sum_{k\in I_r} w_k\Big)^{\mu}$$

and so

$$\frac{1}{h_r}\sum_{k\in I_r} r_k^{\mu_k} \leq \frac{1}{h_r}\sum_{k\in I_r} r_k + \Big(\frac{1}{h_r}\sum_{k\in I_r} w_k\Big)^{\mu}.$$

Hence, $\eta = \{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$. This completes the proof of the theorem. □

**Theorem 2.9.** *(i) If $0 < \inf p_k \leq p_k \leq 1$ for all $k \in \mathbb{N}$, then*

$$c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) \subseteq c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, u, \|.\|).$$

*(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$, for all $k \in \mathbb{N}$, then*

$$c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, u, \|.\|) \subseteq c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|).$$

*Proof.* (*i*) Let $\eta = \{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$. Then

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\|u_k\Delta_n^m\eta_k - L\|}{\rho}\Big)\Big]^{p_k} = 0.$$

Since $0 < \inf p_k \leq p_k \leq 1$. This implies that

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\|u_k\Delta_n^m\eta_k - L\|}{\rho}\Big)\Big] \leq \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\|u_k\Delta_n^m\eta_k - L\|}{\rho}\Big)\Big]^{p_k}.$$

Thus, $\eta = \{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, u, \|.\|)$.

(*ii*) Let $p_k \geq 1$ for each $k$ and $\sup p_k < \infty$. Let $\eta = \{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, u, \|.\|)$. Then for each $0 < \epsilon < 1$, there exists a positive integer $N$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\|u_k\Delta_n^m\eta_k - L\|}{\rho}\Big)\Big] \leq \epsilon < 1 \quad \text{for all } r \geq N.$$

This implies that

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\|u_k\Delta_n^m\eta_k - L\|}{\rho}\Big)\Big]^{p_k} \leq \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\|u_k\Delta_n^m\eta_k - L\|}{\rho}\Big)\Big].$$

Therefore, $\eta = \{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$. This completes the proof. □

**Theorem 2.10.** *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. If $0 < \inf p_k \leq p_k \leq \sup p_k = H < \infty$, for all $k \in \mathbb{N}$, then*

$$c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) = c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, u, \|.\|).$$

*Proof.* It is easy to prove so we omit the details. □

**Proposition 2.11.** *The spaces* $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$, $c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *and* $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *are Banach spaces.*

**Theorem 2.12.** *The spaces* $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$, $c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *and* $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *are not solid in general.*

**Example 2.13.** *Let* $M_k(x) = x$, $(p_k) = (u_k) = 1$ *for all* $k \in \mathbb{N}$, $\rho = 1$, $m = 0$ *and* $\theta = \{1, 2, ..., n\}$. *Consider a sequence* $\{\eta_k\} \in \omega_4$ *given as* $\eta_k = \{\eta_k^{(s)}\} = \{2, 2, 2, ...\}$. *Then* $\{\eta_k^{(s)}\} \in c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$. *Now, let* $\{\alpha_k\} = \{(-1)^k\}$, $\forall k \in \mathbb{N}$. *Then,* $\{\alpha_k \eta_k^{(s)}\} \notin c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$. *Therefore,* $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *is not solid.*
*Let* $\{\eta_k\} \in \omega_4$ *defined as* $\eta_k = \{\eta_k^{(s)}\} = \{k^2, k^2+1, k^2+2, ...\}$, $\forall k, s \in \mathbb{N}$. *Then* $\{\eta_k^{(s)}\} \in c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *as well as* $\{\eta_k^{(s)}\} \in \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$. *Now, let* $\{\alpha_k\} = \{(-1)^k\}$, $\forall k \in \mathbb{N}$. *Then,* $\{\alpha_k \eta_k^{(s)}\} \notin c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *as well as* $\{\alpha_k \eta_k^{(s)}\} \notin \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$. *Hence, the spaces* $c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *and* $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *are not solid.*

**Theorem 2.14.** *The spaces* $c_0(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$, $c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *and* $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *are not symmetric in general.*

To show that the spaces are not symmetric in general, consider the following example.

**Example 2.15.** *Let* $M_k(x) = x$, $(p_k) = 2$, $(u_k) = 1$ *for all* $k \in \mathbb{N}$, $\rho = 1$, $m = 0$ *and* $\theta = \{1, 2, ..., n\}$. *Suppose that* $\{\eta_k\} = \{\eta_k^s\} = \{k^2, k^2+1, k^2+2, ...\}$, $\forall k, s \in \mathbb{N}$. *Then,* $\{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|) \cap \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *Consider the rearranged sequence,* $(\xi_k)$ *of* $(\eta_k)$ *defined as*

$$\{\xi_k\} = \{\eta_1^s, \eta_8^s, \eta_2^s, \eta_{27}^s, \eta_3^s, \eta_{64}^s, \eta_4^s, ...\}.$$

*Then* $\{\xi_k\} \notin c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *as well as* $\{\xi_k\} \notin \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$. *Hence,* $c(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *and* $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta_n^m, p, u, \|.\|)$ *are not symmetric in general. Similarly, we can prove for other space.*

**Theorem 2.16.** *Let* $\mathcal{M}_1 = M_1$ *and* $\mathcal{M}_2 = M_2$ *be the Orlicz functions with* $\Delta_2$ *conditions and* $p = (p_k) \in l^\infty$, *then* $c(\mathbb{C}_2, \theta, \mathcal{M}_1, \Delta_n^m, p, u, \|.\|) \cap c(\mathbb{C}_2, \theta, \mathcal{M}_2, \Delta_n^m, p, u, \|.\|) \subset c(\mathbb{C}_2, \theta, \mathcal{M}_1 + \mathcal{M}_2, \Delta_n^m, p, u, \|.\|)$.

*Proof.* Let $\{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}_1, \Delta_n^m, u, \|.\|) \cap c(\mathbb{C}_2, \theta, \mathcal{M}_2, \Delta_n^m, p, u, \|.\|)$. Then $\exists$ some $L \in \mathbb{C}_2$, $\rho_1 > 0$, $\rho_2 > 0$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} \left[ M_1 \left( \frac{\|u_k \Delta_n^{m-1} \eta_k - L\|}{\rho_1} \right) \right]^{p_k} t_k \to 0 \tag{8}$$

$$\frac{1}{h_r} \sum_{k \in I_r} \left[ M_2 \left( \frac{\|u_k \Delta_n^{m-1} \eta_k - L\|}{\rho_2} \right) \right]^{p_k} t_k \to 0. \tag{9}$$

Let $\rho = \max\{\rho_1, \rho_2\}$. Then,

$$\left\{ \frac{1}{h_r} \sum_{k \in I_r} \left[ \left( M_1 + M_2 \right) \left( \frac{\|u_k \Delta_n^m \eta_k - L\|}{\rho} \right) \right]^{p_k} t_k \right\} \leq \frac{1}{h_r} \sum_{k \in I_r} \left[ M_1 \left( \frac{\|u_k \Delta_n^m \eta_k - L\|}{\rho_1} \right) \right] + \frac{1}{h_r} \sum_{k \in I_r} \left[ M_2 \left( \frac{\|u_k \Delta_n^m \eta_k - L\|}{\rho_2} \right) \right].$$

From (3.4) and (3.5), we get $\{\eta_k\} \in c(\mathbb{C}_2, \theta, \mathcal{M}_1 + \mathcal{M}_2, \Delta_n^m, u, \|.\|)$. □

**Theorem 2.17.** *The sequence space* $\ell^\infty(\mathbb{C}_2, \theta, \Delta_n^m, p, u, \|.\|)$ *is convex.*

*Proof.* Let $\{\eta_k\}, \{\xi_k\} \in \ell^\infty(\mathbb{C}_2, \theta, \Delta_n^m, p, u, \|.\|)$ and $\lambda \in \mathbb{R}$ satisfying $\lambda \in [0, 1]$. Then

$$\left\{\sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\frac{\|u_k \Delta_n^m \eta_k\|}{\rho_1}\right)\right]^{p_k}\right\}$$

and

$$\left\{\sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\frac{\|u_k \Delta_n^m \xi_k\|}{\rho_2}\right)\right]^{p_k}\right\}$$

are finite. Now, let $\rho = \max\{\rho_1, \rho_2\}$ then, we have

$$\left\{\sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\frac{\|u_k \Delta_n^m \lambda\eta_k + u_k \Delta_n^m \xi_k(1-\lambda)\|}{\rho}\right)\right]^{p_k}\right\}$$

$$\begin{aligned}
&\leq \sup_r \frac{1}{h_r} \sum_{k\in I_r} M_k\left(\frac{\|u_k \Delta_n^m \lambda\eta_k\|}{\rho_1}\right)^{p_k} + \sup_r \frac{1}{h_r} \sum_{k\in I_r} M_k\left(\frac{\|u_k \Delta_n^m \xi_k(1-\lambda)\|}{\rho_2}\right)^{p_k} \\
&= \lambda \sup_r \frac{1}{h_r} \sum_{k\in I_r} M_k\left(\frac{\|u_k \Delta_n^m \eta_k\|}{\rho_1}\right)^{p_k} + (1-\lambda) \sup_r \frac{1}{h_r} \sum_{k\in I_r} M_k\left(\frac{\|u_k \Delta_n^m \xi_k\|}{\rho_2}\right)^{p_k}
\end{aligned}$$

which implies $\lambda\eta_k + (1-\lambda)\xi_k \in \ell^\infty(\mathbb{C}_2, \theta, \Delta_n^m, p, u, \|.\|)$. Thus, $\ell^\infty(\mathbb{C}_2, \theta, \Delta_n^m, p, u, \|.\|)$ is convex. □

## 3. Modified complex Banach Algebra

From many years a lot of results has been published on modified complex Banach Algebra by various mathematicians. Recently Nilay Sager and Birsen Sağır [17] worked on completeness of bicomplex sequence space. By using modified complex Banach Algebra they have proved bicomplex Hölder's Inequality and several other interesting results.
The norm of the product of two bicomplex numbers and the product of their norms are connected by means of the following inequality:

$$\|\eta\xi\| \leq \sqrt{2}\|\eta\|\|\xi\|. \tag{10}$$

The inequality given in (10) is the best possible relation. For this reason, we call $(\mathbb{C}_2, +, \cdot, \|\cdot\|)$ as modified complex Banach algebra.

**Justification of** (10)**:** Let $\eta,\ \xi \in \mathbb{C}_2$. Then $\|\eta\xi\| \leq \sqrt{2}\|\eta\|\|\xi\|$.
Let $\eta = (z_1 + jz_2) \in \mathbb{C}_2$ and $\xi = z_3 + jz_4 \in \mathbb{C}_2$. Then

$$\eta\xi = (z_1 + jz_2)(z_3 + jz_4) = z_1(z_3 + jz_4) + jz_2(z_3 + jz_4).$$

Moreover, $\|z_1(z_3 + jz_4)\| = \|z_1\|\|z_3 + jz_4\|$ and $\|jz_2(z_3 + jz_4)\| = |j|\|z_2\|\|z_3 + jz_4\| = \|z_2\|\|z_3 + jz_4\|$. Therefore, from the triangle inequality, we have

$$\begin{aligned}
\|\eta\xi\| = \|(z_1 + jz_2)(z_3 + jz_4)\| &\leq \|z_1\|\|(z_3 + jz_4)\| + \|z_2\|\|(z_3 + jz_4)\| \\
&\leq (\|z_1\| + \|z_2\|)\|z_3 + jz_4\|.
\end{aligned}$$

Since $2\|z_1\|\|z_2\| \leq \|z_1\|^2 + \|z_2\|^2$, then $(\|z_1\| + \|z_2\|)^2 \leq 2(\|z_1\|^2 + \|z_2\|^2)$. Thus, we have

$$(\|z_1\| + \|z_2\|) \leq \sqrt{2}(\|z_1\|^2 + \|z_2\|^2)^{1/2}.$$

Hence, $||\eta\xi|| \leq \sqrt{2}||\eta||||\xi||$.
We note that the constant $\sqrt{2}$ is the best possible one in above justification. Moreover, if we combine the last results with the fact that $(\mathbb{C}_2, +, \cdot, ||\cdot||)$ is a Banach space, we obtain that $(\mathbb{C}_2, +, \cdot, ||\cdot||)$ is a **modified complex Banach algebra**.
But in the usual definition of a **complex Banach algebra**, the norm of the product of two elements is required to be equal to or less than the product of the norms of these elements that is, $||z_1z_2|| \leq ||z_1||||z_2||$. This is the difference between the complex Banach algebra and the modified complex Banach algebra.
Romesh et al. [9] introduced the spectrum of the unilateral shift operator by using $\ell^2_{\mathbb{C}_2}$. Dubey et al. [3] studied the Orlicz bicomplex sequence spaces. They proved that the bicomplex sequence spaces $\ell^M_{\mathbb{C}2}$ is a Banach space and used as Complex Banach Algebra. They studied the different properties of linear operators such as boundedness, compactness etc.
Now we prove some results on modified Complex Banach Algebra.

**Theorem 3.1.** *Let* $\{z_k\}, z, y \in \mathbb{C}_2$.

*(i) If* $z_k \to z$ *then* $yz_k \to yz$ *and* $z_ky \to zy$;
*(ii) If* $z_k \to z$ *and* $y_k \to y$ *then* $z_ky_k \to zy$.

*Proof.* (i) Since $z_k \to z$, $||z_k - z|| \to 0$ in $\mathbb{C}_2$ and hence we have

$$||yz_k - yz|| = ||y(z_k - z)|| \leq \sqrt{2}||y||||z_k - z|| \to \sqrt{2}||y||.0 \in \mathbb{C}_2.$$

Other case can be proved in the similar manner.
(ii) If $z_k \to z$ also $||z_k|| \to ||z||$, hence $||z_k||$ is bounded say by $M$. Now, for given $\epsilon$, let $N_z$ be such that $k \geq N_z \Rightarrow ||z_k - z|| < \frac{\epsilon}{2\sqrt{2}||y||}$ if $y \neq 0$ and arbitrary otherwise, so that in any case $||z_k - z||||y|| < \frac{\epsilon}{2\sqrt{2}}$. Let $N_y$ such that $k \geq N_y \Rightarrow ||y_k - y|| < \frac{\epsilon}{2\sqrt{2}M}$ (choose $M > 0$) for $N = \max(N_z, N_y)$ holds if $k \geq N$, then

$$\begin{aligned}
||z_ky_k - zy|| &= ||z_ky_k - z_ky + z_ky - zy|| \\
&\leq ||z_ky_k - z_ky|| + ||z_ky - zy|| \\
&\leq \sqrt{2}||z_k||||y_k - y|| + \sqrt{2}||y||||z_k - z|| \\
&< \sqrt{2}M \times \frac{\epsilon}{2\sqrt{2}M} + \sqrt{2}\frac{\epsilon}{2\sqrt{2}} \\
&= \epsilon.
\end{aligned}$$

Thus, $z_ky_k \to zy$. □

Now, let us define $\omega_4 = \{\{\eta_k\} : \forall k \in \mathbb{N}, \eta_k \in \mathbb{C}_2\}$. This space of all $\mathbb{C}_2$ sequences forms a $\mathbb{C}_2-$module (see [17]). Also, $\omega_4$ forms a $\mathbb{C}_2-$module with the operations addition and bicomplex scaler multiplication as follows:

$$\oplus : \omega_4 \times \omega_4 \to \omega_4, (\eta, s) \to \eta + s = (\eta_k \oplus s_k),$$

$$\odot : \mathbb{C}_2 \times \omega_4 \to \omega_4, (\vartheta, \eta) \to \vartheta \odot \eta = \vartheta\eta = (\vartheta\eta_k),$$

$$\otimes : \mathbb{C}_2 \otimes \omega_4 \to \omega_4, (\vartheta, \eta) \to \vartheta \cdot \eta = \vartheta\eta = (\vartheta\eta_k),$$

for all $\{\eta_k\}$, $\{s_k\} \in \omega_4$ and $\forall\vartheta \in \mathbb{C}_2$.

**Remark:** $\ell^M_{\mathbb{C}_2}$ is a subspace of $\omega_4$.

*Proof.* It is obvious that $\ell^M_{\mathbb{C}_2} \subset \omega_4$. Let $\{\eta_k\}$, $\{s_k\} \in \ell^M_{\mathbb{C}_2}$. Then $\exists$ $\rho_1, \rho_2$ such that

$$\sum_{k=1}^{\infty} M\Big(\frac{||\eta_k||}{\rho_1}\Big) < \infty$$

and

$$\sum_{k=1}^{\infty} M\Big(\frac{\|s_k\|}{\rho_2}\Big) < \infty.$$

Let $\rho = \max(\rho_1, \rho_2)$, then

$$\sum_{k=1}^{\infty} M\Big(\frac{\|\eta_k + s_k\|}{\rho}\Big) \leq \sum_{k=1}^{\infty} M\Big(\frac{\|\eta_k\|}{\rho_2}\Big) + \sum_{k=1}^{\infty} M\Big(\frac{\|s_k\|}{\rho_2}\Big),$$

which means that $\eta_k \oplus s_k \in \ell^M_{\mathbb{C}_2}$. Now, suppose $\alpha \in \mathbb{R}$ and $\{\eta_k\} \in \ell^M_{\mathbb{C}_2}$. Since

$$\|\alpha\eta_k\| = |\alpha|\|\eta_k\|$$

and

$$\sum_{k=1}^{\infty} M\Big(\frac{\|\eta_k\|}{\rho}\Big) < \infty.$$

We can easily say that

$$\sum_{k=1}^{\infty} M\Big(\frac{\|\alpha\eta_k\|}{\rho}\Big) < \infty \implies |\alpha| \sum_{k=1}^{\infty} M\Big(\frac{\|\eta_k\|}{\rho}\Big) < \infty.$$

So, $\alpha \odot \eta_k \in \ell^M_{\mathbb{C}_2}$. Thus, $\ell^M_{\mathbb{C}_2}$ is a subspace of $\omega_4$. □

**Remark:** $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta^m_n, p, u, \|.\|)$ is a subspace of $\omega_4$.

*Proof.* This remark can be proved in similar manner as proof of above remark. □

**Theorem 3.2.** *$\ell^M_{\mathbb{C}_2}$ is a $\mathbb{C}_2-$submodule of $\omega_4$.*

*Proof.* As $\ell^M_{\mathbb{C}_2}$ is a subspace of $\omega_4$. Also, we obtain that $\{\eta_k\} \in \ell^M_{\mathbb{C}_2}$ and $\vartheta \in \mathbb{C}_2 - \{0\}$.

$$\sum_{k=1}^{\infty} M\Big(\frac{\|\eta_k\vartheta\|}{\rho}\Big) \leq \sum_{k=1}^{\infty} M\frac{(\sqrt{2})\|\eta_k\|\|\vartheta\|}{\rho} = (\sqrt{2})\|\vartheta\| \sum_{k=1}^{\infty} M\frac{\|\eta_k\|}{\rho} < \infty.$$

Thus, $\forall\ \{\eta_k\} \in \ell^M_{\mathbb{C}_2},\ \vartheta \in \mathbb{C}_2$ implies $\eta_k\vartheta \in \ell^M_{\mathbb{C}_2}$. □

**Theorem 3.3.** *$\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta^m_n, p, u, \|.\|)$ is a $\mathbb{C}_2-$submodule of $\omega_4$.*

*Proof.* As $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta^m_n, p, u, \|.\|)$ is a subspace of $\omega_4$. Now, $\forall\vartheta \in \mathbb{C}_2$ and $\forall\ \{\eta_k\} \in \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta^m_n, p, u, \|.\|)$ we have

$$\begin{aligned}
&\sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big(\Big[M_k\Big(\frac{\|u_k\Delta^m_n\eta_k\vartheta\|}{\rho}\Big)\Big]^{p_k}\Big)^{\frac{1}{p_k}} \\
&\leq \sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big(\Big[M_k\Big(\frac{\sqrt{2}\|u_k\Delta^m_n\eta_k\|\|\vartheta\|}{\rho}\Big)\Big]^{p_k}\Big)^{\frac{1}{p_k}} \\
&= \sqrt{2}\ \|\vartheta\|\ \sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(\frac{\|u_k\Delta^m_n\eta_k\|}{\rho}\Big)\Big] \\
&< \infty.
\end{aligned}$$

Thus, $\forall\vartheta \in \mathbb{C}_2$ and $\forall\ \{\eta_k\} \in \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta^m_n, p, u, \|.\|)$, we have $\vartheta\eta_k \in \ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta^m_n, p, u, \|.\|)$. Hence $\ell^\infty(\mathbb{C}_2, \theta, \mathcal{M}, \Delta^m_n, p, u, \|.\|)$ is a $\mathbb{C}_2-$submodule of $\omega_4$. □

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