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Lipschitz functions class for the generalized Dunkl transform

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Abstract. This paper is intented to establish the analogue of Titchmarsh's theorem for the Dunkl generalized transform on the real line.

1. Introduction

Consider the first-order singular differential-difference operator on $\mathbb{R}$

$$\mathrm{D}f(x) = \frac{df(x)}{dx} + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x} - 2n\frac{f(-x)}{x},$$

where $\alpha > -1/2$ and $n = 0, 1, ...$ For $n = 0$, we obtain the classical Dunkl operator with parameter $\alpha + \frac{1}{2}$ associated with the reflection group $\mathbb{Z}_2$ on the real line.

$$\mathrm{D}_\alpha f(x) = \frac{df(x)}{dx} + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x}.$$

These operators D have been generalized the calssical theory of Dunkl harmonics. The one-dimensional Dunkl introduced by Dunkl [6–8] and plays an important role in the study of quantum harmonic oscillators governed by Wigner's commutation rules ([9]).

We construct in this paper class of Lipschitz functions in the Hilbet space $\mathrm{L}^2(\mathbb{R}, |x|^{2\alpha+1}dx)$, where $\alpha > -1/2$, and we define the relationship between these classes.

Titchmarsh's [[10], Theorem 85] characterized the set of functions in $\mathrm{L}^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1. *[10] Let $\alpha \in (0, 1)$ and assume that $f \in \mathrm{L}^2(\mathbb{R})$. Then the following are equivalents:*

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1. $\|f(.+h)-f(.)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$

2. $\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow \infty$

where $\widehat{f}$ *stands for the Fourier transform of* f.

Using essentially the properties of the Dunkl generalized transform associated to D, we establish the analogue of Titchmarsh's theorem.

2. Preliminaries

In this section, we collect some notations and results on Dunkl generalized operator and Dunkl generalized transform (see [2, 3]).
In all what follows assume that $\alpha > -1/2$ and $n = 0, 1, ...$ Let $j_\alpha(z)$ is the normalized spherical Bessel function of index α, i.e.,

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{j=0}^{+\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{j!\Gamma(j+\alpha+1)}, \ (z \in \mathbb{C}) \tag{1}$$

The function j_α is infinitely differentiable and even, in addition $j_\alpha(0) = 1$. Moreover from formula (1) we see that

$$\lim_{z\to 0} \frac{j_\alpha(z)-1}{z^2} \neq 0. \tag{2}$$

The one-dimensional Dunkl kernel is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz) \ \ (z \in \mathbb{C}).$$

The function $y = e_\alpha(x)$ satisfies the equation $D_\alpha y = iy$ with initial condition $y(0) = 1$. If $\alpha = -1/2$ the one-dimentional Dunkl kernel coincides with the usual exponential function e^{ix}.

Using the correlation

$$j'_\alpha(x) = -\frac{x j_{\alpha+1}(x)}{2(\alpha+1)}.$$

We conclude that the function $e_\alpha(x)$ admits the representation

$$e_\alpha(x) = j_\alpha(ix) - i j'_\alpha(ix) \tag{3}$$

For all $x \in \mathbb{R}$, we have ([2])

$$|e_\alpha(ix)| \leq 1 \tag{4}$$

Lemma 2.1. *For* $x \in \mathbb{R}$ *the following inequalities are fulfilled.*

1. $|j_\alpha(x)| \leq 1$,
2. $|1 - j_\alpha(x)| \geq c$ *with* $|x| \geq 1$, *where* $c > 0$ *is a certain constant which depends only on* α.

Proof. (analog of Lemma 2.9 in [4]). □

In the terms of $j_\alpha(x)$, we have (see [1])

$$1 - j_\alpha(x) = O(1),\ x \geq 1, \tag{5}$$

$$1 - j_\alpha(x) = O(x^2),\ 0 \leq x \leq 1. \tag{6}$$

We denote by

• S(ℝ) the space of C^∞ functions f on ℝ, which are rapidly decreasing together with their devatives, i.e., such that for all $m, n = 0, 1, ...$

$$p_{n,m}(f) = \sup_{x\in\mathbb{R}}(1+|x|)^m \left|\frac{d^n}{dx^n}f(x)\right| < \infty.$$

The topology of S(ℝ) is defined by the semi-norms $p_{n,m}$.

• $S_n(\mathbb{R})$ the subspace of S(ℝ) consisting of functions f such that

$$f(0) = ... = f^{(2n-1)}(0) = 0$$

• $L^2_\alpha(\mathbb{R})$ the class of measurable functions f on ℝ for which

$$\|f\|_{2,\alpha} = \left(\int_{-\infty}^{+\infty} |f(x)|^2 |x|^{2\alpha+1} dx\right)^{1/2} < \infty.$$

From [2], we have

Definition 2.2. *The Dunkl generalized transform of a function* $f \in S_n(\mathbb{R})$ *is defined by*

$$\mathcal{F}(f)(\lambda) = \int_{-\infty}^{+\infty} f(x) e_{\alpha+2n}(-i\lambda x)|x|^{2\alpha+2n+1} dx,\ \lambda \in \mathbb{R}$$

If $n = 0$ then $\mathcal{F}$ reduces to Dunkl transform classical associated with reflection group $\mathbb{Z}_2$ on the real line.

Theorem 2.3. *The Dunkl generalized transform* $\mathcal{F}$ *is a to topological isomorphism* $S_n(\mathbb{R})$ *onto* S(ℝ). *The inverse transform is given by*

$$f(x) = m_{\alpha+2n} x^{2n} \int_{-\infty}^{+\infty} \mathcal{F}(f)(\lambda) e_{\alpha+2n}(i\lambda x)|\lambda|^{2\alpha+4n+1} d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2\alpha+2}\left(\Gamma(\alpha+1)\right)^2}.$$

Theorem 2.4. 1. *For every* $f \in S_n(\mathbb{R})$ *we have the Plancherel formula*

$$\int_{-\infty}^{+\infty} |f(x)|^2 |x|^{2\alpha+1} dx = m_{\alpha+2n} \int_{-\infty}^{+\infty} |\mathcal{F}(f)(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda.$$

2. *The Dunkl generalized transform* $\mathcal{F}$ *extends uniquely to an isometric isomorphism from* $L^2_\alpha(\mathbb{R})$ *onto* $L^2_{\alpha,n} = L^2(\mathbb{R}, m_{\alpha+2n}|\lambda|^{2\alpha+4n+1}d\lambda)$.

Definition 2.5. *The generalized translation operators* T_x, $x \in \mathbb{R}$, *tied to* D *are defined by*

$$\begin{aligned} T_x f(y) &= \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f\left(\sqrt{x^2+y^2-2xyt}\right)}{(x^2+y^2-2xyt)^n}\left(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt \\ &+ \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f\left(-\sqrt{x^2+y^2-2xyt}\right)}{(x^2+y^2-2xyt)^n}\left(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt, \end{aligned}$$

where

$$A(t) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}(1+t)(1-t^2)^{\alpha+2n-1/2}.$$

Proposition 2.6. *[2] Let $x \in \mathbb{R}$ and $f \in L^2_\alpha(\mathbb{R})$. Then $T_x f \in L^2_\alpha(\mathbb{R})$ and*

$$\|T_x f\|_{2,\alpha} \leq 2x^{2n}\|f\|_{2,\alpha}.$$

Furthermore

$$\mathcal{F}(T_x f)(\lambda) = x^{2n} e_{\alpha+2n}(i\lambda x)\mathcal{F}(f)(\lambda). \tag{7}$$

Lemma 2.7. *Let $f \in L^2_\alpha(\mathbb{R})$. Then*

$$\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|^2_{2,\alpha} = 4m_{\alpha+2n}h^{4n}\int_{-\infty}^{+\infty}|1 - j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda.$$

Proof. From formula (7), we have $\mathcal{F}(T_h f)(\lambda) = h^{2n}e_{\alpha+2n}(i\lambda h)\mathcal{F}(f)(\lambda)$ and $\mathcal{F}(T_{-h} f)(\lambda) = h^{2n}e_{\alpha+2n}(-i\lambda h)\mathcal{F}(f)(\lambda)$. Then

$$\mathcal{F}\left(T_h f + T_{-h} f - 2h^{2n} f\right)(\lambda) = h^{2n}(e_{\alpha+2n}(i\lambda h) + e_{\alpha+2n}(-i\lambda h) - 2)\mathcal{F}(f)(\lambda)$$

By formula (3) and the function $j_{\alpha+2n}$ is even, we obtain

$$\mathcal{F}\left(T_h f + T_{-h} f - 2h^{2n} f\right)(\lambda) = 2h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}(f)(\lambda).$$

Invoking Plancherel identity gives

$$\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|^2_{2,\alpha} = 4m_{\alpha+2n}h^{4n}\int_{-\infty}^{+\infty}|1 - j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda.$$

which ends the proof. □

3. Lipschitz class Functions

Definition 3.1. *Let $f \in L^2_\alpha(\mathbb{R})$, and let*

$$\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|_{2,\alpha} \leq Ch^\alpha, \ \alpha > 0,$$

i.e

$$\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|_{2,\alpha} = O(h^\alpha)$$

for all x in $\mathbb{R}$ and for all sufficiently small h, C being a positive constant. Then we say that f satisfies a Dunkl generalized Lipschitz of order α, or f belongs to $Lip(\alpha)$.

Definition 3.2. *If however*

$$\frac{\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|_{2,\alpha}}{h^\alpha} \to 0 \text{ as } h \to 0.$$

i.e

$$\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|_{2,\alpha} = o(h^\alpha) \text{ as } h \to 0, \ \alpha > 0$$

then f is said to be belong to the little Dunkl generalized Lipschitz class $lip(\alpha)$.

Remark It follows immediately from these definitions that

$$lip(\alpha) \subset Lip(\alpha) \text{ and } Lip(\alpha + \gamma) \subset lip(\alpha), \ \gamma > 0.$$

Theorem 3.3. *Let $\alpha > 1$. If $f \in Lip(\alpha)$, then $f \in lip(1)$.*

Proof. For h small, $x \in \mathbb{R}$ and $f \in Lip(\alpha)$ we have

$$\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|_{2,\alpha} \leq Ch^{\alpha}.$$

Then

$$0 \leq \frac{\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|_{2,\alpha}}{h} \leq Ch^{\alpha-1}$$

since $\lim_{h\to 0} h^{\alpha-1} = 0 \ (\alpha > 1)$. Thus

$$\frac{\|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|_{2,\alpha}}{h} \to 0 \ as \ h \to 0.$$

Then $f \in lip(1)$. □

Definition 3.4. *A function $f \in L^2_\alpha(\mathbb{R})$ is said to be in the ψ-Dunkl generalized Lipschitz class, denoted by $Lip_n(\psi)$, if*

$$\|T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha} \leq K\psi(h)$$

i.e.,

$$\|T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha} = O\left(\psi(h)\right) \ as \ h \to 0$$

for all $x \in \mathbb{R}$, C being a positive constant and

1. *$\psi(t)$ is continuous function in $[0, \infty[$,*
2. *$\psi(0) = 0$,*
3. *$\psi(t)$ is derivable and $\psi'(0) = 0$.*

Theorem 3.5. *Let $f \in L^2_\alpha(\mathbb{R})$ and let ψ be a fixed function satisfying the condition of Definition 3.4. If $f \in Lip_n(\psi)$, then $f \in lip(1)$.*

Proof. For $x \in \mathbb{R}$ and h small. If $f \in Lip_n(\psi)$ we have

$$\|T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha} = O\left(\psi(h)\right) \ as \ h \to 0.$$

Then

$$\frac{\|T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha}}{h} \leq C\frac{\psi(h)}{h}$$

i.e.,

$$0 \leq \frac{\|T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha}}{h} \leq C\frac{\psi(h) - \psi(0)}{h}$$

since, $\lim_{h\to 0} \frac{\psi(h)-\psi(0)}{h} = \psi'(0) = 0$. Thus

$$\frac{\|T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha}}{h} \to 0 \ as \ h \to 0$$

Then $f \in lip(1)$. □

Theorem 3.6. *If $\alpha < \beta$, then $Lip(\alpha) \supset Lip(\beta)$ and $lip(\alpha) \supset lip(\beta)$.*

Proof. We have $0 \leq h \leq 1$ and $\alpha < \beta$, then $h^\beta \leq h^\alpha$. Thus the proof of this theorem. □

Theorem 3.7. *Let $f \in L^2_\alpha(\mathbb{R})$. If f belong to $Lip(\alpha)$ then $T_h f \in Lip(\alpha + 2n)$.*

Proof. Assume that $f \in Lip(\alpha)$. Then

$$\|T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha} \leq Ch^{\alpha},$$

i.e.,

$$4m_{\alpha+2n} h^{4n} \int_{-\infty}^{+\infty} |1 - j_{\alpha+2n}(\lambda h|^2 |\mathcal{F}(f)(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda \leq C^2 h^{2\alpha}.$$

Since $\mathcal{F}(T_h f)(\lambda) = h^{2n} e_{\alpha+2n}(i\lambda h)\mathcal{F}(f)(\lambda)$, we have

$$\mathcal{F}(T_h(T_h f))(\lambda) = h^{2n} e_{\alpha+2n}(i\lambda h)\mathcal{F}(T_h f)(\lambda) = h^{4n} e^2_{\alpha+2n}(i\lambda h)\mathcal{F}(f)(\lambda).$$

and

$$\mathcal{F}(T_{-h}(T_h f))(\lambda) = h^{2n} e_{\alpha+2n}(-i\lambda h)\mathcal{F}(T_h f)(\lambda) = h^{4n} e_{\alpha+2n}(-i\lambda h) e_{\alpha+2n}(i\lambda h)\mathcal{F}(f)(\lambda).$$

Then

$$\begin{aligned}
&\mathcal{F}(T_h(T_h f) + T_{-h}(T_h f) - 2h^{2n} T_h f)(\lambda) \\
= \;& \left(\left(h^{4n} e^2_{\alpha+2n}(i\lambda) + h^{4n} e_{\alpha+2n}(-i\lambda h) e_{\alpha+2n}(i\lambda h) - 2h^{4n} e_{\alpha+2n}(i\lambda h)\right)\mathcal{F}(f)(\lambda)\right. \\
= \;& h^{4n} e_{\alpha+2n}(i\lambda h)\left(e_{\alpha+2n}(i\lambda h) + e_{\alpha+2n}(-i\lambda h) - 2\right)\mathcal{F}(f)(\lambda) \\
= \;& 2h^{4n} e_{\alpha+2n}(i\lambda h)\left(j_{\alpha+2n}(\lambda h) - 1\right)\mathcal{F}(f)(\lambda).
\end{aligned}$$

By Plancherel identity, we obtain

$$\begin{aligned}
&\|T_h(T_h f)(.) + T_{-h}(T_h f(.)) - 2h^{2n} T_h f(.)\|^2_{2,\alpha} \\
= \;& 4m_{\alpha+2n} h^{8n} \int_{-\infty}^{+\infty} |e_{\alpha+2n}(i\lambda h)|^2 |1 - j_{\alpha+2n}(\lambda h)|^2 |\mathcal{F}(f)(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda
\end{aligned}$$

From formula (4), we have

$$\begin{aligned}
&\|T_h(T_h f)(.) + T_{-h}(T_h f(.)) - 2h^{2n} T_h f(.)\|^2_{2,\alpha} \\
\leq \;& 4m_{\alpha+2n} h^{8n} \int_{-\infty}^{+\infty} |1 - j_{\alpha+2n}(\lambda h)|^2 |\mathcal{F}(f)(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda \\
= \;& 4m_{\alpha+2n} h^{8n} . \frac{1}{4m_{\alpha+2n} h^{4n}} \|T_h f(.) + T_{-h} f(.) - 2h^{2n} f(.)\|^2_{2,\alpha} \\
\leq \;& C^2 h^{4n} h^{2\alpha} = Ch^{2\alpha+4n}.
\end{aligned}$$

which completes the proof. □

Theorem 3.8. *Let $\alpha > 2$. If f belong to Dunkl generalized Lipschitz class, i.e.,*

$$f \in Lip(\alpha).$$

Then f is equal to the null function in $\mathbb{R}$.

Proof. Assume that $f \in Lip(\alpha)$. Then

$$\|T_h f(x) + T_{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha} \leq Ch^{\alpha}$$

So

$$4m_{\alpha+2n} h^{4n} \int_{-\infty}^{+\infty} |1 - j_{\alpha+2n}(\lambda h)|^2 |\mathcal{F}(f)(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda \leq C^2 h^{2\alpha}$$

Then

$$\frac{4m_{\alpha+2n} h^{4n} \int_{-\infty}^{+\infty} |1 - j_{\alpha+2n}(\lambda h)|^2 |\mathcal{F}(f)(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda}{h^4} \leq C^2 h^{2\alpha-4}$$

Since $\alpha > 2$, we have $\lim\limits_{h\to 0} h^{2\alpha-4} = 0$.
Therefore

$$\lim_{h\to 0} 4m_{\alpha+2n}h^{4n}\int_{-\infty}^{+\infty}\left(\frac{|1-j_{\alpha+2n}(\lambda h)|}{\lambda^2h^2}\right)^2|\lambda|^4|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda = 0$$

From this, (2) and Fatou's theorem we get

$$\||\lambda|^2\mathcal{F}(f)(\lambda)\|_{\mathrm{L}^2_{\alpha,n}} = 0.$$

Thus $|\lambda|^2\mathcal{F}(f)(\lambda) = 0$ for all $\lambda \in \mathbb{R}$, then $f(x)$ is the null function. □

Analog of theorem 3.8 we obtai these theorems

Theorem 3.9. *Let $f \in \mathrm{L}^2_\alpha(\mathbb{R})$ and ψ be a fixed function satisfying the conditions of Definition 3.4. If*

$$\|\mathrm{T}_hf(x) + \mathrm{T}_{-h}f(x) - 2h^{2n}f(x)\|_{2,\alpha} \le Ch^\beta\psi(h),$$

where C a positive constant and $\beta \ge 3$. Then f is equal to the null function in $\mathbb{R}$.

Theorem 3.10. *Let $f \in \mathrm{L}^2_\alpha(\mathbb{R})$. If f belong to lip(4), i.e.,*

$$\|\mathrm{T}_hf(x) + \mathrm{T}_{-h}f(x) - 2h^{2n}f(x)\|_{2,\alpha} = o(h^4) \text{ as } h \to 0.$$

Then f is equal to null function in $\mathbb{R}$.

4. Analog of Titchmarsh's theorem

Now, we give another the main result of this paper analog of theorem 1.1.

Theorem 4.1. *Let $\alpha \in (0,1)$ and $f \in \mathrm{L}^2_\alpha(\mathbb{R})$. The following are equivalent*

1. $f \in Lip(\alpha + 2n)$,
2. $\int_{|\lambda|\ge s}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda = O(s^{-2\alpha})$ *as* $s \to +\infty$

Proof. 1) $\Longrightarrow$ 2) Assume that $f \in Lip(\alpha + 2n)$. Then

$$\|\mathrm{T}_hf(.) + \mathrm{T}_{-h}f(.) - 2h^{2n}f(.)\|_{2,\alpha} = O(h^{\alpha+2n}) \text{ as } h \to 0.$$

By Lemma 2.7, we obtain

$$\|\mathrm{T}_hf(.) + \mathrm{T}_{-h}f(.) - 2h^{2n}f(.)\|^2_{2,\alpha} = 4m_{\alpha+2n}h^{4n}\int_{-\infty}^{+\infty}|1-j_{\alpha+2n}|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda.$$

If $|\lambda| \in \left[\frac{1}{h}, \frac{2}{h}\right]$, then $|\lambda h| \ge 1$ and (2) of Lemma 2.1 implies that

$$1 \le \frac{1}{c^2}|1-j_{\alpha+2n}(\lambda h)|^2.$$

Then

$$\begin{aligned}
\int_{\frac{1}{h}\le|\lambda|\le\frac{2}{h}}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda &\le \frac{1}{c^2}\int_{\frac{1}{h}\le|\lambda|\le\frac{2}{h}}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda\\
&\le \frac{1}{c^2}\int_{-\infty}^{+\infty}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda\\
&\le \frac{1}{c^2}\frac{1}{4m_{\alpha+2n}h^{4n}}\|\mathrm{T}_hf(.) + \mathrm{T}_{-h}f(.) - 2h^{2n}f(.)\|^2_{2,\alpha}\\
&= O(h^{2\alpha}).
\end{aligned}$$

We obtain

$$\int_{s\leq|\lambda|\leq 2s}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda = O(s^{-2\alpha})\ as\ s\to+\infty.$$

There exists a positive constant $K>0$ such that

$$\int_{s\leq|\lambda|\leq 2s}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda \leq Ks^{-2\alpha}.$$

So that

$$\begin{aligned}\int_{|\lambda|\geq s}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda &= \left(\int_{s\leq|\lambda|\leq 2s}+\int_{2s\leq|\lambda|\leq 4s}+\int_{4s\leq|\lambda|\leq 8s}+.....\right)|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda\\ &\leq Ks^{-2\alpha}+K(2s)^{-2\alpha}+K(4s)^{-2\alpha}+.....\\ &\leq Ks^{-2\alpha}\left(1+2^{-2\alpha}+(2^{-2\alpha})^2+(2^{-2\alpha})^3+......\right)\\ &\leq K_\alpha s^{-2\alpha}.\end{aligned}$$

where $K_\alpha = K(1-2^{-2\alpha})^{-1}$ since $2^{-2\alpha}<1$.

This proves that

$$\int_{|\lambda|\geq s}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda = O(s^{-2\alpha})\ as\ s\to+\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_{|\lambda|\geq s}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda = O(s^{-2\alpha})\ as\ s\to+\infty.$$

We write

$$\int_{-\infty}^{+\infty}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda = \mathrm{I}_1+\mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_{|\lambda|<\frac{1}{h}}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda$$

and

$$\mathrm{I}_2 = \int_{|\lambda|\geq\frac{1}{h}}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda.$$

Estimate the summands I_1 and I_2.

From inequality (1) of Lemma 2.1, we have

$$\begin{aligned}\mathrm{I}_2 &= \int_{|\lambda|\geq\frac{1}{h}}|1-j_{\alpha+2n}(\lambda h|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda\\ &\leq 4\int_{|\lambda|\geq\frac{1}{h}}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda\\ &= O(h^{2\alpha})\end{aligned}$$

Set

$$\phi(x) = \int_x^{+\infty}|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda.$$

An integration by parts, we obtain

$$\begin{aligned}\int_0^x \lambda^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda &= \int_0^x -\lambda^2\phi'(\lambda)d\lambda \\ &= -x^2\phi(x)+2\int_0^x \lambda\phi(\lambda)d\lambda \\ &\leq 2\int_0^x O(\lambda^{1-2\alpha})d\lambda \\ &= O(x^{2-2\alpha}).\end{aligned}$$

We use the formula (6)

$$\begin{aligned}&\int_{-\infty}^{+infty} |1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda \\ = \;& O\left(h^2\int_{|\lambda|<\frac{1}{h}} \lambda^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda\right)+O(h^{2\alpha}) \\ = \;& O(h^2h^{-2+2\alpha})+O(h^{2\alpha}) \\ = \;& O(h^{2\alpha}).\end{aligned}$$

Therefore

$$4m_{\alpha+2n}h^{4n}\int_{-\infty}^{+infty} |1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}(f)(\lambda)|^2|\lambda|^{2\alpha+4n+1}d\lambda = O(h^{2\alpha+4n}).$$

Then

$$\|\mathrm{T}_hf(.)+\mathrm{T}_{-h}f(.)-2h^{2n}f(.)\|^2_{2,\alpha} = O(h^{2\alpha+2n}) \; as \; h\to 0.$$

and this ends the proof. □

Theorem 4.1 in the case $n = 0$ can be found in the work of [5].

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