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Unified Massera type theorems for dynamic equations on time scales

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Abstract. In this paper, we aim to obtain Massera type theorems for both linear and nonlinear dynamic equations by using a generalized periodicity notion, namely (T, λ)-periodicity, on time scales. To achieve this task, first we define a new boundedness concept so-called λ-boundedness, and then we establish a linkage between the existence of λ-bounded solutions and (T, λ)-periodic solutions of dynamic equations in both linear and nonlinear cases. In our analysis, we assume that the time scale $\mathbb{T}$ is periodic in shifts $\delta_{\pm}$ which does not need to be translation invariant. Thus, outcomes of this work are valid for a large class of time-domains not restricted to $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.

1. Introduction

The theory of time scales is still a very popular research area for mathematicians since the time scale calculus prevents separate studies of continuous and discrete time mathematical models. By a quick literature review, it is possible to find very interesting applications of time scales in different fields such as mathematical analysis [8, 36], fractals and fractional calculus [9, 29], biology [24, 34, 37], economics [10, 19], and optimization [32]. Qualitative analysis of dynamic equations is one of the fundamental research paths in the theory of time scales. Since utilization of hybrid time domains enables researchers to study differential and difference equations under one roof, scholars have constructed unification and generalization of already established theories on time scales in the last three decades. Stability theory, oscillation theory, asymptotic theory, and existence of solutions of dynamic equations on time scales have become very fruitfull research areas in the applied mathematics, and undoubtedly, there is a vast literature on these titles.

Research of periodic structures and the existence of periodic solutions of dynamic equations on time scales have taken remarkable attention in applied mathematics. Initially, periodic functions are defined on translation invariant (additively periodic) time scales. That is, a time scale $\mathbb{T}$ is said to be translation invariant (additively periodic) if there exists a $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [22]). Then in [1], it is underlined that addition is not the only way for defining forward and backward motion on time scales, but instead the shift operators are proposed to step forward and backward on a time scale. In the sequel, Adıvar [2] introduced the new periodicity concept on time scales based on shift operators. We refer to readers [3, 4, 16–18, 21] as relevant papers.

In the existing literature, numerous types of periodicities are given for function classes which are presented as a relaxation or a generalization of conventional periodicity notion. As a relaxation of conventional periodicity, we can address almost periodicity and almost automorphy notions which have been introduced

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in 20^{th} century. Moreover, a generalization of conventional periodicity is introduced as (ω, c)-periodicity in recent papers [5, 6, 35], and (ω, c)-periodic solutions of differential equations are studied. Additionally, the discrete counter part of (ω, c)-periodicity is defined in [7]. A function f is said to be (ω, c)-periodic if

$$f(t+\omega) = cf(t)$$

for $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. One may easily deduce that (ω, c)-periodicity coincides with periodicity, anti-periodicity, and Bloch periodicity when c is particularly chosen as $c = 1$, $c = -1$, and $c = e^{ikN}$, respectively.

In the qualitative theory of dynamic equations, Massera type theorems are very popular since they establish a bridge between the existence of periodic solutions and the existence of bounded solutions. The original result was proved by Jose Luis Massera in [28]. In the past few decades, Massera type theorems are studied, reconstructed, generalized, and unified by several scholars for both linear and nonlinear dynamic equations on continuous, discrete, and hybrid time domains. Since, periodicity is a relaxable and generalizable condition for function classes, these type of theorems have been obtained also for almost periodic, almost automorphic, anti-periodic, quasi-periodic, and affine periodic solutions of dynamic equations. We refer to readers [13–15, 20, 23, 25, 27, 30, 31] as related papers.

The objective of this manuscript is two-fold:

- As the first task, we aim to exhibit (T, λ)-periodicity, (T, λ)-Δ-periodicity, (T, λ)-symmetry, and (T, λ)-Δ-symmetry notions for functions defined on time scales inspired by the papers [5–7]. In our design, we assume that the time scale $\mathbb{T}$ is periodic with respect to shift operators, thus our definitions and results also cover time scales which do not need to be translation invariant, for instance $q^{\mathbb{N}_0}$ or $q^{\mathbb{Z}} \cup \{0\}$. Provided definitions and results can be regarded as a relaxation and generalization of the new periodicity concept proposed in [2].

- As the second task, we aspire to obtain Massera type theorems regarding (T, λ)-periodic solutions of both linear and nonlinear dynamic equations on time scales. To achieve this objective, first we present a new boundedness definition, namely λ-boundedness. Then, the linkage between the existence of (T, λ)-periodic solutions and the existence of λ-bounded solutions are identified by employing fixed point theory and an asymptotic approach for linear and nonlinear dynamic equations, respectively. We provide some examples as an implementation of our results.

The organization of the paper is as follows: The next section is devoted to presentation of preliminaries for time scale calculus, shift operators, and the new periodicitiy concept on time scales. In Section 3, we illustrate (T, λ)-periodicity notions on time scales by utilizing shift operators. The last section aims to present the generalization of Massera type theorems.

2. Time Scale Preliminaries and Essentials of New Periodicity Concept Based on Shift Operators

2.1. Time Scale Calculus

Throughout the manuscript, we assume a familiarity with the theory of time scales. In this section, we just give a brief information about the basics of time scale calculus. Given definitions, results and examples can be found in pioneering books [11] and [12].

A time scale, denoted by $\mathbb{T}$, is an arbitrary, nonempty and closed subset of real numbers. The operator $\sigma : \mathbb{T} \to \mathbb{T}$ is called forward jump operator and defined by $\sigma(t) := \inf\{s \in \mathbb{T}, s > t\}$. The step size function $\mu : \mathbb{T} \to \mathbb{R}$ is given by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right dense if $\mu(t) = 0$, and right scattered if $\mu(t) > 0$. Moreover, a point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) := \sup\{s \in \mathbb{T}, s < t\} = t$ and left scattered if $\rho(t) < t$. We use the notation $[s, t)_{\mathbb{T}}$ to indicate the intersection $[s, t) \cap \mathbb{T}$ and the intervals $[s, t]_{\mathbb{T}}$, $(s, t)_{\mathbb{T}}$, and $(s, t]_{\mathbb{T}}$ are defined similarly.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous if it is continuous at right dense points and its left sided limits exists at left dense points. Moreover, the notation C_{rd} is used in order to represent all *rd*-continuous functions on $\mathbb{T}$. The set $\mathbb{T}^k$ is defined in the following way: If $\mathbb{T}$ has a left-scattered maximum m, then

$\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.
The delta-derivative of a function $f : \mathbb{T} \to \mathbb{C}$ at $t \in \mathbb{T}$ is given by

$$f^{\Delta}(t) = \begin{cases} \lim_{s\to t} \frac{f(t)-f(s)}{t-s}, & \mu(t) = 0 \\ \frac{f(\sigma(t))-f(t)}{\mu(t)}, & \mu(t) > 0 \end{cases}$$

provided the limit exists. For $f \in C_{rd}$ and $s, t \in \mathbb{T}$, we define delta-integral as

$$\int_s^t f(\tau)\,\Delta\tau = F(t) - F(s),$$

where $F^{\Delta} = f$ on $\mathbb{T}^k$.

Theorem 2.1 (Substitution). *Assume $\upsilon : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \upsilon(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and υ is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b g(s)\,\upsilon^{\Delta}(s)\,\Delta s = \int_{\upsilon(a)}^{\upsilon(b)} g\left(\upsilon^{-1}(s)\right)\widetilde{\Delta}s.$$

Definition 2.2. *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. We denote the set of all regressive functions by $\mathcal{R}$. Also, $\mathcal{R}^+$ stands for the set of all positively regressive elements of $\mathcal{R}$ defined by*

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Definition 2.3 (Exponential function). *For $h > 0$, set $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -1/h\}$, $\mathbb{J}_h := \{z \in \mathbb{C} : -\pi/h < Im(z) \leq \pi/h\}$ and $\mathbb{C}_0 := \mathbb{J}_0 := \mathbb{C}$. For $h \geq 0$ and $z \in \mathbb{C}_h$, the* cylinder transformation *$\xi_h : \mathbb{C}_h \to \mathbb{J}_h$ is given by*

$$\xi_h(z) := \begin{cases} z, & h = 0 \\ \frac{1}{h}\,\mathrm{Log}(1 + zh), & h > 0. \end{cases}$$

Then, the exponential function on $\mathbb{T}$ is presented in the form

$$e_p(t, s) := \exp\left\{\int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big)\Delta\tau\right\} \quad \text{for } s, t \in \mathbb{T}.$$

Lemma 2.4. *Let $p, q \in \mathcal{R}$. Then*

i. $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;

ii. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;

iii. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;

iv. $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;

v. $e_p(t, s)e_p(s, r) = e_p(t, r)$;

vi. $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$;

vii. $e_p(t, s)e_q(s, r) = e_{p\oplus q}(t, r)$, *where* $p \oplus q = p(t) + q(t) + p(t)q(t)\mu(t)$.

Theorem 2.5 (Variation of constants). *Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the regressive initial value problem*

$$\begin{cases} y^{\Delta}(t) = p(t)\, y(t) + f(t) \\ y(t_0) = y_0 \end{cases}$$

is given by

$$y(t) = e_p(t, t_0)\, y_0 + \int_{t_0}^{t} e_p(t, \sigma(s))\, f(s)\, \Delta s.$$

We refer to readers [11] for further reading on time scale calculus.

2.2. Shift operators and periodic time scales in shifts

In this part of the manuscript, we present the centerpieces of shift operators and the new periodicity concept on time scales constructed due to the shift operators. The given content can be found in [1] and [2].

Definition 2.6. *Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exists operators $\delta_\pm : [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:*

1. The functions $\delta_\pm$ are strictly increasing with respect to their second arguments, if

   $$(T_0, t), (T_0, u) \in \mathcal{D}_\pm := \{(s, t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_\pm(s, t) \in \mathbb{T}^*\},$$

   then

   $$T_0 \le t \le u \text{ implies } \delta_\pm(T_0, t) \le \delta_\pm(T_0, u);$$

2. If $(T_1, u), (T_2, u) \in \mathcal{D}_-$ with $T_1 < T_2$, then $\delta_-(T_1, u) > \delta_-(T_2, u)$ and if $(T_1, u), (T_2, u) \in \mathcal{D}_+$ with $T_1 < T_2$, then $\delta_+(T_1, u) < \delta_+(T_2, u)$;
3. If $t \in [t_0, \infty)_{\mathbb{T}}$, then $(t, t_0) \in \mathcal{D}_+$ and $\delta_+(t, t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0, t) \in \mathcal{D}_+$ and $\delta_+(t_0, t) = t$;
4. If $(s, t) \in \mathcal{D}_\pm$, then $(s, \delta_\pm(s, t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s, \delta_\pm(s, t)) = t$;
5. If $(s, t) \in \mathcal{D}_\pm$ and $(u, \delta_\pm(s, t)) \in \mathcal{D}_\mp$, then $(s, \delta_\mp(u, t)) \in \mathcal{D}_\pm$ and $\delta_\mp(u, \delta_\pm(s, t)) = \delta_\pm(s, \delta_\mp(u, t))$.

Then the operators δ_+ and δ_- are called forward and backward shift operators associated with the initial point t_0 on $\mathbb{T}^*$ and the sets $\mathcal{D}_+$ and $\mathcal{D}_-$ are domains of the operators, respectively.

Table 1 illustrates examples of some shift operators $\delta_\pm(s, t)$ defined on significant time scales:

$\mathbb{T}$	t_0	$\mathbb{T}^*$	$\delta_-(s,t)$	$\delta_+(s,t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t-s$	$t+s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t-s$	$t+s$
$q^{\mathbb{Z}} \cup \{0\}$	1	$q^{\mathbb{Z}}$	$\frac{t}{s}$	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$\left(t^2 - s^2\right)^{1/2}$	$\left(t^2 + s^2\right)^{1/2}$

Table 1

Lemma 2.7. *Let $\delta_\pm$ be the shift operators associated with the initial point t_0. Then we have the following:*

1. $\delta_-(t, t) = t_0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$;
2. $\delta_-(t_0, t) = t$ for all $t \in \mathbb{T}^*$;
3. If $(s, t) \in \mathcal{D}_+$, then $\delta_+(s, t) = u$ implies $\delta_-(s, u) = t$ and if $(s, u) \in \mathcal{D}_-$, then $\delta_-(s, u) = t$ implies $\delta_+(s, t) = u$;
4. $\delta_+(t, \delta_-(s, t_0)) = \delta_-(s, t)$ for all $(s, t) \in \mathcal{D}_+$ with $t \ge t_0$;
5. $\delta_+(u, t) = \delta_+(t, u)$ for all $(u, t) \in ([t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}) \cap \mathcal{D}_+$;
6. $\delta_+(s, t) \in [t_0, \infty)_{\mathbb{T}}$ for all $(s, t) \in \mathcal{D}_+$ with $t \ge t_0$;

7. $\delta_-(s,t) \in [t_0,\infty)_{\mathbb{T}}$ for all $(s,t) \in ([t_0,\infty)_{\mathbb{T}} \times [s,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$;
8. If $\delta_+(s,.)$ is Δ-differentiable in its second variable, then $\delta_+^{\Delta_t}(s,.) > 0$;
9. $\delta_+(\delta_-(u,s), \delta_-(s,v)) = \delta_-(u,v)$ for all $(s,v) \in ([t_0,\infty)_{\mathbb{T}} \times [s,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$ and $(u,s) \in ([t_0,\infty)_{\mathbb{T}} \times [u,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$;
10. If $(s,t) \in \mathcal{D}_-$ and $\delta_-(s,t) = t_0$, then $s = t$.

Definition 2.8 (Periodicity in shifts). *Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$, then $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$, if there exists a $p \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t) \in \mathcal{D}_\mp$ for all $t \in \mathbb{T}^*$. P is called the period of $\mathbb{T}$ if*

$$P = \inf\{p \in (t_0,\infty)_{\mathbb{T}^*} : (p,t) \in \mathcal{D}_\mp \text{ for all } t \in \mathbb{T}^*\} > t_0.$$

Remark 2.9. *Observe that an additive periodic time scale must be unbounded. However, unlike additive periodic time scales, a time scale, which is periodic in shifts, may be bounded.*

Example 2.10. *The following time scales are not additive periodic but periodic in shifts $\delta_\pm$.*

1. $\mathbb{T}_1 = \left\{\pm n^2 : n \in \mathbb{Z}\right\}$, $\delta_\pm(P,t) = \begin{cases} \left(\sqrt{t} \pm \sqrt{P}\right)^2 & \text{if } t > 0 \\ \pm P & \text{if } t = 0 \\ -\left(\sqrt{-t} \pm \sqrt{P}\right)^2 & \text{if } t < 0 \end{cases}$, $P = 1$, $t_0 = 0$,
2. $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = q$, $t_0 = 1$,
3. $\mathbb{T}_3 = \cup_{n\in\mathbb{Z}} \left[2^{2n}, 2^{2n+1}\right]$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = 4$, $t_0 = 1$,
4. $\mathbb{T}_4 = \left\{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\right\} \cup \{0,1\}$,

$$\delta_\pm(P,t) = \frac{q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}{1 + q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}, \quad P = \frac{q}{1+q}.$$

Notice that the time scale $\mathbb{T}_4$ in Example 2.10 is bounded above and below, and

$$\mathbb{T}_4^* = \left\{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\right\}.$$

Corollary 2.11. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Then we have*

$$\delta_\pm(P,\sigma(t)) = \sigma(\delta_\pm(P,t)) \text{ for all } t \in \mathbb{T}^*. \tag{1}$$

Definition 2.12 (Periodic function in shifts $\delta_\pm$). *Let $\mathbb{T}$ be a time scale P-periodic in shifts. We say that a real valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in \mathcal{D}_\pm \text{ and } f\left(\delta_\pm^T(t)\right) = f(t) \text{ for all } t \in \mathbb{T}^*, \tag{2}$$

where $\delta_\pm^T(t) = \delta_\pm(T,t)$. T is called period of f, if it is the smallest number satisfying (2).

Definition 2.13 (Δ-periodic function in shifts $\delta_\pm$). *Let $\mathbb{T}$ be a time scale P-periodic in shifts. A real valued function f defined on $\mathbb{T}^*$ is Δ-periodic function in shifts if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in \mathcal{D}_\pm \text{ for all } t \in \mathbb{T}^* \tag{3}$$

$$\text{the shifts } \delta_\pm^T \text{ are } \Delta\text{-differentiable with rd-continuous derivatives} \tag{4}$$

and

$$f\left(\delta_\pm^T(t)\right)\delta_\pm^{\Delta T}(t) = f(t) \tag{5}$$

for all $t \in \mathbb{T}^*$, where $\delta_\pm^T(t) = \delta_\pm(T,t)$. The smallest number T satisfying (3-5) is called period of f.

Lemma 2.14 ([18]). *Let $\mathbb{T}$ be a time scale that is periodic in shift operators $\delta_{\pm}$ with period P. Suppose that the shift operators $\delta_{\pm}^{T}$ are Δ-differentiable on $t \in \mathbb{T}^{*}$ where $T \in [P,\infty)_{\mathbb{T}^{*}}$. Then the graininess function $\mu : \mathbb{T} \to [0,\infty)$ satisfies*

$$\mu\left(\delta_{\pm}^{T}(t)\right) = \delta_{\pm}^{\Delta T}(t)\,\mu(t).$$

Lemma 2.15 ([16]). *Let $\mathbb{T}$ be a time scale P-periodic in shifts and the shift operators $\delta_{\pm}^{T}$ are Δ-differentiable on $t \in \mathbb{T}^{*}$ where $T \in [P,\infty)_{\mathbb{T}^{*}}$. Suppose that $p \in \mathcal{R}$ is a Δ-periodic function in shifts $\delta_{\pm}$ with period $T \in [P,\infty)_{\mathbb{T}^{*}}$. Then*

$$e_{p}\left(\delta_{\pm}^{T}(t), \delta_{\pm}^{T}(t_0)\right) = e_{p}(t,t_0)$$

for all $t, t_0 \in \mathbb{T}^{*}$.

3. (T,λ)-periodicity with respect to shift operators

The main objective of this section is to introduce (T,λ)-periodicity concept on time scales which is initiated in [26]. Provided definitions, results, and examples of this section can be found in [26].
Henceforth, we suppose that $\mathbb{T}$ is a P-periodic time scale in shifts $\delta_{\pm}$, and the shift operators $\delta_{\pm}$ are Δ-differentiable with rd-continuous derivatives. We use the phrase "periodic in shifts" to indicate periodicity in shifts $\delta_{\pm}$. Moreover, by $\delta_{\pm}^{(k)}(T,t)$, $k \in \mathbb{N}$ we denote k-times composition of shifts of $\delta_{\pm}^{T}$ with itself, namely,

$$\delta_{\pm}^{(k)}(T,t) := \underbrace{\delta_{\pm}^{T} \circ \delta_{\pm}^{T} \circ \ldots \circ \delta_{\pm}^{T}}_{k-times}(t).$$

Inspired by [4], we provide the following setup. Define

$$P(t_0) := \left\{\delta_{+}^{(k)}(T,t_0),\ k = 0,1,2,\ldots\right\}, \tag{6}$$

$$m(t) := \max\left\{k \in \mathbb{N} : \delta_{+}^{(k)}(T,t_0) \le t\right\}, \tag{7}$$

and accordingly any point $t \ge t_0$ of $\mathbb{T}^{*}$ can be decomposed as

$$t = \delta_{+}^{(m(t))}(T,t_0) + t_r, \tag{8}$$

where

$$t_r := \begin{cases} 0 & \text{if } t \in P(t_0) \\ \delta_{-}\left(\delta_{+}^{(m(t))}(T,t_0), t\right) & \text{if } t \notin P(t_0) \end{cases}. \tag{9}$$

Definition 3.1. *A function f defined on $\mathbb{T}^{*}$ is said to be (T,λ)-periodic in shifts if for a fixed $\lambda \in \mathbb{C}\setminus\{0\}$ there exists a $T \in [P,\infty)_{\mathbb{T}^{*}}$ such that*

$$(T,t) \in \mathcal{D}_{\pm} \text{ and } f\left(\delta_{\pm}^{T}(t)\right) = \lambda^{\pm 1} f(t) \text{ for all } t \in \mathbb{T}^{*}.$$

Lemma 3.2. *Let f be a (T,λ)-periodic function in shifts. Then, f can be represented as*

$$f(t) = \begin{cases} \lambda^{m(t)} f(t_0) & \text{if } t \in P(t_0) \\ \lambda^{m(t)} f(t_r) & \text{if } t \notin P(t_0) \end{cases}.$$

The proof of the above result is omitted since it is a direct consequence of Definition 3.1 and (6-9).

Example 3.3. *Let $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$ which is a q-periodic time scale with shift operators $\delta_{\pm}(P,t) = P^{\pm 1}t$. Then the function*

$$f(t) = (-2)^{\log_q t}$$

is $\left(q^2, 4\right)$-periodic in shifts. To see this, we write

$$f\left(\delta_+\left(q^2,t\right)\right) = (-2)^{\log_q q^2 t} = (-1)^{2+\log_q t}(2)^{2+\log_q t} = 4(-2)^{\log_q t} = 4f(t).$$

Remark 3.4. *In [2], it is highlighted that set of real numbers $\mathbb{R}$ is not only an additively periodic time scale but also a time scale periodic in shifts with $t_0 = 1$, where*

$$\delta_-(s,t) = \begin{cases} t/s & \text{if } t \geq 0 \\ st & \text{if } t < 0 \end{cases}, \text{ for } s \in [1,\infty), \tag{10}$$

and

$$\delta_+(s,t) = \begin{cases} st & \text{if } t \geq 0 \\ t/s & \text{if } t < 0 \end{cases}, \text{ for } s \in [1,\infty). \tag{11}$$

In the next example, we slightly modify [2, Example 5] in order to give an example of (T,λ)-periodic function on $\mathbb{R}$ corresponding to shift operator given in (11).

Example 3.5. *Let $\mathbb{T} = \mathbb{R}$ with shift operators given in (10-11). We define the function*

$$f(t) = \sin\left(-\frac{\ln t}{\ln 3}\pi\right)3^{-\log_3 t}, \ t > 0$$

as a $\left(9, \frac{1}{9}\right)$-periodic function on the half line since

$$\begin{aligned} f(\delta_+(9,t)) &= \sin\left(-\frac{\ln 9t}{\ln 3}\pi\right)3^{-\log_3 9t} \\ &= \sin\left(-\frac{\ln 9 + \ln t}{\ln 3}\right)3^{-2-\log_3 t} \\ &= \frac{1}{9}\sin\left(-\frac{\ln t}{\ln 3}\pi\right)3^{-\log_3 t} \\ &= \frac{1}{9}f(t). \end{aligned}$$

Next, we introduce (T,λ)-Δ-periodic function in shifts in a similar fashion with [2, Definition 6].

Definition 3.6. *A function f defined on $\mathbb{T}^*$ is said to be (T,λ)-Δ-periodic function in shifts if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in \mathcal{D}_{\pm} \text{ and } f\left(\delta_{\pm}^{T}(t)\right)\delta_{\pm}^{\Delta T}(t) = \lambda^{\pm 1}f(t) \text{ for all } t \in \mathbb{T}^*.$$

Example 3.7. *Let $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$ with shift operators $\delta_{\pm}(P,t) = P^{\pm 1}t$. Then, the function*

$$f(t) = \frac{2^{-\log_q t}}{t}$$

is $\left(q, \frac{1}{2}\right)$-$\Delta$-periodic in shifts; that is,

$$f\left(\delta_+^{q}(t)\right)\delta_+^{\Delta q}(t) = \frac{2^{-\log_q qt}}{qt}q = \frac{1}{2}\frac{2^{-\log_q t}}{t} = \frac{1}{2}f(t).$$

In the following lemma, we give a remarkable property regarding time scale exponential function:

Lemma 3.8. *Let $p \in \mathcal{R}$ be a (T, λ)-Δ-periodic function in shifts on $\mathbb{T}$ and suppose that also $\lambda p \in \mathcal{R}$. Then*

$$e_p\left(\delta_+^T(t), \delta_+^T(t_0)\right) = e_{\lambda p}(t, t_0) \text{ for } t, t_0 \in \mathbb{T}^*.$$

Proof. We assume p is a (T, λ)-Δ-periodic function and $p, \lambda p \in \mathcal{R}$. Then, we present the time scale exponential function as

$$e_p\left(\delta_+^T(t), \delta_+^T(t_0)\right) = \begin{cases} \exp\left(\int\limits_{\delta_+^T(t_0)}^{\delta_+^T(t)} \frac{1}{\mu(z)} \mathrm{Log}(1 + \mu(z)p(z))\,\Delta z\right) & \text{if } \mu(z) \neq 0 \\ \exp\left(\int\limits_{\delta_+^T(t_0)}^{\delta_+^T(t)} p(z)\,\Delta z\right) & \text{if } \mu(z) = 0 \end{cases}.$$

By using Theorem 2.1, Lemma 2.14, and (T, λ)-Δ-periodicity of p in shifts, we obtain

$$\begin{aligned} e_p\left(\delta_+^T(t), \delta_+^T(t_0)\right) &= \begin{cases} \exp\left(\int\limits_{t_0}^{t} \frac{\delta_+^{\Delta T}(z)}{\mu(\delta_+^T(z))} \mathrm{Log}(1 + \mu(\delta_+^T(z))p(\delta_+^T(z)))\,\Delta z\right) & \text{if } \mu(z) \neq 0 \\ \exp\left(\int\limits_{t_0}^{t} p\left(\delta_+^T(z)\right) \delta_+^{\Delta T}(z)\,\Delta z\right) & \text{if } \mu(z) = 0 \end{cases} \\ &= \begin{cases} \exp\left(\int\limits_{t_0}^{t} \frac{\delta_+^{\Delta T}(z)}{\mu(\delta_+^T(z))} \mathrm{Log}(1 + \frac{\delta_+^{\Delta T}(z)}{\delta_+^{\Delta T}(z)}\mu(\delta_+^T(z))p(\delta_+^T(z)))\,\Delta z\right) & \text{if } \mu(z) \neq 0 \\ \exp\left(\int\limits_{t_0}^{t} \lambda p(z)\ \Delta z\right) & \text{if } \mu(z) = 0 \end{cases} \\ &= \begin{cases} \exp\left(\int\limits_{t_0}^{t} \frac{1}{\mu(z)} \mathrm{Log}(1 + \lambda\mu(z)p(z))\,\Delta z\right) & \text{if } \mu(z) \neq 0 \\ \exp\left(\int\limits_{t_0}^{t} \lambda p(z)\ \Delta z\right) & \text{if } \mu(z) = 0 \end{cases} \\ &= e_{\lambda p}(t, t_0), \end{aligned}$$

which proves our assertion. □

We conclude this section by introducing (T, λ)-symmetry on time scales.

Definition 3.9. *A function $f : \mathbb{T}^* \times \mathbb{R} \to \mathbb{R}$ is said to be (T, λ)-symmetric in shifts if for a fixed $\lambda \in \mathbb{C}\backslash\{0\}$ there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that*

$$(T, t) \in \mathcal{D}_\pm \text{ and } f\left(\delta_\pm^T(t), x\right) = \lambda^{\pm 1} f\left(t, \lambda^{\mp 1}x\right) \text{ for all } t \in \mathbb{T}^*.$$

Definition 3.10. *A function $f : \mathbb{T}^* \times \mathbb{R} \to \mathbb{R}$ is said to be (T, λ)-Δ-symmetric in shifts if for a fixed $\lambda \in \mathbb{C}\backslash\{0\}$ there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that*

$$(T, t) \in \mathcal{D}_\pm \text{ and } f\left(\delta_\pm^T(t), x\right) = \lambda^{\pm 1} f\left(t, \lambda^{\mp 1}x\right) \delta_\pm^{\Delta T}(t) \text{ for all } t \in \mathbb{T}^*.$$

4. Unified Massera type theorems

This section is devoted to presentation of Massera's theorems for both linear and nonlinear dynamic equations constructed on time scales periodic in shifts. Hereafter, we consider (T, λ)-periodicity only depending on forward motion on time scales, i.e., $f\left(\delta_+^T(t)\right) = \lambda f(t)$ for the sake of brevity. Also, (T, λ)-Δ-periodic, (T, λ)-symmetric, and (T, λ)-Δ-symmetric functions in shifts are going to be utilized in a similar fashion.

4.1. *Linear Case*

Let $\mathbb{T}$ be a time scale periodic in shifts. Consider the following dynamic equation

$$\begin{cases} x^\Delta(t) = a(t)x(t) + f(t) \\ \quad x(t_0) = x_0 \end{cases}, \ t \in \mathbb{T}, \tag{12}$$

where a is Δ-periodic in shifts, f is (T, λ)-Δ-periodic function, $a \in \mathcal{R}$, and $f \in C_{rd}$.

We give the following result which is crucial in the construction of the Massera's theorem.

Theorem 4.1. *Consider the dynamic equation (12) with functions a and f which are Δ-periodic and (T, λ)-Δ-periodic functions in shifts, respectively. If at least one solution x of (12) satisfies the condition $\frac{1}{\lambda}x\left(\delta_+^T(t_0)\right) = x(t_0)$ for $t_0 \in \mathbb{T}^*$, then x is (T, λ)-periodic solution of (12).*

Proof. Assume that x is a solution of (12) with $\frac{1}{\lambda}x\left(\delta_+^T(t_0)\right) = x(t_0)$. We introduce the function $\xi(t) = \frac{1}{\lambda}x\left(\delta_+^T(t)\right) - x(t)$, and obviously $\xi(t_0) = 0$. Next, we consider

$$\begin{aligned} \xi^\Delta(t) &= \frac{1}{\lambda}x^\Delta\left(\delta_+^T(t)\right) - x^\Delta(t) \\ &= \frac{1}{\lambda}\left(a\left(\delta_+^T(t)\right)x\left(\delta_+^T(t)\right) + f\left(\delta_+^T(t)\right)\right)\delta_+^{\Delta T}(t) - a(t)x(t) - f(t) \\ &= a\left(\delta_+^T(t)\right)\delta_+^{\Delta T}(t)\frac{1}{\lambda}x\left(\delta_+^T(t)\right) - a(t)x(t) \\ &= a(t)\left(\frac{1}{\lambda}x\left(\delta_+^T(t)\right) - x(t)\right) \\ &= a(t)x(t) \end{aligned}$$

which indicates $\xi(t) \equiv 0$. Thus $x\left(\delta_+^T(t)\right) = \lambda x(t)$, and this proves the assertion. □

In this study, we focus on the relationship between the existence of a bounded solution and a (T, λ)-periodic solution of (12). Inspired by [25, Definition 1], we introduce an alternative boundedness concept called λ-boundedness for a function defined on a time scale.

Definition 4.2. *Let $T \in [P, \infty)_{\mathbb{T}^*}$ be fixed constant, where P is the period of the time scale. A function $x : \mathbb{T}^* \to \mathbb{R}$ is said to be λ-bounded if*

$$\left|\lambda^{-m(t)}x(t)\right| \le M \text{ for all } t \in \mathbb{T}^*,$$

where λ is a fixed nonzero constant and $m(t)$ is as in (7).

Next, we recall the following fixed point theorem as the fundamental tool for the proof of the unified Massera's theorem based on (T, λ)-periodicity.

Theorem 4.3 ([33]). *[Brouwer's fixed point theorem]Every continuous function that maps a compact convex subset of a Euclidian space into itself has a fixed point.*

Now, we are ready to present the first main result of this study.

Theorem 4.4. *The dynamical equation (12) has a (T,λ)-periodic solution in shifts if and only if it has a λ-bounded solution.*

Proof. Suppose that the dynamical equation in (12) has a (T,λ)-periodic solution in shifts, and fix

$$M = \sup_{k\in[t_0,T)} |x(k)|.$$

Then, we write

$$\begin{aligned}\left|\lambda^{-m(t)}x(t)\right| &= \begin{cases}\left|\lambda^{-m(t)}x\left(\delta_+^{(m(t))}(T,t_0)\right)\right| & \text{if } t\in P(t_0)\\ \left|\lambda^{-m(t)}x\left(\delta_+^{(m(t))}(T,t_r)\right)\right| & \text{if } t\notin P(t_0)\end{cases}\\ &= \begin{cases}\left|\lambda^{-m(t)}\lambda^{m(t)}x(t_0)\right| & \text{if } t\in P(t_0)\\ \left|\lambda^{-m(t)}\lambda^{m(t)}x(t_r)\right| & \text{if } t\notin P(t_0)\end{cases}\\ &\le M,\end{aligned}$$

where we employ the representation given in (9) and use the fact $t_0<t_r<T$ for the case $t\notin P(t_0)$.

On the other hand, suppose that (12) has a λ-bounded solution. By Theorem 2.5, the unique solution of (12) is given by

$$x(t,x_0) = e_a(t,t_0)x_0 + \int_{t_0}^{t} e_a(t,\sigma(s))f(s)\Delta s.$$

In the sequel, we define the following set

$$\Phi := \left\{x_0\in\mathbb{R} : |x_0|\le w \text{ and } \left|\lambda^{-m(t)}x(t,x_0)\right|\le w\right\}, \tag{13}$$

which is a nonempty, bounded and closed; therefore, a compact subset of $\mathbb{R}$. By letting $x_1,x_2\in\Phi$ and $0\le\alpha\le1$, we verify convexity of Φ as follows:

$$|\alpha x_1 + (1-\alpha)x_2| \le \alpha|x_1| + (1-\alpha)|x_2| \le w,$$

and

$$\begin{aligned}\left|\lambda^{-m(t)}x(t,\alpha x_1+(1-\alpha)x_2)\right| &= \left|\lambda^{-m(t)}\left[e_a(t,t_0)(\alpha x_1+(1-\alpha)x_2) + \int_{t_0}^{t} e_a(t,\sigma(s))f(s)\Delta s\right]\right|\\ &= \left|\lambda^{-m(t)}\left[\alpha\left(e_a(t,t_0)x_1 + \int_{t_0}^{t} e_a(t,\sigma(s))f(s)\Delta s\right)\right.\right.\\ &\quad\left.\left. + (1-\alpha)\left(e_a(t,t_0)x_2 + \int_{t_0}^{t} e_a(t,\sigma(s))f(s)\Delta s\right)\right]\right|\\ &= \left|\alpha\lambda^{-m(t)}x(t,x_1) + \lambda^{-m(t)}(1-\alpha)x(t,x_2)\right|\\ &\le w.\end{aligned}$$

As the core part of our proof, we define the continuous operator $H:\Phi\to\mathbb{R}$ by

$$H(x_0) = \lambda^{-1}x\left(\delta_+^T(t_0),x_0\right).$$

For all $x_0 \in \Phi$ we have

$$|H(x_0)| = \left|\lambda^{-1} x\left(\delta_+^T(t_0), x_0\right)\right| \le w.$$

Moreover, we consider

$$\begin{aligned}
\left|\lambda^{-m(t)} x(t, H(x_0))\right| &= \left|\lambda^{-m(t)}\left[e_a(t, t_0) H(x_0) + \int_{t_0}^{t} e_a(t, \sigma(s)) f(s)\Delta s\right]\right| \\
&= \lambda^{-m(t)}\left[\lambda^{-1} e_a(t, t_0)\left(e_a\left(\delta_+^T(t_0), t_0\right) x_0 + \int_{t_0}^{\delta_+^T(t_0)} e_a\left(\delta_+^T(t_0), \sigma(s)\right) f(s)\Delta s\right)\right. \\
&\quad \left. + \int_{t_0}^{t} e_a(t, \sigma(s)) f(s)\Delta s\right].
\end{aligned}$$

Then,

$$\begin{aligned}
\left|\lambda^{-m(t)} x(t, H(x_0))\right| &= \lambda^{-m(t)}\left[\lambda^{-1} e_a\left(\delta_+^T(t), \delta_+^T(t_0)\right) e_a\left(\delta_+^T(t_0), t_0\right) x_0\right. \\
&\quad + e_a\left(\delta_+^T(t), \delta_+^T(t_0)\right) \int_{t_0}^{\delta_+^T(t_0)} e_a\left(\delta_+^T(t_0), \sigma(s)\right) f(s)\Delta s \\
&\quad \left. + \int_{\delta_+^T(t_0)}^{\delta_+^T(t)} e_a\left(t, \sigma\left(\delta_-^T(s)\right)\right) f\left(\delta_-^T(s)\right) \delta_-^{\Delta T}(s)\Delta s\right] \\
&= \left|\lambda^{-m(t)}\left[\lambda^{-1} e_a\left(\delta_+^T(t), t_0\right) x_0 + \lambda^{-1}\int_{t_0}^{\delta_+^T(t)} e_a\left(\delta_+^T(t), t_0\right) f(s)\Delta s\right]\right| \\
&= \left|\lambda^{-m(t)-1} x\left(\delta_+^T(t), x_0\right)\right| \le w.
\end{aligned}$$

Thus H maps Φ into itself. Brouwer's fixed point theorem implies H has a fixed point in Φ; that is, there exists $x_0^* \in \Phi$ such that $H\left(x_0^*\right) = \lambda^{-1} x\left(\delta_+^T(t_0), x_0^*\right) = x_0^*$. Consequently, (12) has a (T, λ)-periodic solution in shifts by Theorem 4.1. □

Remark 4.5. *In the linear dynamic equation (12), we necessarily assume that a is Δ T-periodic in shifts. This assumption enables us to use crucial property of time scale exponential function given in Lemma 2.15. Therefore, we obtain (T, λ)-periodicity of solution for (12) by using a mapping in the light of Theorem 2.5. Unfortunately, we are unable to relax Δ-periodicity condition on the function a by replacing it with (T, λ)-Δ-periodicity, since Lemma 3.8 illustrates $e_p\left(\delta_+^T(t), \delta_+^T(t_0)\right) = e_{\lambda p}(t, t_0)$ for $t, t_0 \in \mathbb{T}^*$ $\left(e_p\left(\delta_+^T(t), \delta_+^T(t_0)\right) \neq \lambda e_p(t, t_0)\right)$.*

Example 4.6. *Set $\mathbb{T} = \mathbb{R}$ with the shift operators $\delta_\pm(s, t) = s^{\pm 1} t$, and $t_0 = 1$. We focus on the following linear differential equation*

$$x'(t) = a(t)x(t) + f(t), \ t \in \mathbb{R}, \tag{14}$$

where a is Δ-periodic in shifts, and f is (T, λ)-Δ-periodic in shifts. Then, the linear differential equation (14) has a solution of the following form

$$x(t) = \int_{-\infty}^{t} e^{\int_s^t a(\tau)d\tau} f(s)\,ds.$$

We write

$$\begin{aligned}
\frac{1}{\lambda}x(Tt) &= \frac{1}{\lambda}\int_{-\infty}^{Tt} e^{\int_s^{Tt} a(\tau)d\tau} f(s)\,ds \\
&= \frac{1}{\lambda}\int_{-\infty}^{t} e^{\int_{Ts}^{Tt} a(\tau)d\tau} f(Ts)\,Tds \\
&= \frac{1}{\lambda}\int_{-\infty}^{t} e^{\int_s^{t} a(\tau)d\tau} \lambda f(s)\,ds \\
&= x(t),
\end{aligned}$$

which results in x being a λ-bounded solution. Then Theorem 4.4 implies that the linear differential equation (14) has a (T,λ)-periodic solution.

Next, we give the following specific example which is constructed on a quantum domain.

Example 4.7. *Let $\mathbb{T} = \overline{2^{\mathbb{Z}}}$ with shift operators $\delta_{\pm}(s,t) = s^{\pm 1}t$, and consider the initial value problem*

$$\begin{cases} x^{\Delta}(t) = \frac{x(t)}{t} + \frac{3^{\log_2 t}}{t} \\ x(1) = 1 \end{cases}, t \in 2^{\mathbb{Z}}, \tag{15}$$

where

$$x^{\Delta}(t) = \frac{x(2t) - x(t)}{t}. \tag{16}$$

If we compare (12) with (15), then clearly $a(t) = \frac{1}{t}$ and $f(t) = \frac{3^{\log_2 t}}{t}$. It is obvious that a is Δ 2-periodic in shifts. Moreover, we write

$$f\left(\delta_+^2(t)\right)\delta_+^{\Delta 2}(t) = 2f(2t) = \frac{3^{\log_2 2t}}{t} = 3f(t),$$

and this shows that f is $(2,3)$-Δ-periodic in shifts. Next, we get

$$x(2) = 2x(1) + 1 = 3$$

by (16), and $\frac{1}{3}x\left(\delta_+^2(1)\right) = x(1)$. Consequently, Theorem 4.1 implies that the solution of the initial value problem (15) is $(2,3)$-periodic in shifts, and accordingly it is 3-bounded by Theorem 4.4.

4.2. Nonlinear Case

In this part, we consider the following functional dynamic equation defined on a periodic time scale $\mathbb{T}$ in shifts

$$x^{\Delta}(t) = f(t, x(t)), \tag{17}$$

where $f : \mathbb{T}^* \times \mathbb{R} \to \mathbb{R}$ is continuous, and (T,λ)-Δ-symmetric in shifts. The main objective of this section is obtaining a Massera type theorem for the nonlinear equation (17).

Before presentation of the second fundamental result, we list the following outcomes.

Lemma 4.8. *If x is a solution of the (T,λ)-symmetric equation (17), then $y(t) = \lambda^{-1}x\left(\delta_+^T(t)\right)$ is also a solution for (17).*

Proof. Suppose that x is a solution of (17), and set $y(t) = \frac{1}{\lambda} x\left(\delta_+^T(t)\right)$. Then

$$\begin{aligned} y^{\Delta}(t) &= \lambda^{-1} x^{\Delta}\left(\delta_+^T(t)\right) \\ &= \lambda^{-1} f\left(\delta_+^T(t), x\left(\delta_+^T(t)\right)\right) \delta_+^{\Delta T}(t) \\ &= f\left(t, \lambda^{-1} x\left(\delta_+^T(t)\right)\right) \\ &= f(t, y). \end{aligned}$$

This completes the proof. □

Lemma 4.9. *The (T, λ)-symmetric equation (17) has a (T, λ)-periodic solution in shifts if and only if $\lambda^{-1} x\left(\delta_+^T(t_0)\right) = x(t_0)$.*

Proof. If x is a (T, λ)-periodic solution of (17), then clearly, the equation $\lambda^{-1} x\left(\delta_+^T(t_0)\right) = x(t_0)$ holds. On the contrary, suppose that x is a solution of (T, λ)-symmetric equation (17) with $\lambda^{-1} x\left(\delta_+^T(t_0)\right) = x(t_0)$. Then, $y(t) = \lambda^{-1} x\left(\delta_+^T(t)\right)$ is also a solution of (17) with $y(t_0) = x(t_0)$ due to Lemma 4.8. This yields to $y(t) = x(t)$ for every $t \in \mathbb{T}^*$ by uniqueness of the solutions. Consequentially, x is (T, λ)-periodic in shifts. □

Lemma 4.10. *Consider the (T, λ)-symmetric equation (17) and suppose that the condition*

$$x < y \text{ implies } x + \mu(t) f(t, x) < y + \mu(t) f(t, y) \tag{18}$$

for all $t \in \mathbb{T}^*$. Then

$$x(t_0) < y(t_0) \text{ implies } x(t) < y(t) \text{ for all } t \in \mathbb{T}^*.$$

Proof. Let $x(t_0) < y(t_0)$ and (18) holds. By induction principle, we suppose that $x(t) < y(t)$. Then, we write

$$\begin{aligned} x(\sigma(t)) &= x(t) + \mu(t) x^{\Delta}(t) \\ &= x(t) + \mu(t) f(t, x) \\ &< y(t) + \mu(t) f(t, y) \\ &= y(t) + \mu(t) y^{\Delta}(t) \\ &= y(\sigma(t)). \end{aligned}$$

This proves the assertion. □

Theorem 4.11. *Suppose that the condition (18) holds. Then the (T, λ)-symmetric equation (17) has a λ-bounded solution if and only if it has a (T, λ)-periodic solution.*

Proof. If (17) has a (T, λ)-periodic solution in shifts, then obviously it is λ-bounded. Thus, the one side of the proof is straightforward.

Conversely, suppose that the condition (18) is satisfied, and the (T, λ)-symmetric equation (17) has a λ-bounded solution, i.e., $\left|\lambda^{-m(t)} x(t)\right| \le M$ for all $t \in \mathbb{T}^*$. We define the sequence x_n as

$$x_n(t) = \lambda^{-m\left(\delta_+^{(n)}(T,t)\right)} x\left(\delta_+^{(n)}(T, t)\right) = \lambda^{-m(t)-n} x\left(\delta_+^{(n)}(T, t)\right)$$

for $n \in \mathbb{N}_0$. Lemma 4.8 implies that x_n is a solution for (17) for each $n \in \mathbb{N}$. It should be pointed out that the sequence x_n is bounded since x is λ-bounded, that is

$$|x_n(t)| = \left|\lambda^{-m\left(\delta_+^{(n)}(T,t)\right)} x\left(\delta_+^{(n)}(T, t)\right)\right| \le M$$

for all $t \in \mathbb{T}^*$.

Now assume that $x(t_0) = x_1(t_0)$, which results in $x(t_0) = \lambda^{-1} x\left(\delta_+^T(t_0)\right)$. Then Lemma 4.9 indicates x is (T,λ)-periodic. Also, suppose that $x(t_0) < x_1(t_0)$, which yields to $x(t) < x_1(t)$ for all $t \in \mathbb{T}^*$ by Lemma 4.10. In the sequel, one may easily write that

$$x\left(\delta_+^{(n)}(T,t)\right) < x_1\left(\delta_+^{(n)}(T,t)\right) \text{ for all } t \in \mathbb{T}^*,$$

and

$$x_n(t) = \lambda^{-m\left(\delta_+^{(n)}(T,t)\right)} x\left(\delta_+^{(n)}(T,t)\right) < \lambda^{-m\left(\delta_+^{(n)}(T,t)\right)} x_1\left(\delta_+^{(n)}(T,t)\right) < x_{n+1}(t)$$

for all $t \in \mathbb{T}^*$. Thus, x_n is increasing and bounded. Consequentially, we have the pointwise limit $\lim_{n\to\infty} x_n(t) = \bar{x}(t)$ for $t \in \mathbb{T}^*$. Thus

$$\bar{x}^\Delta(t) = \lim_{n\to\infty} x_n^\Delta(t) = \lim_{n\to\infty} f(t, x_n(t)) = f(t, x(t)) \text{ for each } t \in \mathbb{T}^*,$$

and this shows $\bar{x}$ is a solution for (T,λ)-symmetric equation (17). Next, we verify (T,λ)-periodicity of $\bar{x}$ by writing

$$\lambda^{-1}\bar{x}\left(\delta_+^T(t)\right) = \lim_{n\to\infty} \lambda^{-1} x_n\left(\delta_+^T(t)\right) = \lim_{n\to\infty} x_{n+1}(t) = \bar{x}(t) \text{ for each } t \in \mathbb{T}^*.$$

Besides, the case $x(t) > x_1(t)$ can be proved in a similar fashion, therefore it is omitted. The proof is complete.
□

Example 4.12. *Let $\mathbb{T} = \mathbb{Z}$ with the shift operators $\delta_\pm(s,t) = s \pm t$. It should be highlighted that (T,λ)-Δ-symmetry coincides with (T,λ)-symmetry with the particular choice of shift operators on $\mathbb{T} = \mathbb{Z}$ since $\delta_\pm^\Delta(s,t) = 1$. Consider the following discrete initial value problem*

$$\begin{cases} \Delta x(t) = \frac{\cos(\pi t)}{2^t}(x(t))^2 \\ x(0) = -3 \end{cases}, \ t \in \mathbb{Z}, \tag{19}$$

where $\Delta x(t) = x(t+1) - x(t)$. If we compare the nonlinear difference equation in (19) with (17), then it is obvious that $f(t,x) = \frac{\cos(\pi t)}{2^t}x^2$. The difference equation given in (19) is $(2,4)$-symmetric ($(2,4)$-Δ-symmetric). To see this, we write

$$f(t+2,x) = \frac{\cos(\pi(t+2))}{2^{t+2}}x^2 = \frac{1}{4}\frac{\cos(\pi t)}{2^t}x^2 = 4f\left(t, \frac{1}{4}x\right).$$

Furthermore, the condition (18) holds. It can be easily verified that $\frac{1}{4}x(2) = x(0) = -3$, and consequently Lemma 4.9 implies that the $(2,4)$-symmetric equation in (19) has a $(2,4)$-periodic solution. Subsequently, (19) has a 4-bounded solution due to Theorem 4.11.

5. Conclusion

In this study, we employed (T,λ)-periodic functions on time scales defined with respect to the shift operators $\delta_\pm$ in order to obtain unified Massera type theorems for linear and nonlinear dynamic equations on hybrid time domains. Since (T,λ)-periodic functions coincide with periodic, anti-periodic, or Bloch periodic functions in particular choices of λ, our results may cover some existing results in the literature for certain time scales. Additionally, we define a new symmetry property, namely (T,λ)-symmetry, for functions of two variables, and we use this notion to present a Massera type criteria for nonlinear functional dynamic equations on time scales. Some examples are provided as an implementation of the main results for the readership.

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